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Triebel-Lizorkin space boundedness of rough singular integrals associated to surfaces

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Abstract

In the present paper, we consider the boundedness of the rough singular integral operator $T_{\Omega,h,\phi}$ along a surface $\Gamma = \{x = \phi(|y|)y/|y|\}$ on the Triebel-Lizorkin space $\dot{F}^{\alpha}_{p,q}(\mathbb{R}^n)$ for $\Omega \in H^1(S^{n-1})$ and Ω belonging to some class $W\mathcal{F}_\alpha(S^{n-1})$, which relates to the Grafakos-Stefanov class.

MSC: Primary 42B20; secondary 42B25; 47G10

Keywords: singular integrals; Triebel-Lizorkin spaces; rough kernel

1 Introduction

Let $\mathbb{R}^n$ $(n \geq 2)$ be the n-dimensional Euclidean space and S^{n-1} be the unit sphere in $\mathbb{R}^n$ equipped with the induced Lebesgue measure $d\sigma = d\sigma(\cdot)$. Suppose that $\Omega \in L^1(S^{n-1})$ satisfies the cancelation condition

$$\int_{S^{n-1}} \Omega(y')\, d\sigma(y') = 0. \tag{1.1}$$

For a suitable function ϕ and a measurable function h on $[0,\infty)$, we denote by $T_{\Omega,\phi,h}$ the singular integral operator along the surface

$$\Gamma = \{x = \phi(|y|)y' : y \in \mathbb{R}^n\}$$

defined as follows:

$$T_{\Omega,h,\phi}f(x) = \text{p.v.} \int_{\mathbb{R}^n} \frac{h(|y|)\Omega(y')}{|y|^n} f(x - \phi(|y|)y')\, dy \tag{1.2}$$

for f in the Schwartz class $\mathcal{S}(\mathbb{R}^n)$. If $\phi = 1$, then $T_{\Omega,h,\phi}$ is the classical singular integral operator $T_{\Omega,h}$, which is defined by

$$T_{\Omega,h}f(x) = \text{p.v.} \int_{\mathbb{R}^n} \frac{h(|y|)\Omega(y')}{|y|^n} f(x-y)\, dy. \tag{1.3}$$

When $h \equiv 1$, we denote simply $T_{\Omega,h,\phi}$ and $T_{\Omega,h}$ by $T_{\Omega,\phi}$ and T_Ω, respectively.

The L^p boundedness of singular integrals along the surface has attracted the attention of many authors [1–3], *etc.* There are several papers concerning rough kernels associated to surfaces as above [4–6]. As one of them, we count the following one.

Theorem A ([5]) *Let $h \in \Delta_\gamma$ for some $\gamma \geq 2$, $1 < p < \infty$, $\Omega \in H^1(S^{n-1})$. Let ϕ be a nonnegative C^1 function on $(0,\infty)$ satisfying*

(i) *$\phi(t)$ is strictly increasing and $\phi(2t) \geq \lambda\phi(t)$ for all $t > 0$ and some $\lambda > 1$,*

(ii) *$\phi(t)$ satisfies a doubling condition $\phi(2t) \leq c\phi(t)$ for all $t > 0$ and some $c > 1$,*

(iii) *$\phi'(t) \geq C_1\phi(t)/t$ for all $t > 0$ and some C_1.*

Then $T_{\Omega,h,\phi}$ is bounded on $L^p(\mathbb{R}^n)$.

This is, in fact, stated in the more general setting, *i.e.*, for a weighted case (Theorem 1 and Corollary 1 in [5]), but we state this as above for our purpose and for the sake of simplicity. We note here that condition (i) follows from (iii).

On the other hand, Triebel-Lizorkin space boundedness of rough singular integrals was also investigated by many authors, see [7, 8] and [9].

Before stating the following result, let us recall the definitions of some function spaces. First we give the definition of the *Hardy space* $H^1(S^{n-1})$:

$$H^1(S^{n-1}) = \left\{\omega \in L^1(S^{n-1}) \;\middle|\; \|f\|_{H^1(S^{n-1})} = \left\| \sup_{0 \leq r < 1} \left| \int_{S^{n-1}} \omega(y') P_{r(\cdot)}(y')\, d\sigma(y') \right| \right\|_{L^1(S^{n-1})} < \infty \right\},$$

where $P_{ry'}(x')$ denotes the Poisson kernel on S^{n-1} defined by

$$P_{ry'}(x') = \frac{1-r^2}{|ry' - x'|^n}, \quad 0 \leq r < 1 \text{ and } x', y' \in S^{n-1}.$$

For $1 \leq \gamma \leq \infty$, $\Delta_\gamma(\mathbb{R}_+)$ is the collection of all measurable functions $h : [0,\infty) \to \mathbb{C}$ satisfying

$$\|h\|_{\Delta_\gamma} = \sup_{R>0} \left(\frac{1}{R} \int_0^R |h(t)|^\gamma\, dt \right)^{1/\gamma} < \infty.$$

Note that

$$L^\infty(\mathbb{R}_+) = \Delta_\infty(\mathbb{R}_+) \subset \Delta_\beta(\mathbb{R}_+) \subset \Delta_\alpha(\mathbb{R}_+) \quad \text{for } \alpha < \beta,$$

and all these inclusions are proper.

As a result of boundedness on Triebel-Lizorkin spaces, we cite the following one, which is somewhat different from our setting, but closely related.

Theorem B ([9]) *Let $\Omega \in H^1(S^{n-1})$ satisfy the cancelation condition* (1.1) *and $h \in \Delta_\gamma$ for some $1 < \gamma \leq \infty$. Let $P = (P_1, P_2, \ldots, P_d)$ be real polynomials in y. Then, for the singular integral*

$$T_{\Omega,P,h}f(x) = \text{p.v.} \int_{\mathbb{R}^n} \frac{h(|y|)\Omega(y')}{|y|^n} f\bigl(x - P(y)\bigr)\, dy$$

(i) *for $\alpha \in \mathbb{R}$ and $|\frac{1}{p} - \frac{1}{2}| < \min(\frac{1}{2}, \frac{1}{\gamma'})$ and $|\frac{1}{q} - \frac{1}{2}| < \min(\frac{1}{2}, \frac{1}{\gamma'})$, there exists a constant $C > 0$ such that $\|T_{\Omega,P,h}f\|_{\dot{F}^\alpha_{p,q}(\mathbb{R}^d)} \leq C\|f\|_{\dot{F}^\alpha_{p,q}(\mathbb{R}^d)}$;*

(ii) *for* $\alpha \in \mathbb{R}$ *and* $|\frac{1}{p} - \frac{1}{2}| < \min(\frac{1}{2}, \frac{1}{\gamma'})$ *and* $1 < q < \infty$, *there exists a constant* $C > 0$ *such that* $\|T_{\Omega,P,h}f\|_{\dot{B}^{\alpha}_{p,q}(\mathbb{R}^d)} \leq C\|f\|_{\dot{B}^{\alpha}_{p,q}(\mathbb{R}^d)}$.

Remark 1 We think that there is a gap in the proof of part (i) in the above theorem. Their proof works in the same region as in our Theorem 1.1 below.

Besides $H^1(S^{n-1})$, there is another class of kernels which leads to L^p and Triebel-Lizorkin space boundedness of singular integral operators $T_{\Omega,h}$. It is closely related to the class $\mathcal{F}_\alpha$ introduced by Grafakos and Stefanov [10]. We say $\Omega \in W\mathcal{F}_\beta = W\mathcal{F}_\beta(S^{n-1})$ if

$$\|\Omega\|_{W\mathcal{F}_\beta} := \sup_{\xi' \in S^{n-1}} \left(\int_{S^{n-1}} \int_{S^{n-1}} \left|\Omega(y')\Omega(z')\right| \log^\beta \frac{2e}{|(y'-z')\cdot \xi'|} \, d\sigma(y') \, d\sigma(z') \right)^{\frac{1}{2}} < \infty. \tag{1.4}$$

We note that $\bigcup_{r>1} L^r(S^{n-1}) \subset W\mathcal{F}_{\beta_2}(S^{n-1}) \subset W\mathcal{F}_{\beta_1}(S^{n-1})$ for $0 < \beta_1 < \beta_2 < \infty$.

About the inclusion relation between $\mathcal{F}_{\beta_1}(S^{n-1})$ and $W\mathcal{F}_{\beta_2}(S^{n-1})$, the following is known: when $n = 2$, Lemma 1 in [11] shows $\mathcal{F}_\beta(S^1) \subset W\mathcal{F}_\beta(S^1)$. It is also known that $W\mathcal{F}_{2\alpha}(S^1) \setminus (\mathcal{F}_\alpha(S^1) \cup H^1(S^1)) \neq \emptyset$, *cf.* [12].

Theorem C ([12]) *Let* $h \in \Delta_\gamma$ *for some* $1 < \gamma \leq \infty$. *Suppose that* $\Omega \in W\mathcal{F}_\beta = W\mathcal{F}_\beta(S^{n-1})$ *for some* $\beta > \max(\gamma', 2)$, *and it satisfies the cancelation condition* (1.1). *Then the singular integral operator* $T_{\Omega,h}$ *is bounded on* $\dot{F}^{\alpha}_{p,q}(\mathbb{R}^n)$ *if* $\alpha \in \mathbb{R}$, *and* $(1/p, 1/q)$ *belongs to the interior of the parallelogram* $P_1P_2P_3P_4$, *where* $P_1 = (\frac{\max(\gamma',2)}{2\beta}, \frac{\max(\gamma',2)}{2\beta})$, $P_2 = (\frac{1}{\gamma'} + \frac{\max(\gamma',2)}{2\beta}(\frac{1}{\gamma} - \frac{1}{\gamma'}), \frac{\max(\gamma',2)}{2\beta})$, $P_3 = (1 - \frac{\max(\gamma',2)}{2\beta}, 1 - \frac{\max(\gamma',2)}{2\beta})$, *and* $P_4 = (\frac{1}{\gamma} - \frac{\max(\gamma',2)}{2\beta}(\frac{1}{\gamma} - \frac{1}{\gamma'}), 1 - \frac{\max(\gamma',2)}{2\beta})$.

Let us recall the definitions of the *homogeneous Triebel-Lizorkin spaces* $\dot{F}^{\alpha}_{p,q} = \dot{F}^{\alpha}_{p,q}(\mathbb{R}^n)$ and the *homogeneous Besov spaces* $\dot{B}^{\alpha}_{p,q} = \dot{B}^{\alpha}_{p,q}(\mathbb{R}^n)$. For $0 < p, q \leq \infty$ ($p \neq \infty$) and $\alpha \in \mathbb{R}$, $\dot{F}^{\alpha}_{p,q}(\mathbb{R}^n)$ is defined by

$$\dot{F}^{\alpha}_{p,q}(\mathbb{R}^n) = \left\{ f \in \mathcal{S}'(\mathbb{R}^n) : \|f\|_{\dot{F}^{\alpha}_{p,q}} = \left\| \left(\sum_{k\in\mathbb{Z}} 2^{k\alpha q} |\Psi_k * f|^q \right)^{1/q} \right\|_{L^p} < \infty \right\} \tag{1.5}$$

and $\dot{B}^{\alpha,q}_{p}(\mathbb{R}^n)$ is defined by

$$\dot{B}^{\alpha}_{p,q}(\mathbb{R}^n) = \left\{ f \in \mathcal{S}'(\mathbb{R}^n) : \|f\|_{\dot{B}^{\alpha}_{p,q}} = \left(\sum_{k\in\mathbb{Z}} 2^{k\alpha q} \|\Psi_k * f\|^q_{L^p} \right)^{1/q} < \infty \right\}, \tag{1.6}$$

where $\mathcal{S}'(\mathbb{R}^n)$ denotes the tempered distribution class on $\mathbb{R}^n$, $\widehat{\Psi}_k(\xi) = \Phi(2^{-k}\xi)$ for $k \in \mathbb{Z}$ and $\Phi \in C^\infty_c(\mathbb{R}^n)$ is a radial function satisfying the following conditions:

$$\begin{aligned} &\text{(i)} \quad 0 \leq \Phi \leq 1; \\ &\text{(ii)} \quad \operatorname{supp} \Phi \subset \{\xi : 1/2 \leq |\xi| \leq 2\}; \\ &\text{(iii)} \quad \Phi > c > 0 \quad \text{if } 3/5 \leq |\xi| \leq 5/3; \\ &\text{(iv)} \quad \sum_{j\in\mathbb{Z}} \Phi(2^{-j}\xi) = 1 \quad (\xi \neq 0). \end{aligned} \tag{1.7}$$

The inhomogeneous versions of Triebel-Lizorkin space and Besov space, which are denoted by $F^{\alpha}_{p,q}(\mathbb{R}^n)$ and $B^{\alpha}_{p,q}(\mathbb{R}^n)$ respectively, are obtained by adding the term $\|\Phi_0 * f\|_p$ to the right-hand side of (1.5) or (1.6) with $\sum_{k\in\mathbb{Z}}$ replaced by $\sum_{k=0}^{\infty}$, where $\Phi_0 \in \mathcal{S}(\mathbb{R}^n)$, $\operatorname{supp}\widehat{\Phi}_0 \subset \{\xi : |\xi| \le 2\}$, and $\widehat{\Phi}_0(\xi) > c > 0$ if $|\xi| \le 5/3$.

The following properties of the Triebel-Lizorkin space and the Besov space are well known. Let $1 < p, q < \infty$, $\alpha \in \mathbb{R}$, and $1/p + 1/p' = 1$, $1/q + 1/q' = 1$:

$$\begin{aligned}
&\text{(a)}\quad \dot{F}^0_{2,2} = \dot{B}^0_{2,2} = L^2, \qquad \dot{F}^0_{p,2} = L^p \quad \text{and} \\
&\qquad \dot{F}^{\alpha}_{p,p} = \dot{B}^{\alpha}_{p,p} \quad \text{for } 1 < p < \infty \quad \text{and} \quad \dot{F}^0_{\infty,2} = \mathrm{BMO}; \\
&\text{(b)}\quad F^{\alpha}_{p,q} \sim \dot{F}^{\alpha}_{p,q} \cap L^p \quad \text{and} \quad \|f\|_{F^{\alpha}_{p,q}} \sim \|f\|_{\dot{F}^{\alpha}_{p,q}} + \|f\|_{L^p} \quad (\alpha > 0); \\
&\text{(c)}\quad B^{\alpha}_{p,q} \sim \dot{B}^{\alpha}_{p,q} \cap L^p \quad \text{and} \quad \|f\|_{B^{\alpha}_{p,q}} \sim \|f\|_{\dot{B}^{\alpha}_{p,q}} + \|f\|_{L^p} \quad (\alpha > 0); \\
&\text{(d)}\quad \big(\dot{F}^{\alpha}_{p,q}\big)^* = \dot{F}^{-\alpha}_{p',q'} \quad \text{and} \quad \big(F^{\alpha}_{p,q}\big)^* = F^{-\alpha}_{p',q'}; \\
&\text{(e)}\quad \big(\dot{B}^{\alpha}_{p,q}\big)^* = \dot{B}^{-\alpha}_{p',q'} \quad \text{and} \quad \big(B^{\alpha}_{p,q}\big)^* = B^{-\alpha}_{p',q'}; \\
&\text{(f)}\quad \big(\dot{F}^{\alpha_1}_{p,q_1}, \dot{F}^{\alpha_2}_{p,q_2}\big)_{\theta,q} = \dot{B}^{\alpha}_{p,q} \\
&\qquad \big(\alpha_1 \ne \alpha_2, 0 < p < \infty, 0 < q, q_1, q_2 \le \infty, \alpha = (1-\theta)\alpha_1 + \theta\alpha_2, 0 < \theta < 1\big).
\end{aligned} \tag{1.8}$$

See [13] and [14] for more properties of $\dot{F}^{\alpha}_{p,q}$ and $\dot{B}^{\alpha}_{p,q}$. See Triebel [14], p.64 and p.244, for (f).

