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Lyapunov type inequalities for even order differential equations with mixed nonlinearities

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Abstract
In the case of oscillatory potentials, we present Lyapunov and Hartman type inequalities for even order differential equations with mixed nonlinearities: $x^{(2n)}(t) + (-1)^{n-1}\sum_{i=1}^{m} q_i(t)|x(t)|^{\alpha_i-1}x(t) = 0$, where $n, m \in \mathbb{N}$ and the nonlinearities satisfy $0 < \alpha_1 < \cdots < \alpha_j < 1 < \alpha_{j+1} < \cdots < \alpha_m < 2$.

MSC: 34C10

Keywords: Lyapunov type inequality; mixed nonlinear; sub-linear; super-linear

1 Introduction

Consider the Hill equation

$$x''(t) + q(t)x(t) = 0; \quad a \le t \le b, \tag{1.1}$$

where $q(t) \in L^1[a,b]$ is a real-valued function. If there exists a nontrivial solution $x(t)$ of (1.1) satisfying the Dirichlet boundary conditions

$$x(a) = x(b) = 0, \tag{1.2}$$

where $a, b \in \mathbb{R}$ with $a < b$ and $x(t) \not\equiv 0$ for $t \in (a,b)$, then the inequality

$$\int_a^b |q(t)|\,dt > 4/(b-a) \tag{1.3}$$

holds. This striking inequality was first proved by Lyapunov [1] and it is known as the Lyapunov inequality. Later Wintner [2] and thereafter some more authors achieved the replacement of the function $|q(t)|$ in (1.3) by the function $q^+(t)$, *i.e.* they obtained the following inequality:

$$\int_a^b q^+(t)\,dt > 4/(b-a), \tag{1.4}$$

where $q^+(t) = \max\{q(t), 0\}$, and the constant 4 in the right hand side of inequalities (1.3) and (1.4) is the best possible largest number (see [1] and [3], Theorem 5.1).

In [3], Hartman obtained an inequality sharper than both (1.3) and (1.4):

$$\int_a^b (b-t)(t-a)q^+(t)\,\mathrm{d}t > (b-a). \tag{1.5}$$

Clearly, (1.5) implies (1.4), since

$$(b-t)(t-a) \le (b-a)^2/4 \tag{1.6}$$

for all $t \in (a,b)$, and equality holds when $t = (a+b)/2$.

It appears that the first generalization of Hartman's result was obtained by Das and Vatsala [4], Theorem 3.1.

Theorem 1.1 (Hartman type inequality) *If $x(t)$ is a nontrivial solution of the equation*

$$x^{(2n)}(t) + (-1)^{n-1}q(t)x(t) = 0; \quad a \le t \le b, \tag{1.7}$$

satisfying the 2-point boundary conditions

$$x^{(k)}(a) = x^{(k)}(b) = 0; \quad k = 0,1,\ldots,n-1, \tag{1.8}$$

where $a, b \in \mathbb{R}$ with $a < b$ are consecutive zeros, then the inequality

$$\int_a^b (b-t)^{2n-1}(t-a)^{2n-1}q^+(t)\,\mathrm{d}t > (2n-1)(n-1)!^2(b-a)^{2n-1} \tag{1.9}$$

holds.

Note that (1.9) generalizes the Lyapunov type inequality

$$\int_a^b q^+(t)\,\mathrm{d}t > 4^{2n-1}(2n-1)(n-1)!^2(b-a)^{1-2n} \tag{1.10}$$

by (1.6) (see also [5], Corollary 3.3).

The proof of Theorem 1.1 is based on the Green's function $\mathcal{G}_n(t,s)$ of the 2-point boundary value problem

$$-x^{(2n)}(t) = 0; \quad a \le t \le b, \tag{1.11}$$

satisfying (1.8), obtained in [4] as follows:

$$\mathcal{G}_n(t,s) = \frac{(-1)^{n-1}}{(2n-1)!}\left(\frac{(t-a)(b-s)}{(b-a)}\right)^n \sum_{j=0}^{n-1}\binom{n-1+j}{j}(s-t)^{n-1-j} \times \left(\frac{(b-t)(s-a)}{(b-a)}\right)^j, \quad t \le s \le b, \tag{1.12}$$

and

$$\mathcal{G}_n(t,s) = \frac{(-1)^{n-1}}{(2n-1)!}\left(\frac{(s-a)(b-t)}{(b-a)}\right)^n \sum_{j=0}^{n-1}\binom{n-1+j}{j}(t-s)^{n-1-j} \\ \times \left(\frac{(t-a)(b-s)}{(b-a)}\right)^j, \quad a \leq s \leq t. \tag{1.13}$$

Note that $(-1)^{n-1}\mathcal{G}_n(t,s) \geq 0$ and

$$\max_{a\leq s\leq b}(-1)^{n-1}\mathcal{G}_n(t,s) = (-1)^{n-1}\mathcal{G}_n(t,t) \tag{1.14}$$

for all $t \in [a,b]$ (see [5]). In fact, in view of the symmetry of $\mathcal{G}_n(t,s)$, (1.14) also implies that

$$\max_{a\leq t\leq b}(-1)^{n-1}\mathcal{G}_n(t,s) = (-1)^{n-1}\mathcal{G}_n(s,s). \tag{1.15}$$

In view of the alternating term $(-1)^{n-1}$ in the Green's function $\mathcal{G}_n(t,s)$, Hartman and Lyapunov type inequalities for the 2-point boundary value problem

$$x^{(2n)}(t) + q(t)x(t) = 0; \quad a \leq t \leq b, \tag{1.16}$$

satisfying the boundary conditions (1.8) can be obtained by replacing the function $q^+(t)$ by $|q(t)|$ in (1.9) and (1.10), respectively.

The Lyapunov inequality and its generalizations have been used successfully in connection with oscillation and Sturmian theory, asymptotic theory, disconjugacy, eigenvalue problems, and various properties of the solutions of (1.1) and related equations; see for instance [2, 3, 6–23] and the references cited therein. For some of its extensions to Hamiltonian systems, higher order differential equations, nonlinear and half-linear differential equations, difference and dynamic equations, and functional and impulsive differential equations, we refer in particular to [10, 11, 24–43].

The aim of our paper is to extend the well-known Lyapunov and Hartman type inequalities for even order nonlinear equations of the form

$$x^{(2n)}(t) + (-1)^{n-1}\sum_{i=1}^{m} q_i(t)\left|x(t)\right|^{\alpha_i-1}x(t) = 0, \tag{1.17}$$

where $n,m \in \mathbb{N}$, the potentials $q_i(t)$, $i = 1,\dots,m$, are real-valued functions and no sign restrictions are imposed on them. Further, the exponents in (1.17) satisfy

$$0 < \alpha_1 < \cdots < \alpha_j < 1 < \alpha_{j+1} < \cdots < \alpha_m < 2. \tag{1.18}$$

It is clear that the two special cases of (1.17) are the even order sub-linear equation

$$x^{(2n)}(t) + (-1)^{n-1}q(t)\left|x(t)\right|^{\gamma-1}x(t) = 0, \quad 0 < \gamma < 1, \tag{1.19}$$

and the even order super-linear equation

$$x^{(2n)}(t) + (-1)^{n-1}p(t)\left|x(t)\right|^{\beta-1}x(t) = 0, \quad 1 < \beta < 2. \tag{1.20}$$

Further, we note that letting $\alpha_i \to 1^-$, $i = 1, \ldots, j$, and $\alpha_i \to 1^+$, $i = j+1, \ldots, m$, in (1.17) results in (1.7) with $q(t) = \sum_{i=1}^m q_i(t)$, *i.e.*,

$$x^{(2n)}(t) + (-1)^{n-1}\left(\sum_{i=1}^m q_i(t)\right)x(t) = 0, \tag{1.21}$$

and as a consequence, our results extend and improve the main results of Das and Vatsala [4], *i.e.* Theorem 1.1, and in particular the classical Lyapunov [1] and Hartman's [3] results.

We further remark that the Lyapunov type inequalities have been studied by many authors, see for instance the survey paper [44] and the references therein, but to the best of our knowledge there are no results in the literature for (1.17), and in particular for (1.19) and (1.20).

2 Main results

Throughout this paper we shall assume that $q_i(t) \in \mathrm{L}^1[a,b]$, $i = 1, \ldots, m$.

We will need the following lemma.