Now we can state our first result.

Theorem 1.1 *Let ϕ be a positive increasing function on $(0,\infty)$ satisfying*

$$\phi(2t) \le c_1\phi(t) \quad (t > 0) \text{ for some } c_1 > 1 \tag{1.9}$$

and

$$\varphi(t) = \phi(t)/\big(t\phi'(t)\big) \in L^{\infty}(0,\infty). \tag{1.10}$$

Let $h \in \Delta_\gamma$ for some $1 < \gamma \le \infty$. Suppose $\Omega \in H^1(S^{n-1})$ satisfying the cancelation condition (1.1). *Then*

- (i) *$T_{\Omega,h,\phi}$ is bounded on $\dot{F}^{\alpha}_{p,q}(\mathbb{R}^n)$ for $\alpha \in \mathbb{R}$ and p, q with $(\frac{1}{p}, \frac{1}{q})$ belonging to the interior of the octagon $P_1P_2R_2P_3P_4P_5R_4P_6$ (hexagon $P_1P_2P_3P_4P_5P_6$ in the case $1 < \gamma \le 2$), where $P_1 = (\frac{1}{2} - \frac{1}{\max\{2,\gamma'\}}, \frac{1}{2} - \frac{1}{\max\{2,\gamma'\}})$, $P_2 = (\frac{1}{2}, \frac{1}{2} - \frac{1}{\max\{2,\gamma'\}})$, $P_3 = (\frac{1}{2} + \frac{1}{\max\{2,\gamma'\}}, \frac{1}{2})$, $P_4 = (\frac{1}{2} + \frac{1}{\max\{2,\gamma'\}}, \frac{1}{2} + \frac{1}{\max\{2,\gamma'\}})$, $P_5 = (\frac{1}{2}, \frac{1}{2} + \frac{1}{\max\{2,\gamma'\}})$, $P_6 = (\frac{1}{2} - \frac{1}{\max\{2,\gamma'\}}, \frac{1}{2})$, $R_2 = (1 - \frac{1}{2\gamma}, \frac{1}{2\gamma})$, and $R_4 = (\frac{1}{2\gamma}, 1 - \frac{1}{2\gamma})$;*
- (ii) *$T_{\Omega,h,\phi}$ is bounded on $\dot{B}^{\alpha}_{p,q}(\mathbb{R}^n)$ for $\alpha \in \mathbb{R}$ and p, q satisfying $|\frac{1}{2} - \frac{1}{p}| < \min\{\frac{1}{2}, \frac{1}{\gamma'}\}$ and $1 < q < \infty$.*

See Figures 1 and 2 for the conclusion (i) of Theorem 1.1.

Example 1 As typical examples of ϕ satisfying conditions (1.9) and (1.10), we list the following three: $t^{\alpha}\log^{\beta}(1+t)$ $(\alpha > 0, \beta \ge 0)$, $(2t^2 - 2t + 1)t^{1+\alpha}$ $(\alpha \ge 0)$, and $\phi(t) = 2t^2 + t$ $(0 < t < \frac{\pi}{2})$, $\phi(t) = 2t^2 + t\sin t$ $(t \ge \frac{\pi}{2})$. Note that linear combinations with positive coefficients of functions ϕ's satisfying the above two conditions also satisfy them, *cf.* [15].

Figure 1 (1 < γ < 2).

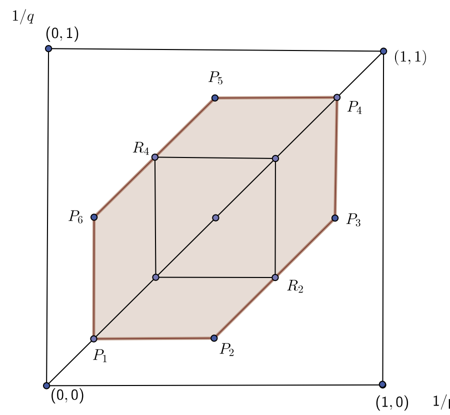
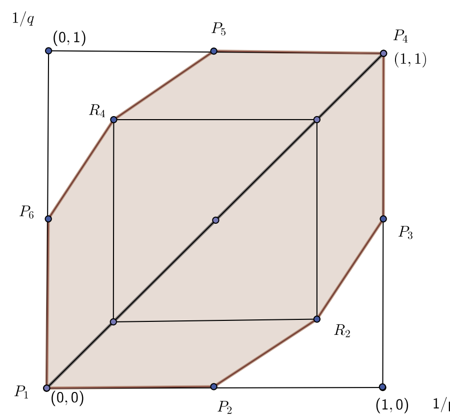


Figure 2 (2 ≤ γ ≤ ∞).



We shall state our following result, which relates to two function spaces $L(\log L)(S^{n-1})$ and the block spaces $B_q^{(0,0)}(S^{n-1})$. Let $L(\log L)^\alpha(S^{n-1})$ (for $\alpha > 0$) denote the class of all measurable functions Ω on S^{n-1} which satisfy

$$\|\Omega\|_{L(\log L)^\alpha(S^{n-1})} = \int_{S^{n-1}} \bigl|\Omega(y')\bigr| \log^\alpha\bigl(2 + \bigl|\Omega(y')\bigr|\bigr)\, d\sigma(y') < \infty.$$

Denote by $L(\log L)(S^{n-1})$ for $L(\log L)^1(S^{n-1})$. A well-known fact is $L(\log L)(S^{n-1}) \subset H^1(S^{n-1})$.

Next, we turn to the block space $B_q^{(0,\nu)}(S^{n-1})$. A q-block on S^{n-1} is an $L^q(S^{n-1})$ $(1 < q \le \infty)$ function b which satisfies

$$\begin{aligned} &\text{(i)} \quad \operatorname{supp} b \subset I; \\ &\text{(ii)} \quad \|b\|_q \le |I|^{-1/q'}, \end{aligned} \tag{1.11}$$

where $|I| = \sigma(I)$, and $I = B(x_0', \theta_0) \cap S^{n-1}$ is a cap on S^{n-1} for some $x_0' \in S^{n-1}$ and $\theta_0 \in (0,1]$. For $1 < q \le \infty$ and $\nu > -1$, the *block space* $B_q^{(0,\nu)}(S^{n-1})$ is defined by

$$B_q^{(0,\nu)}\bigl(S^{n-1}\bigr) = \left\{\Omega \in L^1\bigl(S^{n-1}\bigr); \Omega = \sum_{j=1}^\infty \lambda_j b_j, M_q^{(0,\nu)}\bigl(\{\lambda_j\}\bigr) < \infty\right\}, \tag{1.12}$$

where $\lambda_j \in \mathbb{C}$ and b_j is a q-block supported on a cap I_j on S^{n-1}, and

$$M_q^{(0,\nu)}(\{\lambda_j\}) = \sum_{j=1}^{\infty} |\lambda_j| \{1 + \log^{(\nu+1)}(|I_j|^{-1})\}. \tag{1.13}$$

For $\Omega \in B_q^{(0,\nu)}(S^{n-1})$, denote

$$\|\Omega\|_{B_q^{(0,\nu)}(S^{n-1})} = \inf\left\{ M_q^{(0,\nu)}(\{\lambda_j\}); \Omega = \sum_{j=1}^{\infty} \lambda_j b_j, b_j \text{ is a } q\text{-block} \right\}.$$

Then $\|\cdot\|_{B_q^{(0,\nu)}(S^{n-1})}$ is a norm on the space $B_q^{(0,\nu)}(S^{n-1})$, and $(B_q^{(0,\nu)}(S^{n-1}), \|\cdot\|_{B_q^{(0,\nu)}(S^{n-1})})$ is a Banach space.

Historically, the block spaces in $\mathbb{R}^n$ originated in the work of Taibleson and Weiss on the convergence of the Fourier series in connection with the developments of the real Hardy spaces. The block spaces on S^{n-1} were introduced by Jiang and Lu [16] in studying the homogeneous singular integral operators. For further information about the theory of spaces generated by blocks and its applications to harmonic analysis, see the book [17] and survey article [18]. The following inclusion relations are known:

$$\begin{aligned}
&\text{(a)}\quad B_q^{(0,\nu_1)}(S^{n-1}) \subset B_q^{(0,\nu_2)}(S^{n-1}) \quad \text{if } \nu_1 > \nu_2 > -1;\\
&\text{(b)}\quad B_{q_1}^{(0,\nu)}(S^{n-1}) \subset B_{q_2}^{(0,\nu)}(S^{n-1}) \quad \text{if } 1 < q_2 < q_1 \text{ for any } \nu > -1;\\
&\text{(c)}\quad \bigcup_{p>1} L^p(S^{n-1}) \subset B_q^{(0,\nu)}(S^{n-1}) \quad \text{for any } q > 1, \nu > -1;\\
&\text{(d)}\quad \bigcup_{q>1} B_q^{(0,\nu)}(S^{n-1}) \not\subset \bigcup_{q>1} L^q(S^{n-1}) \quad \text{for any } \nu > -1;\\
&\text{(e)}\quad B_q^{(0,\nu)}(S^{n-1}) \subset H^1(S^{n-1}) + L(\log L)^{1+\nu}(S^{n-1}) \quad \text{for any } q > 1, \nu > -1;\\
&\text{(f)}\quad \bigcup_{q>1} B_q^{(0,0)}(S^{n-1}) \subset H^1(S^{n-1}).
\end{aligned} \tag{1.14}$$

The following theorem shows that if Ω belongs to $L \log L(S^{n-1})$ or block spaces, then we can get better results than Theorem 1.1.

Theorem 1.2 *Let ϕ be a positive increasing function on $(0,\infty)$ satisfying the same condition as in Theorem* 1.1. *Let $h \in \Delta_\gamma$ for some $1 < \gamma \le \infty$, and $\Omega \in L^1(S^{n-1})$ satisfy the cancelation condition* (1.1). *Then if $\Omega \in L(\log L)(S^{n-1}) \cup (\bigcup_{1<q<\infty} B_q^{(0,0)}(S^{n-1}))$, then*

(i) *$T_{\Omega,h,\phi}$ is bounded on $\dot{F}^\alpha_{p,q}(\mathbb{R}^n)$ for $\alpha \in \mathbb{R}$ and p, q with $(\frac{1}{p}, \frac{1}{q})$ belonging to the interior of the octagon $Q_1Q_2R_2P_3Q_3Q_4R_4P_6$ (hexagon $Q_1Q_2P_3Q_3Q_4P_6$ in the case $1 < \gamma \le 2$), where $Q_1 = (0,0)$, $Q_2 = (\frac{1}{\gamma'}, 0)$, $Q_3 = (1,1)$, $Q_4 = (\frac{1}{\gamma}, 1)$, $P_3 = (\frac{1}{2} + \frac{1}{\max\{2,\gamma'\}}, \frac{1}{2})$, $P_6 = (\frac{1}{2}1\frac{1}{\max\{2,\gamma'\}}, \frac{1}{2})$, $R_2 = (1 - \frac{1}{2\gamma}, \frac{1}{2\gamma})$, and $R_4 = (\frac{1}{2\gamma}, 1 - \frac{1}{2\gamma})$;*

(ii) *$T_{\Omega,h,\phi}$ is bounded on $\dot{B}^\alpha_{p,q}(\mathbb{R}^n)$ for $\alpha \in \mathbb{R}$ and $1 < p, q < \infty$.*

See Figures 3 and 4 for the conclusion of Theorem 1.2 for the cases $1 < \gamma < 2$ and $2 \le \gamma < \infty$, respectively.

As a corresponding result to Theorem C, we have the following theorem.

Figure 3 (1 < γ < 2).

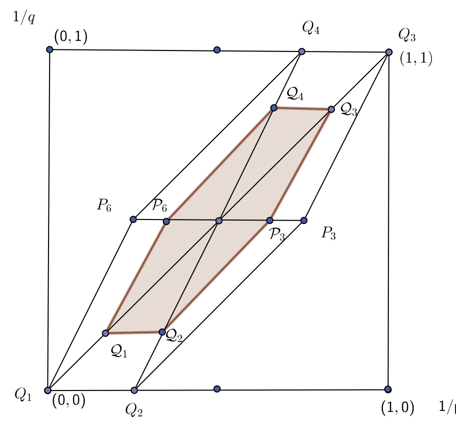
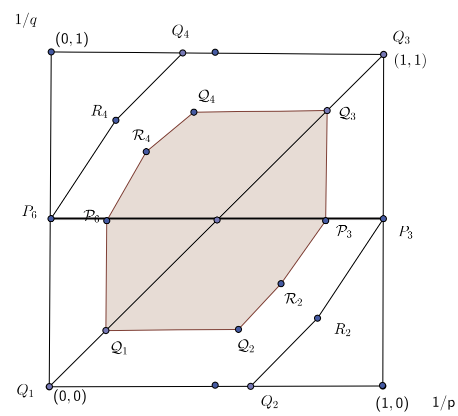


Figure 4 (2 ≤ γ ≤ ∞).