Lemma 2.1 *If A is positive, and B, z are nonnegative, then*

$$Az^2 - Bz^\mu + (2-\mu)\mu^{\mu/(2-\mu)}2^{2/(\mu-2)}A^{-\mu/(2-\mu)}B^{2/(2-\mu)} \geq 0 \tag{2.1}$$

for any $\mu \in (0,2)$ with equality holding if and only if $B = z = 0$.

Proof Let

$$\mathcal{H}(z) = Az^2 - Bz^\mu, \quad z \geq 0, \tag{2.2}$$

where $A > 0$ and $B \geq 0$. Clearly, when $z = 0$ or $B = 0$, (2.1) is obvious. On the other hand, if $B > 0$, then it is easy to see that $\mathcal{H}$ attains its minimum at $z_0 = (\mu A^{-1}B/2)^{1/(2-\mu)}$ and

$$\mathcal{H}_{\min} = -(2-\mu)\mu^{\mu/(2-\mu)}2^{2/(\mu-2)}A^{-\mu/(2-\mu)}B^{2/(2-\mu)}.$$

Thus, (2.1) holds. Note that if $B > 0$, then (2.1) is strict. □

Now we state and prove our first result.

Theorem 2.1 (Hartman type inequality) *If $x(t)$ is a nontrivial solution of* (1.17) *satisfying the* 2*-point boundary conditions* (1.8), *where $a, b \in \mathbb{R}$ with $a < b$ are consecutive zeros, then the inequality*

$$\begin{aligned}&\left(\int_a^b (b-t)^{2n-1}(t-a)^{2n-1}\widehat{Q}_m(t)\,\mathrm{d}t\right)\left(\int_a^b (b-t)^{2n-1}(t-a)^{2n-1}\widetilde{Q}_m(t)\,\mathrm{d}t\right)\\&\quad > (2n-1)^2(n-1)!^4(b-a)^{4n-2}/4\end{aligned} \tag{2.3}$$

holds, where

$$\widehat{Q}_m(t) = \sum_{i=1}^m q_i^+(t) \quad \textit{and} \quad \widetilde{Q}_m(t) = \sum_{i=1}^m \theta_i q_i^+(t)$$

with

$$\theta_i = (2-\alpha_i)\alpha_i^{\alpha_i/(2-\alpha_i)}2^{2/(\alpha_i-2)}. \tag{2.4}$$

Proof Let $x(t)$ be a nontrivial solution of (1.17) satisfying the boundary conditions (1.8), where $a, b \in \mathbb{R}$ with $a < b$ are consecutive zeros. Without loss of generality, we may assume that $x(t) > 0$ for $t \in (a, b)$. In fact, if $x(t) < 0$ for $t \in (a, b)$, then we can consider $-x(t)$, which is also a solution. Then, by using the Green's function of (1.11)-(1.8), $x(t)$ can be expressed as

$$x(t) = \int_a^b (-1)^{n-1}\mathcal{G}_n(t,s)\sum_{i=1}^m q_i(s)x^{\alpha_i}(s)\,\mathrm{d}s. \tag{2.5}$$

Let $x(c) = \max_{t\in(a,b)} x(t)$. Then by (2.1) in Lemma 2.1 with $A = B = 1$, we have

$$x^{\alpha_i}(c) < x^2(c) + \theta_i.$$

Using this in (2.5), we obtain

$$\begin{aligned} x(c) &= \int_a^b (-1)^{n-1}\mathcal{G}_n(c,s)\sum_{i=1}^m q_i(s)x^{\alpha_i}(s)\,\mathrm{d}s \\ &< \int_a^b (-1)^{n-1}\mathcal{G}_n(s,s)\sum_{i=1}^m q_i^+(s)\big[x^2(c)+\theta_i\big]\,\mathrm{d}s, \end{aligned} \tag{2.6}$$

which implies the quadratic inequality

$$\Theta_1 x^2(c) - x(c) + \Theta_2 > 0, \tag{2.7}$$

where

$$\Theta_1 = \int_a^b (-1)^{n-1}\mathcal{G}_n(s,s)\widehat{Q}_m(t)\,\mathrm{d}s$$

and

$$\Theta_2 = \int_a^b (-1)^{n-1}\mathcal{G}_n(s,s)\widetilde{Q}_m(t)\,\mathrm{d}s.$$

But inequality (2.7) is possible if and only if $\Theta_1\Theta_2 > 1/4$. Finally, we note that

$$(-1)^{n-1}\mathcal{G}_n(s,s) = \frac{(b-s)^{2n-1}(s-a)^{2n-1}}{(2n-1)(n-1)!^2(b-a)^{2n-1}}.$$

This completes the proof of Theorem 2.1. □

Next, we prove the following result.

Theorem 2.2 (Lyapunov type inequality) *If $x(t)$ is a nontrivial solution of* (1.17) *satisfying the* 2*-point boundary conditions* (1.8) *where $a, b \in \mathbb{R}$ with $a < b$ are consecutive zeros, then* *the inequality*

$$\left(\int_a^b \widehat{Q}_m(t)\,\mathrm{d}t\right)\left(\int_a^b \widetilde{Q}_m(t)\,\mathrm{d}t\right) > \frac{4^{4n-3}(2n-1)^2(n-1)!^4}{(b-a)^{4n-2}} \tag{2.8}$$

holds, where the functions $\widehat{Q}_m$ *and* $\widetilde{Q}_m$ *are defined in Theorem* 2.1.

Proof In the view of (1.6), (2.3) immediately implies (2.8). □

Remark 1 Since

$$\lim_{\substack{\alpha_i\to 1^+\\(i>j)}} \theta_i = \lim_{\substack{\alpha_i\to 1^-\\(i\le j)}} \theta_i = 1/4,$$

where θ_i is defined in (2.4), it is easy to see that inequalities (2.3) and (2.8) reduce to inequalities (1.9) and (1.10), respectively, with $q^+(t) = \sum_{i=1}^m q_i^+(t)$. Thus, Theorems 2.1 and 2.2 reduce to Theorem 3.1 of Das and Vatsala [4], and Corollary 3.3 of Yang [5], respectively. Moreover, when $n = 1$, they reduce to the classical Lyapunov (1.4) and Hartman (1.5) inequalities with $q^+(t) = \sum_{i=1}^m q_i^+(t)$.

Remark 2 It is of interest to find analogs of Theorems 2.1 and 2.2 for (1.17)-(1.8) without the term $(-1)^{n-1}$, *i.e.*, for the equation

$$x^{(2n)}(t) + \sum_{i=1}^m q_i(t)\left|x(t)\right|^{\alpha_i-1}x(t) = 0 \tag{2.9}$$

satisfying the 2-point boundary conditions (1.8). We state these results in the following.

Proposition 1 *If* $x(t)$ *is a nontrivial solution of* (2.9) *satisfying the* 2*-point boundary conditions* (1.8) *where* $a, b \in \mathbb{R}$ *with* $a < b$ *are consecutive zeros, then the following hold*:

(i) *Hartman type inequality*;

$$\left(\int_a^b (b-t)^{2n-1}(t-a)^{2n-1}\widehat{P}_m(t)\,\mathrm{d}t\right)\left(\int_a^b (b-t)^{2n-1}(t-a)^{2n-1}\widetilde{P}_m(t)\,\mathrm{d}t\right) > (2n-1)^2(n-1)!^4(b-a)^{4n-2}/4.$$

(ii) *Lyapunov type inequality*;

$$\left(\int_a^b \widehat{P}_m(t)\,\mathrm{d}t\right)\left(\int_a^b \widetilde{P}_m(t)\,\mathrm{d}t\right) > \frac{4^{4n-3}(2n-1)^2(n-1)!^4}{(b-a)^{4n-2}},$$

where

$$\widehat{P}_m(t) = \sum_{i=1}^m \left|q_i(t)\right| \quad \textit{and} \quad \widetilde{P}_m(t) = \sum_{i=1}^m \theta_i\left|q_i(t)\right|$$

and θ_i *is defined in* (2.4).

When $q_i(t) = 0$, for all $i = 2, 3, \dots, m-1$, then (1.17) and (2.9) reduce to the equations

$$x^{(2n)}(t) + (-1)^{n-1}p(t)\left|x(t)\right|^{\beta-1}x(t) + (-1)^{n-1}q(t)\left|x(t)\right|^{\gamma-1}x(t) = 0 \tag{2.10}$$

and

$$x^{(2n)}(t) + p(t)\left|x(t)\right|^{\beta-1}x(t) + q(t)\left|x(t)\right|^{\gamma-1}x(t) = 0, \tag{2.11}$$

respectively, where $p(t) = q_m(t)$, $q(t) = q_1(t)$, $\gamma = \alpha_1 \in (0,1)$, and $\beta = \alpha_m \in (1,2)$.