Theorem 1.3 *Let ϕ be a positive increasing function on $(0,\infty)$ satisfying the same condition as in Theorem* 1.1. *Let $h \in \Delta_\gamma$ for some $1 < \gamma \le \infty$. Suppose $\Omega \in W\mathcal{F}_\beta = W\mathcal{F}_\beta(S^{n-1})$ for some $\beta > \max(\gamma',2)$, and it satisfies the cancelation condition* (1.1). *Then*

(i) *the singular integral operator $T_{\Omega,h,\phi}$ is bounded on $\dot{F}^\alpha_{p,q}(\mathbb{R}^n)$, if $\alpha \in \mathbb{R}$ and $(\frac{1}{p},\frac{1}{q})$ belongs to the interior of the octagon $\mathcal{Q}_1\mathcal{Q}_2\mathcal{R}_2\mathcal{P}_3\mathcal{Q}_3\mathcal{Q}_4\mathcal{R}_4\mathcal{P}_6$ (hexagon $\mathcal{Q}_1\mathcal{Q}_2\mathcal{P}_3\mathcal{Q}_3\mathcal{Q}_4\mathcal{P}_6$ in the case $1 < \gamma \le 2$), where $\mathcal{Q}_1 = (\frac{\max(\gamma',2)}{2\beta}, \frac{\max(\gamma',2)}{2\beta})$,*
$\mathcal{Q}_2 = (\frac{1}{\gamma'} + \frac{\max(\gamma',2)}{\beta}(\frac{1}{2} - \frac{1}{\gamma'}), \frac{\max(\gamma',2)}{2\beta})$, $\mathcal{P}_3 = (\frac{1}{2} + \frac{1}{\max(\gamma',2)} - \frac{1}{\beta}, \frac{1}{2})$,
$\mathcal{Q}_3 = (1 - \frac{\max(\gamma',2)}{2\beta}, 1 - \frac{\max(\gamma',2)}{2\beta})$, $\mathcal{Q}_4 = (\frac{1}{\gamma} - \frac{\max(\gamma',2)}{\beta}(\frac{1}{\gamma} - \frac{1}{2}), 1 - \frac{\max(\gamma',2)}{2\beta})$,
$\mathcal{P}_6 = (\frac{1}{2} - \frac{1}{\max(\gamma',2)} + \frac{1}{\beta}, \frac{1}{2})$, $\mathcal{R}_2 = (1 - \frac{1}{2\gamma} - \frac{\max(\gamma',2)}{2\beta\gamma'}, \frac{1}{2\gamma} + \frac{\max(\gamma',2)}{2\beta\gamma'})$, *and*
$\mathcal{R}_4 = (\frac{1}{2\gamma} + \frac{\max(\gamma',2)}{2\beta\gamma'}, 1 - \frac{1}{2\gamma} - \frac{\max(\gamma',2)}{2\beta\gamma'})$;

(ii) *$T_{\Omega,h,\phi}$ is bounded on $\dot{B}^\alpha_{p,q}(\mathbb{R}^n)$, if $\alpha \in \mathbb{R}$, $\frac{\max(\gamma',2)}{2\beta} < p < 1 - \frac{\max(\gamma',2)}{2\beta}$ and $1 < q < \infty$.*

This improves Theorem C sufficiently. See Figures 3 and 4 for the conclusion (i) of Theorem 1.3.

The proofs of Theorems 1.1 and 1.3 will be given in Sections 2 and 3, respectively, and the proof of Theorem 1.2 will be given in Section 4. The letter C will denote a positive constant that may vary at each occurrence but is independent of the essential variables.

2 Proof of Theorem 1.1

2.1 Some lemmas

In [19], the following atom-decomposition of $H^1(S^{n-1})$ was given. If $\Omega \in H^1(S^{n-1})$ satisfying (1.1), then

$$\Omega = \sum_{j=1}^{\infty} \lambda_j a_j, \tag{2.1}$$

where $\sum_{j=1}^{\infty} |\lambda_j| \le C\|\Omega\|_{H^1(S^{n-1})}$ and each a_j is a regular $H^1(S^{n-1})$ atom. A function a on S^{n-1} is called regular ∞-atom in $H^1(S^{n-1})$ if there exist $\zeta \in S^{n-1}$ and $\rho \in (0,2]$ such that

(i) $\operatorname{supp}(a) \subset S^{n-1} \cap B(\zeta,\rho)$, where $B(\zeta,\rho) = \{y \in R^n : |y-\zeta| < \rho\}$;

(ii) $\|a\|_{L^\infty} \le \rho^{-n+1}$;

(iii) $\int_{S^{n-1}} a(y)\,d\sigma(y) = 0$.

Let a be a regular ∞-atom. When $n \ge 3$, set

$$E_a(s,\xi') = (1-s^2)^{\frac{n-3}{2}} \chi_{(-1,1)}(s) \int_{S^{n-2}} a(s, \sqrt{1-s^2}\tilde{y})\,d\sigma(\tilde{y}), \tag{2.2}$$

and when $n = 2$, set

$$e_a(s,\xi') = \frac{1}{\sqrt{1-s^2}} \chi_{(-1,1)}(s)\bigl[a(s,\sqrt{1-s^2}) + a(s,-\sqrt{1-s^2})\bigr]. \tag{2.3}$$

Next we prepare two lemmas, whose proofs can be found in Fan and Pan [4].

Lemma 2.1 *Let Ω be a regular ∞-atom in $H^1(S^{n-1})$ $(n \ge 3)$. Then there exists a constant $c > 0$, independent of Ω, such that $cE_\Omega(s,\xi')$ is an ∞-atom in $H^1(\mathbb{R})$. That is, $cE_\Omega(s,\xi')$ satisfies*

$$\begin{aligned} &\|cE_\Omega\|_{L^\infty} \le \frac{1}{4r(\xi')}, \qquad \operatorname{supp} E_\Omega \subset \bigl(\xi_1' - 2r(\xi'), \xi_1' + 2r(\xi')\bigr) \quad \text{and} \\ &\int_{\mathbb{R}} E_\Omega(s,\xi')\,ds = 0, \end{aligned} \tag{2.4}$$

where $r(\xi') = |\xi|^{-1}|A_\tau \xi|$ and $A_\tau(\xi) = (\tau^2\xi_1, \tau\xi_2, \dots, \tau\xi_n)$.

Lemma 2.2 *Let Ω be a regular ∞-atom in $H^1(S^1)$. Then, for $1 < q < 2$, there exists a constant $c > 0$, independent of Ω, such that $ce_\Omega(s,\xi')$ is a q-atom in $H^1(\mathbb{R})$, the center of whose support is ξ_1' and the radius $r(\xi') = |\xi|^{-1}(\tau^4\xi_1^2 + \tau^2\xi_2^2)^{1/2}$.*

For $\Omega \in L^1(S^{n-1})$, $h \in \Delta_\gamma$ for some $1 < \gamma \le \infty$, and a suitable function ϕ on $\mathbb{R}_+$, we define the maximal functions $M_{\Omega,h,\phi}$ by

$$M_{\Omega,h,\phi}f(x) = \sup_{k\in\mathbb{Z}} \frac{1}{2^{kn}} \int_{2^{k-1}<|y|\le 2^k} \bigl|\Omega(y')h(|y|)f(x-\phi(|y|)y')\bigr|\,dy. \tag{2.5}$$

Let ϕ be a positive increasing function on $(0,\infty)$ satisfying $\phi(2t) \le c_1\phi(t)$ $(t > 0)$ for some $c_1 > 1$, and $\varphi(t) = \phi(t)/(t\phi'(t)) \in L^\infty(0,\infty)$. Then, as in the proof of Lemma 2.3 in [20], p.246, we have

$$M_{\Omega,h,\phi}f(x) \leq \|h\|_{\Delta_\gamma}\left(\|\Omega\|_{L^1(S^{n-1})}\right)^{\frac{1}{\gamma}} \times \left(\int_{S^{n-1}} \left|\Omega(y')\right| M_{y'}\left(|f|^{\gamma'}\right)(x)\, d\sigma(y')\right)^{\frac{1}{\gamma'}}, \tag{2.6}$$

where $M_{y'}g$ is the directional Hardy-Littlewood maximal function of g defined by

$$M_{y'}g(x) = \sup_{r>0} \frac{1}{2r} \int_{|t|<r} \left|g(x-ty')\right| dt.$$

For this directional maximal function $M_{y'}$, we know that for $1 < p, q < \infty$,

$$\left(\int_{\mathbb{R}^n}\left[\left(\sum_{j\in\mathbb{Z}}\left(M_{y'}(f_j)(x)\right)^q\right)^{\frac{1}{q}}\right]^p dx\right)^{\frac{1}{p}} \leq C_{p,q}\left(\int_{\mathbb{R}^n}\left[\left(\sum_{j\in\mathbb{Z}}|f_j(x)|^q\right)^{\frac{1}{q}}\right]^p dx\right)^{\frac{1}{p}}. \tag{2.7}$$

This is just (2.7) in the proof of Lemma 2.3 of [8], p.496. From (2.6) and (2.7), we get the following lemma.

Lemma 2.3 *Let ϕ be a positive increasing function on $(0,\infty)$ satisfying $\phi(2t) \leq c_1\phi(t)$ $(t > 0)$ for some $c_1 > 1$, and $\varphi(t) = \phi(t)/(t\phi'(t)) \in L^\infty(0,\infty)$. Let $h \in \Delta_\gamma$ for some $1 < \gamma \leq \infty$. For $\gamma' < p, q < \infty$, we have*

$$\left\|\left(\sum_{j\in\mathbb{Z}}|M_{\Omega,h,\phi}f_j|^q\right)^{\frac{1}{q}}\right\|_{L^p(\mathbb{R}^n)} \leq C\left\|\left(\sum_{j\in\mathbb{Z}}|f_j|^q\right)^{\frac{1}{q}}\right\|_{L^p(\mathbb{R}^n)}. \tag{2.8}$$

Proof Let $\{g_j\}_{j\in\mathbb{Z}}$ be a sequence of functions satisfying $\|(\sum_{j\in\mathbb{Z}}|g_j|^{q'})^{1/q'}\|_{L^{p'}(\mathbb{R}^n)} \leq 1$. Then, noting $p, q > \gamma'$ and using (2.6), the duality, and Minkowski's inequality, we see that

$$\begin{aligned}
&\left|\int_{\mathbb{R}^n}\sum_{j\in\mathbb{Z}} M_{\Omega,h,\phi}f_j(x)g_j(x)\, dx\right| \\
&\quad\leq C\int_{\mathbb{R}^n}\sum_{j\in\mathbb{Z}}\left(\int_{S^{n-1}}\left|\Omega(y')\right| M_{y'}\left(|f_j|^{\gamma'}\right)(x)\, d\sigma(y')\right)^{\frac{1}{\gamma'}} \left|g_j(x)\right| dx \\
&\quad\leq C\left(\int_{\mathbb{R}^n}\left(\sum_{j\in\mathbb{Z}}\left(\int_{S^{n-1}}\left|\Omega(y')\right| M_{y'}\left(|f_j|^{\gamma'}\right)(x)\, d\sigma(y')\right)^{\frac{q}{\gamma'}}\right)^{\frac{p}{q}} dx\right)^{\frac{1}{p}} \\
&\qquad\times\left\|\left(\sum_{j\in\mathbb{Z}}|g_j(x)|^{q'}\right)^{\frac{1}{q'}}\right\|_{L^{p'}(\mathbb{R}^n)} \\
&\quad\leq C\left\{\int_{S^{n-1}}\left|\Omega(y')\right|\left(\int_{\mathbb{R}^n}\left[\left(\sum_{j\in\mathbb{Z}}\left(M_{y'}\left(|f_j|^{\gamma'}\right)(x)\right)^{\frac{q}{\gamma'}}\right)^{\frac{\gamma'}{q}}\right]^{\frac{p}{\gamma'}} dx\right)^{\frac{\gamma'}{p}} d\sigma(y')\right\}^{\frac{1}{\gamma'}}.
\end{aligned}$$

Hence by (2.7) we have

$$\left|\int_{\mathbb{R}^n}\sum_{j\in\mathbb{Z}}M_{\Omega,h,\phi}f_j(x)g_j(x)\,dx\right| \le C\left\{\int_{S^{n-1}}\left|\Omega(y')\right|\left(\int_{\mathbb{R}^n}\left(\sum_{j\in\mathbb{Z}}\left|f_j(x)\right|^q\right)^{\frac{p}{q}}dx\right)^{\frac{\gamma'}{p}}d\sigma(y')\right\}^{\frac{1}{\gamma'}}$$
$$\le C\left\|\left(\sum_{j\in\mathbb{Z}}|f_j|^q\right)^{\frac{1}{q}}\right\|_{L^p(\mathbb{R}^n)},$$

which implies our (2.8). □

Now, for $\Omega\in L^1(S^{n-1})$, we define the measures $\sigma_{\Omega,h,\phi,k}$ on $\mathbb{R}^n$ and the maximal operator $\sigma^*_{\Omega,h,\phi}f(x)$ by

$$\int_{\mathbb{R}^n}f(x)\,d\sigma_{\Omega,h,\phi,k}(x)=\int_{\mathbb{R}^n}f\left(\phi\left(|x|\right)x'\right)\frac{\Omega(x')h(|x|)}{|x|^n}\chi_{2^{k-1}<|x|\le 2^k}(x)\,dx, \tag{2.9}$$

$$\sigma^*_{\Omega,h,\phi}f(x)=\sup_{k\in\mathbb{Z}}\left|\left|\sigma_{\Omega,h,\phi,k}\right|*f(x)\right|, \tag{2.10}$$

where $|\sigma_{\Omega,h,\phi,k}|$ is defined in the same way as $\sigma_{\Omega,h,\phi,k}$, but with Ω replaced by $|\Omega|$ and h by $|h|$.

Then we have the following lemma.

Lemma 2.4 *Let ϕ be a positive increasing function on $(0,\infty)$ satisfying $\phi(2t)\le c_1\phi(t)$ ($t>0$) for some $c_1>1$, and $\varphi(t)=\phi(t)/(t\phi'(t))\in L^\infty(0,\infty)$. Let $h\in\Delta_\gamma$ for some $1<\gamma\le\infty$, $\Omega\in L^1(S^{n-1})$. Then*:

(i) *If $(\frac{1}{p},\frac{1}{q})$ belongs to the interior of the octagon $P_1P_2R_2P_3P_4P_5R_4P_6$, there exists $C>0$ such that*

$$\left\|\left(\sum_{j\in\mathbb{Z}}\left(\sum_{k\in\mathbb{Z}}|\sigma_{\Omega,h,\phi,k}*g_{k,j}|^2\right)^{\frac{q}{2}}\right)^{\frac{1}{q}}\right\|_{L^p(\mathbb{R}^n)}$$
$$\le C\|h\|_{\Delta_\gamma}\|\Omega\|_{L^1(S^{n-1})}\left\|\left(\sum_{j\in\mathbb{Z}}\left(\sum_{k\in\mathbb{Z}}|g_{k,j}|^2\right)^{\frac{q}{2}}\right)^{\frac{1}{q}}\right\|_{L^p(\mathbb{R}^n)}, \tag{2.11}$$

where $P_1=(\frac{1}{2}-\frac{1}{\max\{2,\gamma'\}},\frac{1}{2}-\frac{1}{\max\{2,\gamma'\}})$, $P_2=(\frac{1}{2},\frac{1}{2}-\frac{1}{\max\{2,\gamma'\}})$, $P_3=(\frac{1}{2}+\frac{1}{\max\{2,\gamma'\}},\frac{1}{2})$, $P_4=(\frac{1}{2}+\frac{1}{\max\{2,\gamma'\}},\frac{1}{2}+\frac{1}{\max\{2,\gamma'\}})$, $P_5=(\frac{1}{2},\frac{1}{2}+\frac{1}{\max\{2,\gamma'\}})$, $P_6=(\frac{1}{2}-\frac{1}{\max\{2,\gamma'\}},\frac{1}{2})$, $R_2=(1-\frac{1}{2\gamma},\frac{1}{2\gamma})$, and $R_4=(\frac{1}{2\gamma},1-\frac{1}{2\gamma})$.

(*Note that if $1<\gamma\le 2$, the octagon $P_1P_2R_2P_3P_4P_5R_4P_6$ reduces to the hexagon $P_1P_2P_3P_4P_5P_6$.*)

(ii) *If $(\frac{1}{p},\frac{1}{q})$ belongs to the interior of $Q_1Q_2Q_3Q_4$, there exists $C>0$ such that*

$$\left\|\left(\sum_{j\in\mathbb{Z}}\sum_{k\in\mathbb{Z}}|\sigma_{\Omega,h,\phi,k}*g_{k,j}|^q\right)^{\frac{1}{q}}\right\|_{L^p(\mathbb{R}^n)}$$
$$\le C\|h\|_{\Delta_\gamma}\|\Omega\|_{L^1(S^{n-1})}\left\|\left(\sum_{j\in\mathbb{Z}}\sum_{k\in\mathbb{Z}}|g_{k,j}|^q\right)^{\frac{1}{q}}\right\|_{L^p(\mathbb{R}^n)}, \tag{2.12}$$

where $Q_1=(0,0)$, $Q_2=(\frac{1}{\gamma'},0)$, $Q_3=(1,1)$, and $Q_4=(\frac{1}{\gamma},1)$.

Proof (a) Let $1 < \gamma \le \infty$. Since

$$\sup_{k\in\mathbb{Z}} |\sigma_{\Omega,h,\phi,k} * g_{k,j}| \le \sup_{k\in\mathbb{Z}} |\sigma_{\Omega,h,\phi,k}| * \sup_{\ell\in\mathbb{Z}} |g_{\ell,j}| \le M_{\Omega,h,\phi}\Big(\sup_{\ell\in\mathbb{Z}} |g_{\ell,j}|\Big),$$

we get using Lemma 2.3

$$\begin{aligned}\bigg\|\bigg(\sum_{j\in\mathbb{Z}}\Big(\sup_{k\in\mathbb{Z}} |\sigma_{\Omega,h,\phi,k} * g_{k,j}|\Big)^q\bigg)^{\frac{1}{q}}\bigg\|_{L^p(\mathbb{R}^n)} &\le \bigg\|\bigg(\sum_{j\in\mathbb{Z}}\Big(M_{\Omega,h,\phi}\Big(\sup_{k\in\mathbb{Z}} |g_{k,j}|\Big)\Big)^q\bigg)^{\frac{1}{q}}\bigg\|_{L^p(\mathbb{R}^n)} \\ &\le C\bigg\|\bigg(\sum_{j\in\mathbb{Z}}\Big(\sup_{k\in\mathbb{Z}} |g_{k,j}|\Big)^q\bigg)^{\frac{1}{q}}\bigg\|_{L^p(\mathbb{R}^n)}. \end{aligned} \tag{2.13}$$

On the other hand, there exists $\{h_j\} \in L^{p'}(\ell^{q'})$ with $\|\{h_j\}\|_{L^{p'}(\ell^{q'})} = 1$ such that

$$\begin{aligned}&\bigg\|\bigg(\sum_{j\in\mathbb{Z}}\Big(\sum_{k\in\mathbb{Z}} |\sigma_{\Omega,h,\phi,k} * g_{k,j}|\Big)^q\bigg)^{\frac{1}{q}}\bigg\|_{L^p(\mathbb{R}^n)} \\ &\quad = \sum_{j\in\mathbb{Z}}\int_{\mathbb{R}^n}\sum_{k\in\mathbb{Z}}\big|\sigma_{\Omega,h,\phi,k} * g_{k,j}(x)\big|h_j(x)\,dx \\ &\quad \le \sum_{j\in\mathbb{Z}}\int_{\mathbb{R}^n}\sum_{k\in\mathbb{Z}}\big|g_{k,j}(x)\big||\tilde{\sigma}_{\Omega,h,\phi,k}| * h_j(x)\,dx \\ &\quad \le \sum_{j\in\mathbb{Z}}\int_{\mathbb{R}^n}\sum_{k\in\mathbb{Z}}\big|g_{k,j}(x)\big|M_{\tilde{\Omega},h,\phi}h_j(x)\,dx \\ &\quad \le C\bigg\|\bigg(\sum_{j\in\mathbb{Z}}\Big(\sum_{k\in\mathbb{Z}} |g_{k,j}|\Big)^q\bigg)^{\frac{1}{q}}\bigg\|_{L^p(\mathbb{R}^n)}\bigg\|\bigg(\sum_{j\in\mathbb{Z}}\big(M_{\tilde{\Omega},h,\phi}h_j(x)\big)^{q'}\bigg)^{\frac{1}{q'}}\bigg\|_{L^{p'}(\mathbb{R}^n)},\end{aligned}$$

where $\tilde{\Omega}(y') = \Omega(-y')$. So by Lemma 2.3 we obtain for $\gamma' < p', q' < \infty$, *i.e.*, $1 < p, q < \gamma$,

$$\begin{aligned}&\bigg\|\bigg(\sum_{j\in\mathbb{Z}}\Big(\sum_{k\in\mathbb{Z}} |\sigma_{\Omega,h,\phi,k} * g_{k,j}|\Big)^q\bigg)^{\frac{1}{q}}\bigg\|_{L^p(\mathbb{R}^n)} \\ &\quad \le C\bigg\|\bigg(\sum_{j\in\mathbb{Z}}\Big(\sum_{k\in\mathbb{Z}} |g_{k,j}|\Big)^q\bigg)^{\frac{1}{q}}\bigg\|_{L^p(\mathbb{R}^n)}\bigg\|\bigg(\sum_{j\in\mathbb{Z}}\big(\big|h_j(x)\big|\big)^{q'}\bigg)^{\frac{1}{q'}}\bigg\|_{L^{p'}(\mathbb{R}^n)} \\ &\quad \le C\bigg\|\bigg(\sum_{j\in\mathbb{Z}}\Big(\sum_{k\in\mathbb{Z}} |g_{k,j}|\Big)^q\bigg)^{\frac{1}{q}}\bigg\|_{L^p(\mathbb{R}^n)}.\end{aligned} \tag{2.14}$$

Now, let $R_1 = (\frac{1}{2\gamma}, \frac{1}{2\gamma})$, $R_2 = (1 - \frac{1}{2\gamma}, \frac{1}{2\gamma})$, $R_3 = (1 - \frac{1}{2\gamma}, 1 - \frac{1}{2\gamma})$, and $R_4 = (\frac{1}{2\gamma}, 1 - \frac{1}{2\gamma})$. Then, if $(\frac{1}{p}, \frac{1}{q})$ belongs to the interior of the square $R_1R_2R_3R_4$, there are two points $(\frac{1}{p_1}, \frac{1}{q_1})$ and $(\frac{1}{p_2}, \frac{1}{q_2})$ such that

$$\begin{aligned}&\frac{1}{p} = \frac{1}{2}\frac{1}{p_1} + \frac{1}{2}\frac{1}{p_2}, \qquad \frac{1}{q} = \frac{1}{2}\frac{1}{q_1} + \frac{1}{2}\frac{1}{q_2}, \\ &1 < p_1, q_1 < \gamma \quad \text{and} \quad \gamma' < p_2, q_2 < \infty.\end{aligned}$$

Hence, interpolating (2.13) with (2.14), we obtain (2.11) if $(\frac{1}{p},\frac{1}{q})$ belongs to the interior of the square $R_1R_2R_3R_4$.

(b) Let $1<\gamma<2$. Using the Cauchy-Schwarz inequality, we get

$$\begin{aligned}\left|\sigma_{\Omega,h,\phi,k}*g_{k,j}(x)\right| &\le \left(\int_{2^{k-1}\le|y|\le 2^k}\frac{|\Omega(y')||h(|y|)|^{\gamma}}{|y|^n}\,dy\right)^{\frac{1}{2}}\\ &\quad\times\left(\int_{2^{k-1}\le|y|\le 2^k}\left|g_{k,j}\bigl(x-\phi(|y|)y'\bigr)\right|^2\frac{|\Omega(y')||h(|y|)|^{2-\gamma}}{|y|^n}\,dy\right)^{\frac{1}{2}}\\ &\le C\|h\|_{\Delta_\gamma}^{\frac{\gamma}{2}}\|\Omega\|_{L^1(S^{n-1})}^{\frac{1}{2}}\bigl(\sigma_{|\Omega|,|h|^{2-\gamma},\phi,k}*|g_{k,j}|^2\bigr)(x)^{\frac{1}{2}}.\end{aligned}$$

So, we have

$$\begin{aligned}&\left\|\left(\sum_{j\in\mathbb{Z}}\left(\sum_{k\in\mathbb{Z}}|\sigma_{\Omega,h,\phi,k}*g_{k,j}|^2\right)^{\frac{q}{2}}\right)^{\frac{1}{q}}\right\|_{L^p(\mathbb{R}^n)}\\ &\quad\le C\|h\|_{\Delta_\gamma}^{\frac{\gamma}{2}}\|\Omega\|_{L^1(S^{n-1})}^{\frac{1}{2}}\left\|\left(\sum_{j\in\mathbb{Z}}\left(\sum_{k\in\mathbb{Z}}\sigma_{|\Omega|,|h|^{2-\gamma},\phi,k}*|g_{k,j}|^2\right)^{\frac{q}{2}}\right)^{\frac{1}{q}}\right\|_{L^p(\mathbb{R}^n)}.\end{aligned}$$

Hence, noting $|h|^{2-\gamma}\in\Delta_{\gamma/(2-\gamma)}$ and using (2.14) for $\gamma/(2-\gamma)$, $p/2$ and $q/2$ in place of γ, p, q, respectively, we see that (2.11) holds provided $1<p/2,q/2<\gamma/(2-\gamma)$, *i.e.*, $1/2-1/\gamma'<1/p,1/q<1/2$. By duality, it holds also provided $1/2<1/p,1/q<1/2+1/\gamma'$. Interpolating these two cases, we see that (2.11) holds if $(\frac{1}{p},\frac{1}{q})$ belongs to the interior of the hexagon $P_1P_2P_3P_4P_5P_6$.

(c) Noting $\Delta_\gamma\subset\Delta_2$ for $\gamma>2$, and interpolating cases (a) and (b) above, we see that (2.11) holds if $(\frac{1}{p},\frac{1}{q})$ belongs to the interior of the octagon $P_1P_2R_2P_3P_4P_5R_4P_6$. This completes the proof of Lemma 2.4(i).

(d) We shall prove Lemma 2.4(ii). If $(\frac{1}{p},\frac{1}{q})$ belongs to the interior of the parallelogram $Q_1Q_2Q_3Q_4$, there are two points $(\frac{1}{p_1},\frac{1}{q_1})$ and $(\frac{1}{p_2},\frac{1}{q_2})$ such that

$$\begin{aligned}&\frac{1}{p}=\left(1-\frac{1}{q}\right)\frac{1}{p_1}+\frac{1}{q}\frac{1}{p_2},\qquad \frac{1}{q}=\left(1-\frac{1}{q}\right)\frac{1}{q_1}+\frac{1}{q}\frac{1}{q_2},\\ &1<p_1,q_1<\gamma\quad\text{and}\quad\gamma'<p_2,q_2<\infty.\end{aligned}$$

Hence, interpolating (2.13) with (2.14), we obtain (2.12). Thus, we finished the proof of Lemma 2.4. □

About the Fourier transform estimates of $\sigma_{\Omega,h,\phi,k}$ with $\Omega\in H^1(S^{n-1})$, we have the following.

Lemma 2.5 *Let $1<q\le+\infty$ and Ω be a regular ∞-atom in $H^1(S^{n-1})$ supported in $S^{n-1}\cap B(\mathbf{e}_1,\tau)$, where $\mathbf{e}_1=(1,0,\dots,0)$. Let ϕ be a positive increasing function on $(0,\infty)$ satisfying $\varphi(t)=\phi(t)/(t\phi'(t))\in L^\infty(0,\infty)$, and $h\in\Delta_\gamma$ for some $1<\gamma\le\infty$. Then there exist positive constants C's such that*

$$\left|\hat{\sigma}_{\Omega,h,\phi,k}(\xi)\right|\le C\|h\|_{\Delta_1}\|\Omega\|_{L^1(S^{n-1})},\tag{2.15}$$

$$\left|\hat{\sigma}_{\Omega,h,\phi,k}(\xi)\right|\le C\|h\|_{\Delta_1}\phi\bigl(2^k\bigr)\left|A_\tau(\xi)\right|\tag{2.16}$$

and

$$\left|\hat{\sigma}_{\Omega,h,\phi,k}(\xi)\right| \le \frac{C\|h\|_{\Delta_\gamma}}{(\phi(2^{k-1})|A_\tau(\xi)|)^{1/\max\{\gamma',2\}}}. \tag{2.17}$$

These are shown by using Lemmas 2.1 and 2.2 as in the proofs of Lemmas 3.3 and 3.4 in [20], pp.247-248. There these are stated for the case where a parameter ρ of positive number arises, but one sees easily that these hold in our case ($\rho = 0$), too.

To show Theorem 1.1, we need a characterization of the Triebel-Lizorkin space in terms of lacunary sequences. Let $\{a_j\}_{j\in\mathbb{Z}}$ be a lacunary sequence with lacunarity $a > 1$, *i.e.*,

$$\frac{a_{j+1}}{a_j} \ge a \quad \text{for } j \in \mathbb{Z}. \tag{2.18}$$

Let η be a radial function in $C^\infty(\mathbb{R}^n)$ satisfying $\chi_{|\xi|\le 1}(\xi) \le \eta(\xi) \le \chi_{|\xi|\le a}(\xi)$ and $|\partial^\alpha \eta(\xi)| \le c_\alpha(a-1)^{-|\alpha|}$ for $\xi \in \mathbb{R}^n$ and $\alpha \in \mathbb{Z}_+^n$. We define functions ψ_j on $\mathbb{R}^n$ by

$$\psi_j(\xi) = \eta\left(\frac{\xi}{a_{j+1}}\right) - \eta\left(\frac{\xi}{a_j}\right) \quad \bigl(\xi \in \mathbb{R}^n\bigr). \tag{2.19}$$

Then observe that

$$\psi_j(\xi) = \begin{cases} 0, & 0 \le |\xi| \le a_j, |\xi| \ge a a_{j+1}, \\ 1, & a a_j \le |\xi| \le a_{j+1}, \end{cases} \tag{2.20}$$

and that

$$\operatorname{supp}\psi_j \subset \bigl\{a_j \le |\xi| \le a a_{j+1}\bigr\}, \tag{2.21}$$

$$\operatorname{supp}\psi_j \cap \operatorname{supp}\psi_\ell = \emptyset \quad \text{for } |j-\ell| \ge 2, \tag{2.22}$$

$$\left|\xi^\alpha \partial^\alpha \psi_j(\xi)\right| \le C_\alpha \quad \text{for } \alpha \in \mathbb{Z}_+^n, \tag{2.23}$$

$$\sum_{j\in\mathbb{Z}} \psi_j(\xi) = 1 \quad \bigl(\xi \in \mathbb{R}^n \setminus \{0\}\bigr). \tag{2.24}$$

Let Ψ_j be defined on $\mathbb{R}^n$ by $\widehat{\Psi}_j(\xi) = \psi_j(\xi)$ for $\xi \in \mathbb{R}^n$, *i.e.*, $\Psi_j(x) = a_{j+1}^n \check{\eta}(a_{j+1}x) - a_j^n \check{\eta}(a_j x)$.