For these equations we have the following corollaries.

Corollary 2.3 *If* $x(t)$ *is a nontrivial solution of* (2.10) *satisfying the* 2*-point boundary conditions* (1.8) *where* $a, b \in \mathbb{R}$ *with* $a < b$ *are consecutive zeros, then the following hold*:

(i) *Hartman type inequality*;

$$\begin{aligned}&\left(\int_a^b (b-t)^{2n-1}(t-a)^{2n-1}\left[p^+(t) + q^+(t)\right]\mathrm{d}t\right)\\ &\quad\times\left(\int_a^b (b-t)^{2n-1}(t-a)^{2n-1}\left[\beta_0 p^+(t) + \gamma_0 q^+(t)\right]\mathrm{d}t\right)\\ &\quad> (2n-1)^2(n-1)!^4(b-a)^{4n-2}/4.\end{aligned}$$

(ii) *Lyapunov type inequality*;

$$\left(\int_a^b \left[p^+(t) + q^+(t)\right]\mathrm{d}t\right)\left(\int_a^b \left[\beta_0 p^+(t) + \gamma_0 q^+(t)\right]\mathrm{d}t\right) > \frac{4^{4n-3}(2n-1)^2(n-1)!^4}{(b-a)^{4n-2}},$$

where

$$\beta_0 = (2-\beta)\beta^{\beta/(2-\beta)}2^{2/(\beta-2)} \quad \text{and} \quad \gamma_0 = (2-\gamma)\gamma^{\gamma/(2-\gamma)}2^{2/(\gamma-2)}.$$

Corollary 2.4 *If* $x(t)$ *is a nontrivial solution of* (2.11) *satisfying the* 2*-point boundary conditions* (1.8) *where* $a, b \in \mathbb{R}$ *with* $a < b$ *are consecutive zeros, then the following hold*:

(i) *Hartman type inequality*;

$$\begin{aligned}&\left(\int_a^b (b-t)^{2n-1}(t-a)^{2n-1}\left[\left|p(t)\right| + \left|q(t)\right|\right]\mathrm{d}t\right)\\ &\quad\times\left(\int_a^b (b-t)^{2n-1}(t-a)^{2n-1}\left[\beta_0\left|p(t)\right| + \gamma_0\left|q(t)\right|\right]\mathrm{d}t\right)\\ &\quad> (2n-1)^2(n-1)!^4(b-a)^{4n-2}/4.\end{aligned}$$

(ii) *Lyapunov type inequality*;

$$\left(\int_a^b \left[\left|p(t)\right| + \left|q(t)\right|\right]\mathrm{d}t\right)\left(\int_a^b \left[\beta_0\left|p(t)\right| + \gamma_0\left|q(t)\right|\right]\mathrm{d}t\right) > \frac{4^{4n-3}(2n-1)^2(n-1)!^4}{(b-a)^{4n-2}},$$

where the constants β_0 *and* γ_0 *are defined in Corollary* 2.3.

Remark 3 Corollary 2.3 is of particular interest since it gives two new results for the even order sub-linear equation (when $p(t) = 0$) and super-linear equation (when $q(t) = 0$), *i.e.*, (1.19) and (1.20). Moreover, classical results can also be obtained by the limiting process $\gamma \to 1^-$ and $\beta \to 1^+$ in inequalities (i) and (ii) given in Corollary 2.3.

3 Some special cases

In this section we consider the situations when the potentials $q_i(t)$, $i = 1, \dots, m$, are either linear, convex, or concave functions.

Corollary 3.1 *Let $q_i(t) = c_i t + d_i$, $i = 1, \dots, m$, in* (1.17) *be positive on $[a, b]$. If $x(t)$ is a nontrivial solution of* (1.17) *satisfying the* 2*-point boundary conditions* (1.8), *where $a, b \in \mathbb{R}$ with $a < b$ are consecutive zeros, then the inequality*

$$\bigl((a+b)\widehat{C}_m + 2\widehat{D}_m\bigr)\bigl((a+b)\widetilde{C}_m + 2\widetilde{D}_m\bigr) > \frac{(2n-1)^2(n-1)!^4(4n-1)!^2}{(2n-1)!^4(b-a)^{4n}} \tag{3.1}$$

holds, where

$$\widehat{C}_m = \sum_{i=1}^m c_i, \qquad \widehat{D}_m = \sum_{i=1}^m d_i, \qquad \widetilde{C}_m = \sum_{i=1}^m \theta_i c_i \quad \textit{and} \quad \widetilde{D}_m = \sum_{i=1}^m \theta_i d_i,$$

and θ_i is the same as in (2.4).