Lemma 2.6 *Define the multiplier S_j by $S_j f = \Psi_j * f$. Then, for $1 < p, q < \infty$, we have*

$$\left\| \left(\sum_{j\in\mathbb{Z}} \left(\sum_{k\in\mathbb{Z}} |S_k f_j|^2 \right)^{q/2} \right)^{1/q} \right\|_{L^p(\mathbb{R}^n)} \le C \left\| \left(\sum_{j\in\mathbb{Z}} |f_j|^q \right)^{1/q} \right\|_{L^p(\mathbb{R}^n)},$$

where C is independent of $\{f_j\}_{j\in\mathbb{Z}}$.

This is a consequence of Proposition 4.6.4 in Grafakos [21]. For the sake of completeness, we will give a proof in the Appendix. From this lemma we have the following lemma with minor change of the proof of Lemma 2.2 in [8].

Lemma 2.7 *Let ψ_j be as in Lemma 2.6. Denote $A_\tau(\xi) = (\tau^2\xi_1, \tau\xi_2, \ldots, \tau\xi_n)$ for $\tau > 0$ and $\xi \in \mathbb{R}^n$. Define the multiplier $S_{j,\tau}$ by $\widehat{S_{j,\tau}f}(\xi) = \psi(a_k A_\tau(\xi))\hat{f}(\xi)$. Then, for $1 < p, q < \infty$, we have*

$$\left\| \left(\sum_{j\in\mathbb{Z}} \left(\sum_{k\in\mathbb{Z}} |S_{k,\tau} f_j|^2 \right)^{q/2} \right)^{1/q} \right\|_{L^p(\mathbb{R}^n)} \le C \left\| \left(\sum_{j\in\mathbb{Z}} |f_j|^q \right)^{1/q} \right\|_{L^p(\mathbb{R}^n)},$$

where C is independent of $\{f_j\}_{j\in\mathbb{Z}}$.

We need one more lemma. If $\{a_k\}_{k\in\mathbb{Z}}$ satisfies furthermore $a_{k+1}/a_k \le d$ for some $d \ge a$, we can characterize Triebel-Lizorkin spaces in terms of this lacunary sequence.

Denote by $\mathcal{P}$ the set of all polynomials in $\mathbb{R}^n$. Let $1 < p, q < \infty$, and $\alpha \in \mathbb{R}$. For $f \in \mathcal{S}'(\mathbb{R}^n)/\mathcal{P}$, we define the norm $\|f\|_{\dot{F}_{pq}^{\alpha,\{\Psi_k\}_{k\in\mathbb{Z}}}(\mathbb{R}^n)}$ by

$$\|f\|_{\dot{F}_{pq}^{\alpha,\{\Psi_k\}_{k\in\mathbb{Z}}}(\mathbb{R}^n)} = \left\| \left(\sum_{k\in\mathbb{Z}} a_k^{\alpha q} |\Psi_k * f|^q \right)^{1/q} \right\|_{L^p(\mathbb{R}^n)}. \tag{2.25}$$

Lemma 2.8 *Let $\alpha \in \mathbb{R}$ and $1 < p, q < \infty$. Let $\{a_k\}_{k\in\mathbb{Z}}$ be a lacunary sequence of positive numbers with $d \ge a_{k+1}/a_k \ge a > 1$ $(k \in \mathbb{Z})$. Then $\|f\|_{\dot{F}_{pq}^{\alpha,\{\Psi_k\}_{k\in\mathbb{Z}}}(\mathbb{R}^n)}$ is equivalent to the usual homogeneous Triebel-Lizorkin space norm $\|f\|_{\dot{F}_{pq}^{\alpha}(\mathbb{R}^n)}$.*

This is stated in Proposition 1 in [22] for $\alpha \ne 0$, but the proof of this part works also for $\alpha = 0$.

2.2 Proof of Theorem 1.1

We have only to show Theorem 1.1 in the case Ω is a regular atom with $\operatorname{supp}\Omega \subset S^{n-1} \cap B(\xi, \tau)$, where $B(\xi, \tau) = \{y \in \mathbb{R}^n; |y - \xi| < \tau\}$. Using the definition of $\sigma_{\Omega,h,\phi,k}$, we see that

$$T_{\Omega,h,\phi} f(x) = \text{p.v.} \int_{\mathbb{R}^n} \frac{h(|y|)\Omega(y')}{|y|^n} f\bigl(x - \phi(|y|)y'\bigr)\, dy = \sum_{k\in\mathbb{Z}} \sigma_{\Omega,h,\phi,k} * f(x). \tag{2.26}$$

Let $a_k = 1/\phi(2^{-k})$, $k \in \mathbb{Z}$. Then as is known, $\{a_k\}_{k\in\mathbb{Z}}$ is a lacunary sequence with lacunarity $a = 2^{1/\|\varphi\|_{L^\infty(\mathbb{R}_+)}}$. This follows from (1.10) (see, for example, [22]). Also, we have $a_{k+1}/a_k \le c_1$, which follows from (1.9).

Let $\psi_k \in C_c^\infty(\mathbb{R}^n)$ be radial functions defined by (2.19). Set $\psi_{k,\tau}(\xi) = \psi_k(A_\tau(\xi))$ and $\widehat{S_{k,\tau}f}(\xi) = \psi_{k,\tau}(\xi)\hat{f}(\xi)$, $\xi \in \mathbb{R}^n$. Then, noting $\sum_{j\in\mathbb{Z}} \psi_j(\xi) = 1$ $(\xi \ne 0)$ and $\sum_{\ell=-1}^{1} \psi_{j+\ell}(\xi) = 1$ on $\operatorname{supp}\psi_j$, we have

$$T_{\Omega,h,\phi} f(x) = \sum_{k\in\mathbb{Z}} \sum_{j\in\mathbb{Z}} \sum_{\ell=-1}^{1} S_{j-k+\ell,\tau}(\sigma_{\Omega,h,\phi,k} * S_{j-k,\tau} f)(x) = \sum_{j\in\mathbb{Z}} Q_j f(x), \tag{2.27}$$

where

$$Q_j f(x) = \sum_{k\in\mathbb{Z}} \sum_{\ell=-1}^{1} S_{j-k+\ell,\tau}(\sigma_{\Omega,h,\phi,k} * S_{j-k,\tau} f)(x). \tag{2.28}$$

We follow the proof of Theorem 1 in [8], using our Lemma 2.7 and Lemma 2.4 in place of Lemma 2.2 and Lemma 2.4 in [8], respectively, and we see that if $\alpha \in \mathbb{R}$ and $(\frac{1}{p}, \frac{1}{q})$ belongs to the interior of the octagon $P_1P_2R_2P_3P_4P_5R_4P_6$, then we have

$$\|Q_j f\|_{\dot{F}^{\alpha}_{p,q}(\mathbb{R}^n)} \le C\|f\|_{\dot{F}^{\alpha}_{p,q}(\mathbb{R}^n)}. \tag{2.29}$$

About L^2 estimate, we have

$$\|Q_j f\|_{\dot{F}^{0}_{2,2}(\mathbb{R}^n)} \le C a^{-|j|/\max(\gamma',2)}\|f\|_{\dot{F}^{0}_{2,2}(\mathbb{R}^n)}. \tag{2.30}$$

In fact, by Lemma 2.5, we get

$$\left|\hat{\sigma}_{\Omega,h,\phi,k}(\xi)\right| \le C\|h\|_{\Delta_1}\|\Omega\|_{L^1(S^{n-1})}, \tag{2.31}$$

$$\left|\hat{\sigma}_{\Omega,h,\phi,k}(\xi,\eta)\right| \le C\|h\|_{\Delta_1}\phi\bigl(2^k\bigr)\left|A_\tau(\xi)\right| \tag{2.32}$$

and

$$\left|\hat{\sigma}_{\Omega,h,\phi,\phi,k}(\xi)\right| \le \frac{C\|h\|_{\Delta_\gamma}}{(\phi(2^{k-1})|A_\tau(\xi)|)^{1/\max(\gamma',2)}}. \tag{2.33}$$

Also, we have

$$\begin{aligned}
\|Q_j f\|_{\dot{F}^{0}_{2,2}(\mathbb{R}^n)} &= \left(\int_{\mathbb{R}^n}\left|\sum_{k\in\mathbb{Z}}\sum_{\ell=-1}^{1} S_{j-k+\ell,\tau}(\sigma_{\Omega,h,\phi,k} * S_{j-k,\tau}f)(x)\right|^2 dx\right)^{1/2} \\
&= \left(\int_{\mathbb{R}^n}\left|\sum_{k\in\mathbb{Z}}\sum_{\ell=-1}^{1} \psi_{j-k+\ell}\bigl(A_\tau(\xi)\bigr)\hat{\sigma}_{\Omega,h,\phi,k}(\xi)\psi_{j-k}\bigl(A_\tau(\xi)\bigr)\hat{f}(\xi)\right|^2 d\xi\right)^{1/2}.
\end{aligned}$$

So, for $j \ge 0$, we have, using (2.33) and $\phi(2^\ell) = 1/a_{-\ell}$ and $a_{\ell+1}/a_\ell \ge a = 2^{1/\|\varphi\|_{L^\infty(\mathbb{R}_+)}}$,

$$\begin{aligned}
&\|Q_j f\|_{\dot{F}^{0}_{2,2}(\mathbb{R}^n)} \\
&\quad\le C\left(\sum_{k\in\mathbb{Z}}\int_{a_{j-k}\le|A_\tau(\xi)|\le a_{j-k+2}}\left|\hat{\sigma}_{\Omega,h,\phi,k}(\xi)\hat{f}(\xi)\right|^2 d\xi\right)^{1/2} \\
&\quad\le C\|h\|_{\Delta_\gamma}\left(\sum_{k\in\mathbb{Z}}\int_{a_{j-k}\le|A_\tau(\xi)|\le a_{j-k+2}}\left(\frac{|A_\tau(\xi)|}{a_{-k+1}}\right)^{-2/\max(\gamma',2)}\left|\hat{f}(\xi)\right|^2 d\xi\right)^{1/2} \\
&\quad\le C\|h\|_{\Delta_\gamma}a^{-(j-1)/\max(\gamma',2)}\left(\sum_{k\in\mathbb{Z}}\int_{a_{j-k}\le|A_\tau(\xi)|\le a_{j-k+2}}\left(\frac{|A_\tau(\xi)|}{a_{j-k}}\right)^{-2/\max(\gamma',2)}\left|\hat{f}(\xi)\right|^2 d\xi\right)^{1/2} \\
&\quad\le C\|h\|_{\Delta_\gamma}a^{-j/\max(\gamma',2)}\left(\int_{\mathbb{R}^n}\left|\hat{f}(\xi)\right|^2 d\xi\right)^{1/2} \\
&\quad\le C a^{-j/\max(\gamma',2)}\|f\|_{\dot{F}^{0}_{2,2}(\mathbb{R}^n)}.
\end{aligned}$$

In the fourth inequality we used $a_{j+1} \le c_1 a_j$.

For $j \le -1$, using (2.32) we get as before

$$\begin{aligned}
\|Q_j f\|_{\dot{F}^0_{2,2}(\mathbb{R}^n)} &\le C\left(\sum_{k\in\mathbb{Z}}\int_{a_{j-k}\le|A_\tau(\xi)|\le a_{j-k+2}} \left|\hat{\sigma}_{\Omega,\psi,h,k}(\xi)\hat{f}(\xi)\right|^2 d\xi\right)^{1/2} \\
&\le C\|h\|_{\Delta_\gamma}\left(\sum_{k\in\mathbb{Z}}\int_{a_{j-k}\le|A_\tau(\xi)|\le a_{j-k+2}} \left(\frac{|A_\tau(\xi)|}{a_{-k}}\right)^2 \left|\hat{f}(\xi)\right|^2 d\xi\right)^{1/2} \\
&\le Ca^j\left(\sum_{k\in\mathbb{Z}}\int_{a_{j-k}\le|A_\tau(\xi)|\le a_{j-k+2}} \left(\frac{|A_\tau(\xi)|}{a_{j-k}}\right)^2 \left|\hat{f}(\xi)\right|^2 d\xi\right)^{1/2} \\
&\le Ca^j\left(\sum_{k\in\mathbb{Z}}\int_{a_{j-k}\le|A_\tau(\xi)|\le a_{j-k+2}} \left|\hat{f}(\xi)\right|^2 d\xi\right)^{1/2} \\
&\le Ca^j\left(\int_{\mathbb{R}^n} \left|\hat{f}(\xi)\right|^2 d\xi\right)^{1/2} \\
&\le Ca^j\|f\|_{L^2(\mathbb{R},\dot{F}^0_{2,2}(\mathbb{R}^n))}.
\end{aligned}$$

Thus we have

$$\|Q_j f\|_{\dot{F}^0_{2,2}(\mathbb{R}^n)} \le Ca^{-|j|/\max(\gamma',2)}\|f\|_{\dot{F}^0_{2,2}(\mathbb{R}^n)},$$

which shows the required estimate (2.30).

Interpolating these two cases (2.29) and (2.30), we see that if $\alpha \in \mathbb{R}$ and $(\frac{1}{p}, \frac{1}{q})$ belongs to the interior of the octagon $P_1P_2Q_2P_3P_4P_5Q_4P_6$, then $T_{\Omega,h,\phi}$ is bounded on $\dot{F}^\alpha_{p,q}(\mathbb{R}^n)$. This completes the proof of Theorem 1.1(i).

Next, we prove (ii). Let $|\frac{1}{2} - \frac{1}{p}| < \min\{\frac{1}{2}, \frac{1}{\gamma'}\}$, $1 < q < \infty$, and $\alpha \in \mathbb{R}$. Then, by Theorem 1.1(ii), $T_{\Omega,h,\phi}$ is bounded on $\dot{F}^{\alpha-1}_{p,p}(\mathbb{R}^n)$ and $\dot{F}^{\alpha+1}_{p,p}(\mathbb{R}^n)$. Since $(\dot{F}^{\alpha-1}_{p,p}(\mathbb{R}^n), \dot{F}^{\alpha+1}_{p,p}(\mathbb{R}^n))_{\frac{1}{2},q} = \dot{B}^\alpha_{p,q}(\mathbb{R}^n)$, we see by interpolation that $T_{\Omega,h,\phi}$ is bounded on $\dot{B}^\alpha_{p,q}(\mathbb{R}^n)$. This shows (ii) and completes the proof of Theorem 1.1.

3 Proof of Theorem 1.3

Let $\sigma_{\Omega,h,\phi,k}$, a_k, ψ_k, and S_k be the same as in the proof of Theorem 1.1. Then, noting $\sum_{j\in\mathbb{Z}}\psi_j(\xi) = 1$ ($\xi \ne 0$) and $\sum_{\ell=-1}^{1}\psi_{j+\ell}(\xi) = 1$ on $\operatorname{supp}\psi_j$, we have

$$T_{\Omega,h,\phi}f(x) = \sum_{k\in\mathbb{Z}}\sum_{j\in\mathbb{Z}}\sum_{\ell=-1}^{1} S_{j-k+\ell}(\sigma_{\Omega,h,\phi,k} * S_{j-k}f)(x) = \sum_{j\in\mathbb{Z}}\tilde{Q}_j f(x), \tag{3.1}$$

where

$$\tilde{Q}_j f(x) = \sum_{k\in\mathbb{Z}}\sum_{\ell=-1}^{1} S_{j-k+\ell}(\sigma_{\Omega,h,\phi,k} * S_{j-k}f)(x). \tag{3.2}$$

Using our Lemma 2.6 and Lemma 2.4(i) in place of Lemma 2.2 and Lemma 2.4 in [8], respectively, we see, as in the proof of Theorem 1.1, that if $\alpha \in \mathbb{R}$ and $(\frac{1}{p}, \frac{1}{q})$ belongs to the interior of the octagon $P_1P_2R_2P_3P_4P_5R_4P_6$, then we have

$$\|\tilde{Q}_j f\|_{\dot{F}^\alpha_{p,q}(\mathbb{R}^n)} \le C\|f\|_{\dot{F}^\alpha_{p,q}(\mathbb{R}^n)}. \tag{3.3}$$

Next, we approach the above estimate (3.3) by another method. We calculate the $\dot{F}^{\alpha}_{p,q}$ norm of $\tilde{Q}_j$ more directly. Considering the support property of ψ_k, we have

$$\begin{aligned}
\|\tilde{Q}_j f\|_{\dot{F}^{\alpha}_{p,q}(\mathbb{R}^n)} &= \left\| \left(\sum_{m\in\mathbb{Z}} a_m^{\alpha q} \left| S_m \sum_{k\in\mathbb{Z}} \sum_{\ell=-1}^{1} S_{j-k+\ell}(\sigma_{\Omega,h,\phi,k} * S_{j-k} f) \right|^q \right)^{\frac{1}{q}} \right\|_{L^p(\mathbb{R}^n)} \\
&\le \left\| \left(\sum_{m\in\mathbb{Z}} a_m^{\alpha q} \left| S_m \sum_{\ell=-1}^{1} S_{m+\ell}(\sigma_{\Omega,h,\phi,j-m} * S_m f) \right|^q \right)^{\frac{1}{q}} \right\|_{L^p(\mathbb{R}^n)} \\
&\quad \times \left\| \left(\sum_{m\in\mathbb{Z}} a_m^{\alpha q} \left| S_m \sum_{\ell=0}^{1} S_{m+\ell}(\sigma_{\Omega,h,\phi,j-m-1} * S_{m+1} f) \right|^q \right)^{\frac{1}{q}} \right\|_{L^p(\mathbb{R}^n)} \\
&\quad \times \left\| \left(\sum_{m\in\mathbb{Z}} a_m^{\alpha q} \left| S_m \sum_{\ell=-1}^{0} S_{m+\ell}(\sigma_{\Omega,h,\phi,j-m-1} * S_{m-1} f) \right|^q \right)^{\frac{1}{q}} \right\|_{L^p(\mathbb{R}^n)}.
\end{aligned}$$

By Fefferman-Stein's vector-valued inequality for maximal functions, Lemma 2.4(ii), and $a_{m+1}/c_1 \le a_m \le a_{m+1}/a$, we get

$$\begin{aligned}
\|\tilde{Q}_j f\|_{\dot{F}^{\alpha}_{p,q}(\mathbb{R}^n)} &\le C \sum_{\ell=-1}^{1} \left\| \left(\sum_{m\in\mathbb{Z}} a_m^{\alpha q} |\sigma_{\Omega,h,\phi,j-m} * S_{m+\ell} f|^q \right)^{\frac{1}{q}} \right\|_{L^p(\mathbb{R}^n)} \\
&\le C \sum_{\ell=-1}^{1} \left\| \left(\sum_{m\in\mathbb{Z}} a_m^{\alpha q} |S_{m+\ell} f|^q \right)^{\frac{1}{q}} \right\|_{L^p(\mathbb{R}^n)} \\
&\le C \left\| \left(\sum_{m\in\mathbb{Z}} a_m^{\alpha q} |S_m f|^q \right)^{\frac{1}{q}} \right\|_{L^p(\mathbb{R}^n)} \le C\|f\|_{\dot{F}^{\alpha}_{p,q}(\mathbb{R}^n)}
\end{aligned} \tag{3.4}$$

if $\alpha \in \mathbb{R}$ and $(\frac{1}{p}, \frac{1}{q})$ belongs to the interior of the parallelogram $Q_1Q_2Q_3Q_4$.

Interpolating (3.3) and (3.4), we obtain

$$\|\tilde{Q}_j f\|_{\dot{F}^{\alpha}_{p,q}(\mathbb{R}^n)} \le C\|f\|_{\dot{F}^{\alpha}_{p,q}(\mathbb{R}^n)} \tag{3.5}$$

if $\alpha \in \mathbb{R}$ and $(\frac{1}{p}, \frac{1}{q})$ belongs to the interior of the octagon $Q_1Q_2R_2P_3Q_3Q_4R_4P_6$ (hexagon $Q_1Q_2P_3Q_3Q_4P_6$ in the case $1 < \gamma \le 2$).

About L^2 estimate, we have

$$\|\tilde{Q}_j f\|_{\dot{F}^{0}_{2,2}(\mathbb{R}^n)} \le C\left(\frac{1}{1+|j|}\right)^{\beta/\max(\gamma',2)}. \tag{3.6}$$

In fact, let $\sigma_k = \sigma_{\Omega,h,\phi,k}$. Then we have

$$\hat{\sigma}_k(\xi) = \int_{2^{k-1}}^{2^k} \int_{S^{n-1}} \Omega(y')h(r)e^{-i\phi(r)y'\cdot\xi}\, d\sigma(y')\frac{dr}{r}.$$

First we have

$$\left|\hat{\sigma}_k(\xi)\right| \le 2\|h\|_{\Delta_\gamma}\|\Omega\|_{L^1(S^{n-1})}. \tag{3.7}$$

Next, using Hölder's inequality and assuming $\|\Omega\|_{L^1(S^{n-1})} \le 1$ without loss of generality, we have

$$
\begin{aligned}
\left|\hat{\sigma}_k(\xi)\right| &\le \left(\int_{2^{k-1}}^{2^k} \left|h(r)\right|^{\gamma} \frac{dr}{r}\right)^{1/\gamma} \left(\int_{2^{k-1}}^{2^k} \left|\int_{S^{n-1}} \Omega(y') e^{-i\phi(r)y'\cdot\xi}\, d\sigma(y')\right|^{\gamma'} \frac{dr}{r}\right)^{1/\gamma'} \\
&\le 2\|h\|_{\Delta_\gamma} \left(\int_{2^{k-1}}^{2^k} \left|\int_{S^{n-1}} \Omega(y') e^{-i\phi(r)y'\cdot\xi}\, d\sigma(y')\right|^{2} \frac{dr}{r}\right)^{\frac{1}{\max(2,\gamma')}} \\
&= 2\|h\|_{\Delta_\gamma} \left(\int_{\phi(2^{k-1})}^{\phi(2^k)} \left|\int_{S^{n-1}} \Omega(y') e^{-iry'\cdot\xi}\, d\sigma(y')\right|^{2} \frac{\phi(\phi^{-1}(r))}{\phi^{-1}(r)\phi'(\phi^{-1}(r))} \frac{dr}{r}\right)^{\frac{1}{\max(2,\gamma')}} \\
&\le 2\|h\|_{\Delta_\gamma} \|\varphi\|_{L^\infty(\mathbb{R}_+)} \left(\int_{\phi(2^{k-1})}^{\phi(2^k)} \left|\int_{S^{n-1}} \Omega(y') e^{-iry'\cdot\xi}\, d\sigma(y')\right|^{2} \frac{dr}{r}\right)^{\frac{1}{\max(2,\gamma')}} \\
&= 2\|h\|_{\Delta_\gamma} \|\varphi\|_{L^\infty(\mathbb{R}_+)} \\
&\quad \times \left(\int_{S^{n-1}}\int_{S^{n-1}} \Omega(y')\overline{\Omega(z')} \int_{\phi(2^{k-1})}^{\phi(2^k)} e^{-ir(y'-z')\cdot\xi} \frac{dr}{r}\, d\sigma(y')\, d\sigma(z')\right)^{\frac{1}{\max(2,\gamma')}}.
\end{aligned}
$$

We see that

$$
\left|\int_{\phi(2^{k-1})}^{\phi(2^k)} e^{-ir(y'-z')\cdot\xi} \frac{dr}{r}\right| \le \log\frac{\phi(2^k)}{\phi(2^{k-1})} \le \log c_1.
$$

We see also

$$
\left|\int_{\phi(2^{k-1})}^{\phi(2^k)} e^{-ir(y'-z')\cdot\xi} \frac{dr}{r}\right| \le \frac{2}{\phi(2^{k-1})|\xi||\xi'\cdot(x'-y')|}.
$$

So, as in [10], p.458 (using Lemma 3.1 in [12]), we have for $\beta > 1$,

$$
\begin{aligned}
&\left|\int_{\phi(2^{k-1})}^{\phi(2^k)} e^{-ir(y'-z')\cdot\xi} \frac{dr}{r}\right| \\
&\quad \le \frac{C}{\log^\beta(\log c_1)e\phi(2^{k-1})|\xi|} \log^\beta \frac{2e}{|(y'-z')\cdot\xi'|} \quad \text{for } \phi(2^k)|\xi| \ge \frac{c_1}{\log c_1}.
\end{aligned}
$$

Hence we have

$$
\left|\hat{\sigma}_k(\xi)\right| \le \frac{2\|h\|_{\Delta_\gamma} (W\mathcal{F}_\beta(\Omega))^{2/\max\{\gamma',2\}} \|\Omega\|_{L^1(S^{n-1})}^{1-2/\max\{\gamma',2\}}}{(\log(e(\log c_1)\phi(2^k)|\xi|/c_1))^{\beta/\max\{\gamma',2\}}} \tag{3.8}
$$

for $\phi(2^k)|\xi| \ge \frac{c_1}{\log c_1} \ge e$.

On the other hand, using the cancelation property of Ω, we get easily

$$
\left|\hat{\sigma}_k(\xi)\right| \le 2\|h\|_{\Delta_1} \|\Omega\|_{L^1(S^{n-1})} \phi(2^k)|\xi|. \tag{3.9}
$$

Now we can estimate the L^2 norm of $\tilde{Q}_j f$:

$$
\begin{aligned}
\|\tilde{Q}_j f\|_{\dot{F}^0_{2,2}(\mathbb{R}^n)} &= \left(\int_{\mathbb{R}^n} \left|\sum_{k\in\mathbb{Z}}\sum_{\ell=-1}^{1} S_{j-k+\ell}(\sigma_{\Omega,h,\phi,k} * S_{j-k}f)(x)\right|^2 dx\right)^{1/2} \\
&= \left(\int_{\mathbb{R}^n} \left|\sum_{k\in\mathbb{Z}}\sum_{\ell=-1}^{1} \psi_{j-k+\ell}(\xi)\hat{\sigma}_{\Omega,h,\phi,k}(\xi)\psi_{j-k}(\xi)\hat{f}(\xi)\right|^2 d\xi\right)^{1/2}.
\end{aligned}
$$

Note that $\phi(2^k)|\xi| = |\xi|/a_{-k} \ge a^j|\xi|/a_{j-k} \ge a^j$ for $a_{j-k} \le |\xi| \le a_{j-k+2}$ and $j \ge 0$, where $a_{\ell+1}/a_\ell \ge a = 2^{1/\|\varphi\|_{L^\infty(\mathbb{R}_+)}}$. So, for $j \ge 2\log_a(c_1/\log c_1)$ and $a_{j-k} \le |\xi| \le a_{j-k+2}$, we have

$$\frac{\log c_1}{c_1}\phi\bigl(2^k\bigr)|\xi| \ge a^{\frac{j}{2}} \ge 1,$$

and hence we have

$$\begin{aligned}
\|\tilde{Q}_j f\|_{\dot{F}^0_{2,2}(\mathbb{R}^n)} &\le C\biggl(\sum_{k\in\mathbb{Z}}\int_{a_{j-k}\le|\xi|\le a_{j-k+2}}\bigl|\hat{\sigma}_{\Omega,h,\phi,k}(\xi)\hat{f}(\xi)\bigr|^2\,d\xi\biggr)^{1/2} \\
&\le C\|h\|_{\Delta_\gamma}\biggl(\sum_{k\in\mathbb{Z}}\int_{a_{j-k}\le|\xi|\le a_{j-k+2}}\biggl(\frac{1}{1+j}\biggr)^{2\beta/\max(\gamma',2)}\bigl|\hat{f}(\xi)\bigr|^2\,d\xi\biggr)^{1/2} \\
&\le C\|h\|_{\Delta_\gamma}\biggl(\frac{1}{1+j}\biggr)^{\beta/\max(\gamma',2)}\biggl(\sum_{k\in\mathbb{Z}}\int_{a_{j-k}\le|\xi|\le a_{j-k+2}}\bigl|\hat{f}(\xi)\bigr|^2\,d\xi\biggr)^{1/2} \\
&\le C\|h\|_{\Delta_\gamma}\biggl(\frac{1}{1+j}\biggr)^{\beta/\max(\gamma',2)}\biggl(\int_{\mathbb{R}^n}\bigl|\hat{f}(\xi)\bigr|^2\,d\xi\biggr)^{1/2} \\
&\le C\biggl(\frac{1}{1+j}\biggr)^{\beta/\max(\gamma',2)}\|f\|_{\dot{F}^0_{2,2}(\mathbb{R}^n)}.
\end{aligned}$$