Proof In this special case, we need to compute the integral

$$I := \int_a^b (b-t)^{2n-1}(t-a)^{2n-1}(ct+d)\,\mathrm{d}t$$

for real constants c and d. Writing $ct + d = c(t-a) + ca + d$ and making the substitution $t = (b-a)z + a$, we obtain

$$\begin{aligned} I &= c\int_a^b (b-t)^{2n-1}(t-a)^{2n}\,\mathrm{d}t + (ca+d)\int_a^b (b-t)^{2n-1}(t-a)^{2n-1}\,\mathrm{d}t \\ &= c(b-a)^{4n}\int_0^1 (1-z)^{2n-1}z^{2n}\,\mathrm{d}z + (ca+d)(b-a)^{4n-1}\int_0^1 (1-z)^{2n-1}z^{2n-1}\,\mathrm{d}z \\ &= c(b-a)^{4n}B(2n, 2n+1) + (ca+d)(b-a)^{4n-1}B(2n, 2n) \\ &= B(2n, 2n)\bigl[c(b-a)/2 + ca + d\bigr](b-a)^{4n-1}, \end{aligned}$$

where $B(\cdot, \cdot)$ is the Beta function. However, since

$$B(2n, 2n) = \frac{\Gamma^2(2n)}{\Gamma(4n)} = \frac{(2n-1)!^2}{(4n-1)!}$$

we have

$$I = \frac{(2n-1)!^2}{(4n-1)!}\bigl[c(a+b)/2 + d\bigr](b-a)^{4n-1}.$$

Using this in (2.3) with $q_i(t) = c_i t + d_i$ the result follows. □

Corollary 3.2 *Let $q_i(t)$, $i = 1, \dots, m$, in* (1.17) *be continuous, positive, and convex on* $[a,b]$. *If $x(t)$ is a nontrivial solution of* (1.17) *satisfying the* 2*-point boundary conditions* (1.8), *where $a, b \in \mathbb{R}$ with $a < b$ are consecutive zeros, then the inequality*

$$\sum_{i=1}^{m} \bigl[q_i(b) + q_i(a)\bigr] > \frac{(2n-1)^2(n-1)!^4(4n-1)!^2}{(2n-1)!^4(b-a)^{4n}} \tag{3.2}$$

holds.

Corollary 3.3 *Let $q_i(t)$, $i = 1, \dots, m$, in* (1.17) *be continuous, positive, and concave on* $[a,b]$. *If $x(t)$ is a nontrivial solution of* (1.17) *satisfying the* 2*-point boundary conditions* (1.8), *where $a, b \in \mathbb{R}$ with $a < b$ are consecutive zeros, then the inequality*

$$\sum_{i=1}^{m} q_i\bigl[(a+b)/2\bigr] > \frac{(2n-1)^2(n-1)!^4(4n-1)!^2}{(2n-1)!^4(b-a)^{4n}} \tag{3.3}$$

holds.

The proofs of Corollaries 3.2 and 3.3 are similar to those of Propositions 4.2 and 4.3 of Das and Vatsala [4], and hence they are omitted.

Finally, we conclude this paper with the following remark. When $n = 1$, the results obtained in this paper for (1.17) (or (2.11)) can easily be extended to the second order equations

$$x''(t) \pm p(t)\bigl|x(t)\bigr|^{\beta-1}x(t) \mp q(t)\bigl|x(t)\bigr|^{\gamma-1}x(t) = 0, \tag{3.4}$$

i.e., for Emden-Fowler sub-linear and Emden-Fowler super-linear equations with positive and negative coefficients. The formulations of these results are left to the reader.

It will be of interest to find similar results for the even order mixed nonlinear equations of the form (1.17) for some $\alpha_k \geq 2$, or the super-linear equation (1.20) for $\beta \in [2, \infty)$. In fact, the case when $n = 1$ (Emden-Fowler super-linear) is of immense interest.

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
All authors contributed equally to the writing of this paper. All authors read and approved the final manuscript.

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