For $j \le -2$, we have $\phi(2^k)|\xi| = |\xi|/a_{-k} \le a^{2+j}$ for $a_{j-k} \le |\xi| \le a_{j-k+2}$. So, using (3.9) we get as before

$$\begin{aligned}
\|\tilde{Q}_j f\|_{\dot{F}^0_{2,2}(\mathbb{R}^n)} &\le C\biggl(\sum_{k\in\mathbb{Z}}\int_{a_{j-k}\le|\xi|\le a_{j-k+2}}\bigl|\hat{\sigma}_{\Omega,\psi,h,k}(\xi)\hat{f}(\xi)\bigr|^2\,d\xi\biggr)^{1/2} \\
&\le C\|h\|_{\Delta_\gamma}\biggl(\sum_{k\in\mathbb{Z}}\int_{a_{j-k}\le|\xi|\le a_{j-k+2}}a^{4+2j}\bigl|\hat{f}(\xi)\bigr|^2\,d\xi\biggr)^{1/2} \\
&\le Ca^j\biggl(\sum_{k\in\mathbb{Z}}\int_{a_{j-k}\le|\xi|\le a_{j-k+2}}\bigl|\hat{f}(\xi)\bigr|^2\,d\xi\biggr)^{1/2} \\
&\le Ca^j\biggl(\int_{\mathbb{R}^n}\bigl|\hat{f}(\xi)\bigr|^2\,d\xi\biggr)^{1/2} \\
&\le Ca^j\|f\|_{\dot{F}^0_{2,2}(\mathbb{R}^n)}.
\end{aligned}$$

For $-2 \le j < 2\log_a(c_1/\log c_1)$, using (3.7), we get

$$\|\tilde{Q}_j f\|_{\dot{F}^0_{2,2}(\mathbb{R}^n)} \le C\biggl(\sum_{k\in\mathbb{Z}}\int_{a_{j-k}\le|\xi|\le a_{j-k+2}}\bigl|\hat{f}(\xi)\bigr|^2\,d\xi\biggr)^{1/2} \le C\|f\|_{\dot{F}^0_{2,2}(\mathbb{R}^n)}.$$

Thus we have (3.6) for $j \in \mathbb{Z}$. Now, let $\mathcal{Q}_1 = (\frac{\max(\gamma',2)}{2\beta}, \frac{\max(\gamma',2)}{2\beta})$, $\mathcal{Q}_2 = (\frac{1}{\gamma'} + \frac{\max(\gamma',2)}{\beta}(\frac{1}{2} - \frac{1}{\gamma'}), \frac{\max(\gamma',2)}{2\beta})$, $\mathcal{P}_3 = (\frac{1}{2} + \frac{1}{\max(\gamma',2)} - \frac{1}{\beta}, \frac{1}{2})$, $\mathcal{Q}_3 = (1 - \frac{\max(\gamma',2)}{2\beta}, 1 - \frac{\max(\gamma',2)}{2\beta})$, $\mathcal{Q}_4 = (\frac{1}{\gamma} - \frac{\max(\gamma',2)}{\beta}(\frac{1}{\gamma} - \frac{1}{2}), 1 - \frac{\max(\gamma',2)}{2\beta})$, $\mathcal{P}_6 = (\frac{1}{2} - \frac{1}{\max(\gamma',2)} + \frac{1}{\beta}, \frac{1}{2})$, $\mathcal{R}_2 = (1 - \frac{1}{2\gamma} - \frac{\max(\gamma',2)}{2\beta\gamma'}, \frac{1}{2\gamma} + \frac{\max(\gamma',2)}{2\beta\gamma'})$, and $\mathcal{R}_4 = (\frac{1}{2\gamma} + \frac{\max(\gamma',2)}{2\beta\gamma'}, 1 - \frac{1}{2\gamma} - \frac{\max(\gamma',2)}{2\beta\gamma'})$. Then, for $(\frac{1}{p}, \frac{1}{q})$ belonging to the interior of the octagon $\mathcal{Q}_1\mathcal{Q}_2\mathcal{R}_2\mathcal{P}_3\mathcal{Q}_3\mathcal{Q}_4\mathcal{R}_4\mathcal{P}_6$ (hexagon $\mathcal{Q}_1\mathcal{Q}_2\mathcal{P}_3\mathcal{Q}_3\mathcal{Q}_4\mathcal{P}_6$ in the case $1 < \gamma \le 2$), we can find $(\frac{1}{p_1}, \frac{1}{q_1})$ in the interior of the octagon $Q_1Q_2R_2P_3Q_3Q_4R_4P_6$ (hexagon $Q_1Q_2P_3Q_3Q_4P_6$

in the case $1 < \gamma \le 2$) such that $\frac{1}{p} = \frac{\theta}{2} + \frac{1-\theta}{p_1}$, $\frac{1}{q} = \frac{\theta}{2} + \frac{1-\theta}{q_1}$, and $1 > \theta > \frac{\max(\gamma',2)}{\beta}$. Hence, for $\alpha \in \mathbb{R}$, taking α_1 with $\alpha = (1-\theta)\alpha_1$ and interpolating between (3.6) and (3.5), we obtain the desired estimate

$$\|\tilde{Q}_j f\|_{\dot{F}^{\alpha}_{p,q}(\mathbb{R}^n)} \le C\bigl(1+|j|\bigr)^{-\theta\beta/\max\{\gamma',2\}} \|f\|_{\dot{F}^{\alpha}_{p,q}(\mathbb{R}^n)}. \tag{3.10}$$

Summing up this with respect to j, we finish the proof of Theorem 1.3(i). The proof of (ii) is the same as in Theorem 1.1(i).

4 Proof of Theorem 1.2

In this section we shall give the proof of Theorem 1.2.

(A) $L\log L$ case. Let $\Omega \in L\log L(S^{n-1})$ satisfying the cancelation property. Then letting $A_m = \|\Omega\chi_{2^{m-1}\le|\Omega(y')|<2^m}\|_{L^1(S^{n-1})}$ and $\Lambda = \{m \in \mathbb{N} : A_m > 2^{-m}\}$, we can construct $\Omega_m \in L^2(S^{n-1})$ ($m \in \Lambda$) and $\Omega_0 \in \bigcap_{1<r<2} L^r(S^{n-1})$ such that

$$\|\Omega_m\|_{L^2(S^{n-1})} \le C2^m, \qquad \|\Omega_m\|_{L^1(S^{n-1})} \le C, \tag{4.1}$$

$$\sum_{m\in\Lambda} mA_m \le C\|\Omega\|_{L\log L(S^{n-1})}, \tag{4.2}$$

$$\int_{S^{n-1}} \Omega_m(y')\,d\sigma(y') = 0 \quad (m = 0, m \in \Lambda), \qquad \Omega = \Omega_0 + \sum_{m\in\Lambda} A_m\Omega_m. \tag{4.3}$$

From the above, we see that

$$T_{\Omega,\psi,h}f = T_{\Omega_0,h,\phi}f + \sum_{m\in\Lambda} A_m T_{\Omega_m,h,\phi}f. \tag{4.4}$$

So, we consider $T_{\Omega_m,h,\phi}$. We use the notations in Section 3 with minor change such as $\tilde{Q}_{m,j}$ for Ω_m instead of $\tilde{Q}_j$ for Ω. Since $\|\Omega_m\|_{L^1(S^{n-1})} \le C$, we have as in Section 3 that

$$\|\tilde{Q}_{m,j}f\|_{\dot{F}^{\alpha}_{p,q}(\mathbb{R}^n)} \le C\|f\|_{\dot{F}^{\alpha}_{p,q}(\mathbb{R}^n)} \tag{4.5}$$

if $\alpha \in \mathbb{R}$ and $(\frac{1}{p}, \frac{1}{q})$ belongs to the interior of the octagon $Q_1Q_2R_2P_3Q_3Q_4R_4P_6$ (hexagon $Q_1Q_2P_3Q_3Q_4P_6$ in the case $1 < \gamma \le 2$).

About L^2 estimate, we have

$$\|\tilde{Q}_{m,j}f\|_{\dot{F}^{0}_{2,2}(\mathbb{R}^n)} \le Ca^{-\frac{\beta}{m}|j|} \|f\|_{\dot{F}^{0}_{2,2}(\mathbb{R}^n)} \tag{4.6}$$

for some β with $0 < \beta < 1/2$. In fact, let $\sigma_{m,k} = \sigma_{\Omega_m,h,\phi,k}$. Since $\|\Omega_m\|_{L^1(S^{n-1})} \le C$ and $\|\Omega_m\|_{L^2(S^{n-1})} \le C2^m$, we get by Lemma 3.1 in [23], p.1567,

$$\bigl|\hat{\sigma}_{m,k}(\xi)\bigr| \le C\|h\|_{\Delta_1}, \tag{4.7}$$

$$\bigl|\hat{\sigma}_{m,k}(\xi)\bigr| \le C\frac{\|h\|_{\Delta_\gamma}(1+\|\varphi\|_\infty)}{|\phi(2^{k-1})\xi|^{\frac{\beta}{m}}}, \tag{4.8}$$

$$\bigl|\hat{\sigma}_{m,k}(\xi)\bigr| \le C\|h\|_{\Delta_1}\bigl|\phi(2^k)\xi\bigr|^{\frac{\beta}{m}}, \tag{4.9}$$

where β is a fixed constant with $0 < \beta < 1/2$. Using Plancherel's theorem, (4.8), the support property of ψ_j, and $a_{k+1}/c_1 \leq a_k \leq a_{k+1}/a$, we get for $j \geq 0$,

$$\begin{aligned}
\|\tilde{Q}_{m,j}f\|_{\dot{F}^0_{2,2}(\mathbb{R}^n)} &\leq C\biggl(\sum_{k\in\mathbb{Z}}\int_{a_{j-k}\leq|\xi|<a_{j-k+2}}\bigl|\hat{\sigma}_{m,k}(\xi)\hat{f}(\xi)\bigr|^2\,d\xi\biggr)^{\frac{1}{2}} \\
&\leq C\biggl(\sum_{k\in\mathbb{Z}}\int_{a_{j-k}\leq|\xi|<a_{j-k+2}}\frac{1}{|\phi(2^{k-1})\xi|^{\frac{\beta}{m}}}\bigl|\hat{f}(\xi)\bigr|^2\,d\xi\biggr)^{\frac{1}{2}} \\
&\leq Ca^{-\frac{\beta}{m}j}\biggl(\sum_{k\in\mathbb{Z}}\int_{a_{j-k}\leq|\xi|<a_{j-k+2}}\bigl|\hat{f}(\xi)\bigr|^2\,d\xi\biggr)^{\frac{1}{2}} \\
&\leq Ca^{-\frac{\beta}{m}j}\|f\|_{\dot{F}^0_{2,2}(\mathbb{R}^n)}.
\end{aligned}$$

For $j < 0$, using (4.9) in place of (4.8), we get

$$\|\tilde{Q}_{m,j}f\|_{\dot{F}^0_{2,2}(\mathbb{R}^n)} \leq Ca^{\frac{\beta}{m}j}\|f\|_{\dot{F}^0_{2,2}(\mathbb{R}^n)}.$$

This shows (4.6). Interpolating (4.6) and (4.5), we obtain for some $0 < \theta < 1$,

$$\|\tilde{Q}_{m,j}f\|_{\dot{F}^\alpha_{p,q}(\mathbb{R}^n)} \leq Ca^{-\frac{\beta\theta}{m}|j|}\|f\|_{\dot{F}^\alpha_{p,q}(\mathbb{R}^n)} \tag{4.10}$$

provided $\alpha \in \mathbb{R}$ and $(\frac{1}{p}, \frac{1}{q})$ belongs to the interior of the octagon $Q_1Q_2R_2P_3Q_3Q_4R_4P_6$ (hexagon $Q_1Q_2P_3Q_3Q_4P_6$ in the case $1 < \gamma \leq 2$).

From (4.10) and the definition of $\tilde{Q}_{m,j}$ it follows

$$\|T_{\Omega_m,h,\phi}f\|_{\dot{F}^\alpha_{p,q}(\mathbb{R}^n)} \leq C\sum_{j\in\mathbb{Z}}a^{-\frac{\beta\theta}{m}|j|}\|f\|_{\dot{F}^\alpha_{p,q}(\mathbb{R}^n)} \leq \frac{C}{1-a^{-\frac{\beta\theta}{m}}}\|f\|_{\dot{F}^\alpha_{p,q}(\mathbb{R}^n)} \leq C\frac{m}{\beta\theta}\|f\|_{\dot{F}^\alpha_{p,q}(\mathbb{R}^n)}.$$

We can see that the same estimate holds for Ω_0. Thus, by (4.4) we have

$$\|T_{\Omega,h,\phi}f\|_{\dot{F}^\alpha_{p,q}(\mathbb{R}^n)} \leq C\biggl(1+\sum_{m\in\Lambda}A_m m\biggr)\|f\|_{\dot{F}^\alpha_{p,q}(\mathbb{R}^n)} \leq C\|f\|_{\dot{F}^\alpha_{p,q}(\mathbb{R}^n)}$$

provided $\alpha \in \mathbb{R}$ and $(\frac{1}{p}, \frac{1}{q})$ belongs to the interior of the octagon $Q_1Q_2R_2P_3Q_3Q_4R_4P_6$ (hexagon $Q_1Q_2P_3Q_3Q_4P_6$ in the case $1 < \gamma \leq 2$). This completes the proof of Theorem 1.2 in the case $\Omega \in L\log L(S^{n-1})$.

(B) Block space case. Let $r > 1$. Then if $\Omega \in B_r^{(0,0)}(S^{n-1})$ and satisfies the cancelation condition, it can be written as $\Omega = \sum_{\ell=1}^{\infty}\lambda_\ell\breve{\Omega}_\ell$, where $\lambda_\ell \in \mathbb{C}$ and $\breve{\Omega}_\ell$ is an r-block supported on a cap $B_\ell = B(x_\ell, \tau_\ell) \cap S^{n-1}$ on S^{n-1} and

$$\sum_{\ell=1}^{\infty}|\lambda_\ell|\bigl\{1+\log\bigl(|B_\ell|^{-1}\bigr)\bigr\} < 2\|\Omega\|_{B_r^{(0,0)}(S^{n-1})} < \infty. \tag{4.11}$$

To each block $\breve{\Omega}_\ell$, we define

$$\Omega_\ell\bigl(y'\bigr) = \breve{\Omega}_\ell\bigl(y'\bigr) - \frac{1}{|S^{n-1}|}\int_{S^{n-1}}\breve{\Omega}_\ell\bigl(x'\bigr)\,d\sigma\bigl(x'\bigr).$$

Let $\Lambda = \{\ell \in \mathbb{N}; |B_\ell| \le 1/2\}$ and set

$$\Omega_0 = \Omega - \sum_{\ell\in\Lambda} \lambda_\ell \Omega_\ell. \tag{4.12}$$

Then there exists a positive constant C such that the following hold for all $\ell \in \Lambda$:

$$\int_{S^{n-1}} \Omega_\ell(x')\,d\sigma(x') = 0, \tag{4.13}$$

$$\|\Omega_\ell\|_{L^r(S^{n-1})} \le C|B_\ell|^{-1/r'}, \tag{4.14}$$

$$\|\Omega_\ell\|_{L^1(S^{n-1})} \le 2, \tag{4.15}$$

$$\Omega = \Omega_0 + \sum_{\ell\in\Lambda} \lambda_\ell \Omega_\ell. \tag{4.16}$$

Moreover, from (4.11) and the definition of Ω_ℓ it follows that

$$\|\Omega_0\|_{L^r(S^{n-1})} \le C \sum_{\ell\in\mathbb{N}\setminus\Lambda} 2^{-1/r'}|\lambda_\ell| \le C\|\Omega\|_{B_r^{(0,0)}(S^{n-1})}, \tag{4.17}$$

$$\int_{S^{n-1}} \Omega_0(x')\,d\sigma(x') = 0. \tag{4.18}$$

By (4.16), we have

$$T_{\Omega,h,\phi}f(x) = \sum_{\ell\in\Lambda\cup 0} \lambda_\ell T_{\Omega_\ell,h,\phi}f(x). \tag{4.19}$$

So, we have only to show the boundedness of $T_{\Omega_\ell,h,\phi}f$. We use the notations in Section 3 with minor change such as $\tilde{Q}_{\ell,j}$ for Ω_ℓ instead of $\tilde{Q}_j$ for Ω. Since $\|\Omega_\ell\|_{L^1(S^{n-1})} \le C$, we have as in Section 3 that

$$\|\tilde{Q}_{\ell,j}f\|_{\dot{F}^{\alpha}_{p,q}(\mathbb{R}^n)} \le C\|f\|_{\dot{F}^{\alpha}_{p,q}(\mathbb{R}^n)} \tag{4.20}$$

if $\alpha \in \mathbb{R}$ and $(\frac{1}{p}, \frac{1}{q})$ belongs to the interior of the octagon $Q_1Q_2R_2P_3Q_3Q_4R_4P_6$ (hexagon $Q_1Q_2P_3Q_3Q_4P_6$ in the case $1 < \gamma \le 2$).

About L^2 estimate, we have

$$\|\tilde{Q}_{\ell,j}f\|_{\dot{F}^{0}_{2,2}(\mathbb{R}^n)} \le Ca^{-\frac{\beta}{m_\ell}|j|}\|f\|_{\dot{F}^{0}_{2,2}(\mathbb{R}^n)} \tag{4.21}$$

for some β with $0 < \beta < r$. In fact, let $\sigma_{\ell,k} = \sigma_{\Omega_\ell,h,\phi,k}$. For $\ell \in \Lambda \cup \{0\}$, we set $m_\ell = [\log_2 |B_\ell|^{-1/r'}] + 1$, where $[\,\cdot\,]$ denotes the greatest integer function.

Since $\|\Omega_\ell\|_{L^1(S^{n-1})} \le 2$ and $\|\Omega_\ell\|_{L^2(S^{n-1})} \le C2^{m_\ell}$, we get by Lemma 3.1 in [23], p.1567

$$|\hat{\sigma}_{\ell,k}(\xi)| \le C\|h\|_{\Delta_1}, \tag{4.22}$$

$$|\hat{\sigma}_{\ell,k}(\xi)| \le C\frac{\|h\|_{\Delta_\gamma}(1 + \|\varphi\|_\infty)}{|\phi(2^{k-1})\xi|^{\frac{\beta}{m_\ell}}}, \tag{4.23}$$

$$|\hat{\sigma}_{\ell,k}(\xi)| \le C\|h\|_{\Delta_1}|\phi(2^k)\xi|^{\frac{\beta}{m_\ell}}, \tag{4.24}$$

where β is a fixed constant with $0 < \beta < r$. Using Plancherel's theorem, (4.23), the support property of ψ_j, and $a_{k+1}/c_1 \le a_k \le a_{k+1}/a$, we get for $j \ge 0$,

$$\begin{aligned}\|\tilde{Q}_{\ell,j}f\|_{\dot{F}^0_{2,2}(\mathbb{R}^n)} &\le C\biggl(\sum_{k\in\mathbb{Z}}\int_{a_{j-k}\le|\xi|<a_{j-k+2}}\bigl|\hat{\sigma}_{\ell,k}(\xi)\hat{f}(\xi)\bigr|^2\,d\xi\biggr)^{\frac{1}{2}}\\ &\le C\biggl(\sum_{k\in\mathbb{Z}}\int_{a_{j-k}\le|\xi|<a_{j-k+2}}\frac{1}{|\phi(2^{k-1})\xi|^{\frac{\beta}{m_\ell}}}\bigl|\hat{f}(\xi)\bigr|^2\,d\xi\biggr)^{\frac{1}{2}}\\ &\le Ca^{-\frac{\beta}{m_\ell}j}\biggl(\sum_{k\in\mathbb{Z}}\int_{a_{j-k}\le|\xi|<a_{j-k+2}}\bigl|\hat{f}(\xi)\bigr|^2\,d\xi\biggr)^{\frac{1}{2}}\\ &\le Ca^{-\frac{\beta}{m_\ell}j}\|f\|_{\dot{F}^0_{2,2}(\mathbb{R}^n)}.\end{aligned}$$

For $j < 0$, using (4.9) in place of (4.8), we get

$$\|\tilde{Q}_{\ell,j}f\|_{\dot{F}^0_{2,2}(\mathbb{R}^n)} \le Ca^{\frac{\beta}{m_\ell}j}\|f\|_{\dot{F}^0_{2,2}(\mathbb{R}^n)}.$$

This shows (4.21). Interpolating (4.21) and (4.20), we obtain for some $0 < \theta < 1$,

$$\|\tilde{Q}_{\ell,j}f\|_{\dot{F}^\alpha_{p,q}(\mathbb{R}^n)} \le Ca^{-\frac{\beta\theta}{m_\ell}|j|}\|f\|_{\dot{F}^\alpha_{p,q}(\mathbb{R}^n)} \tag{4.25}$$

provided $\alpha \in \mathbb{R}$ and $(\frac{1}{p}, \frac{1}{q})$ belongs to the interior of the octagon $Q_1Q_2R_2P_3Q_3Q_4R_4P_6$ (hexagon $Q_1Q_2P_3Q_3Q_4P_6$ in the case $1 < \gamma \le 2$).

From (4.25) and the definition of $\tilde{Q}_{\ell,j}$ it follows

$$\|T_{\Omega_\ell,h,\phi}f\|_{\dot{F}^\alpha_{p,q}(\mathbb{R}^n)} \le C\sum_{j\in\mathbb{Z}}a^{-\frac{\beta\theta}{m_\ell}|j|}\|f\|_{\dot{F}^\alpha_{p,q}(\mathbb{R}^n)} \le \frac{C}{1-a^{-\frac{\beta\theta}{m_\ell}}}\|f\|_{\dot{F}^\alpha_{p,q}(\mathbb{R}^n)} \le C\frac{m_\ell}{\beta\theta}\|f\|_{\dot{F}^\alpha_{p,q}(\mathbb{R}^n)}.$$

We can see that the same estimate holds for Ω_0. Thus, by (4.14) we have

$$\|T_{\Omega,h,\phi}f\|_{\dot{F}^\alpha_{p,q}(\mathbb{R}^n)} \le C\biggl(1+\sum_{\ell\in\Lambda}\lambda_\ell m_\ell\biggr)\|f\|_{\dot{F}^\alpha_{p,q}(\mathbb{R}^n)} \le C\|f\|_{\dot{F}^\alpha_{p,q}(\mathbb{R}^n)}$$

provided $\alpha \in \mathbb{R}$ and $(\frac{1}{p}, \frac{1}{q})$ belongs to the interior of the octagon $Q_1Q_2R_2P_3Q_3Q_4R_4P_6$ (hexagon $Q_1Q_2P_3Q_3Q_4P_6$ in the case $1 < \gamma \le 2$).

This completes the proof of Theorem 1.2.

Appendix

In this section we shall prove Lemma 2.6. Let $\{a_j\}_{j\in\mathbb{Z}}$, ψ_j, Ψ_j, and S_j be the same as in Lemma 2.6. Set $\eta_j(\xi) = \eta(\xi/a_{j+1})$ and $\hat{\Phi}_j(\xi) = \eta_j(\xi)$. Then we have

$$(-ix)^\alpha\Phi_j(x) = c_n\int_{\mathbb{R}^n}\partial^\alpha\eta_j(\xi)e^{ix\cdot\xi}\,d\xi,$$

so we have

$$\bigl|x^\alpha\bigr|\bigl|\Phi_j(x)\bigr| \le C\int_{\mathbb{R}^n}\bigl|\partial^\alpha\eta_j(\xi)\bigr|\,d\xi \le C\int_{\operatorname{supp}\partial^\alpha\eta_j}\bigl|\partial^\alpha\eta_j(\xi)\bigr|\,d\xi.$$

From this and the definition of η_j we get

$$\left|\Phi_j(x)\right| \le Ca^n a_{j+1}^n, \tag{A.1}$$

and for $N \in \mathbb{N}$,

$$|x|^N\left|\Phi_j(x)\right| \le C\frac{1}{((a-1)a_{j+1})^N}\left(\int_{a_{j+1}}^{aa_{j+1}} r^{n-1}\,dr\right) \le C\frac{(aa_{j+1})^n}{((a-1)a_{j+1})^N}. \tag{A.2}$$

Thus, for $N \in \mathbb{N}$ we have

$$\left|\Phi_j(x)\right| \le C\left(\frac{a}{a-1}\right)^n \frac{((a-1)a_{j+1})^n}{(1+|(a-1)a_{j+1}x|)^N}. \tag{A.3}$$

Next we have

$$(-ix)^\alpha \partial_{x_k}\Phi_j(x) = c_n\int_{\mathbb{R}^n} \partial^\alpha(i\xi_k\eta_j)(\xi)e^{ix\cdot\xi}\,d\xi,$$

so we have

$$\left|x^\alpha\right|\left|\partial_{x_k}\Phi_j(x)\right| \le C\int_{\mathbb{R}^n}\left|\partial^\alpha(\xi_k\eta_j)(\xi)\right|d\xi \le C\int_{\operatorname{supp}\partial^\alpha(\xi_k\eta_j)}\left|\partial^\alpha(\xi_k\eta_j)(\xi)\right|d\xi.$$

From this and the definition of η_j we get

$$\left|\nabla\Phi_j(x)\right| \le C(aa_{j+1})^{n+1}, \tag{A.4}$$

and for $N \in \mathbb{N}$,

$$\begin{aligned}|x|^N\left|\nabla\Phi_j(x)\right| &\le C\frac{1}{((a-1)a_{j+1})^{N-1}}\int_{a_{j+1}}^{aa_{j+1}} r^{n-1}\,dr + C\frac{1}{((a-1)a_{j+1})^N}\int_{a_{j+1}}^{aa_{j+1}} r^n\,dr\\ &\le C\frac{(aa_{j+1})^n}{((a-1)a_{j+1})^{N-1}} + C\frac{(aa_{j+1})^{n+1}}{((a-1)a_{j+1})^N}.\end{aligned} \tag{A.5}$$

Thus, for $N \in \mathbb{N}$ we have

$$\left|\nabla\Phi_j(x)\right| \le C\left(\frac{a}{a-1}\right)^{n+1} \frac{((a-1)a_{j+1})^{n+1}}{(1+|(a-1)a_{j+1}x|)^N}. \tag{A.6}$$

Let $b = a-1$ and $B = (\frac{a}{a-1})^{n+1}$. Taking $N = n+1$ and using (A.3) and (A.6), we obtain

$$\begin{aligned}I &:= \int_{|x|>2|y|}\left(\sum_{k\in\mathbb{Z}}\left|\Phi_k(x-y)-\Phi_k(x)\right|^2\right)^{1/2}dx\\ &\le \int_{|x|>2|y|}\sum_{k\in\mathbb{Z}}\left|\Phi_k(x-y)-\Phi_k(x)\right|dx\\ &\le \sum_{a_{k+1}<|by|^{-1}}\int_{|x|>2|y|}\left|\Phi_k(x-y)-\Phi_k(x)\right|dx\end{aligned}$$

$$
\begin{aligned}
&\quad + \sum_{a_{k+1} \geq |by|^{-1}} \int_{|x|>2|y|} \left|\Phi_k(x-y)\right| + \left|\Phi_k(x)\right| dx \\
&\leq \sum_{a_{k+1} < |by|^{-1}} \int_{|x|>2|y|} |y| \left|\nabla \Phi_k(x-\theta y)\right| dx \\
&\quad + \sum_{a_{k+1} \geq |by|^{-1}} \int_{|x|>2|y|} \left|\Phi_k(x-y)\right| + \left|\Phi_k(x)\right| dx \\
&\leq CB \sum_{a_{k+1} < |by|^{-1}} |ba_{k+1}y| \int_{\mathbb{R}^n} \frac{(ba_{k+1})^n}{(1+|ba_{k+1}x|)^{n+1}} dx \\
&\quad + CB \sum_{a_{k+1} \geq |by|^{-1}} \int_{|x|>2|y|} \frac{(ba_{k+1})^n}{(|ba_{k+1}x|)^{n+1}} dx \\
&\leq CB \sum_{a_{k+1} < |by|^{-1}} |by| a_{k+1} \int_{\mathbb{R}^n} \frac{1}{(1+|x|)^{n+1}} dx \\
&\quad + CB \sum_{a_{k+1} \geq |by|^{-1}} \int_{2|ba_{k+1}y|}^{\infty} \frac{1}{r^2} dr \\
&\leq CB \sum_{a_{k+1} < |by|^{-1}} |by| a_{k+1} + CB \sum_{a_{k+1} \geq |by|^{-1}} \frac{1}{|by|} \frac{1}{a_{k+1}}.
\end{aligned}
$$

Let k_0 be the integer satisfying $a_{k_0} < |(a-1)y|^{-1} \leq a_{k_0+1}$. Then we have

$$
I \leq \frac{CB}{a_{k_0}} \sum_{k \leq k_0-1} a_{k+1} + CBa_{k_0+1} \sum_{k \geq k_0} \frac{1}{a_{k+1}}.
$$

From $a_{k+1}/a_k \geq a$ it follows that $a_{k+1} \leq a^{-1} a_{k+2} \leq \cdots \leq a^{k-k_0+1} a_{k_0}$ for $k \leq k_0$. Hence we get

$$
\sum_{k \leq k_0-1} a_{k+1} \leq \sum_{k \leq k_0-1} a_{k_0} a^{k-k_0+1} = a_{k_0} \sum_{k=0}^{\infty} \frac{1}{a^k} = a_{k_0} \frac{a}{a-1}.
$$

From $a_{k+1}/a_k \geq a$ it follows that $a_{k+1} \geq a a_k \geq \cdots \geq a^{k-k_0} a_{k_0+1}$ for $k \geq k_0$. Hence we get

$$
\sum_{k \geq k_0} \frac{1}{a_{k+1}} \leq \sum_{k \geq k_0} \frac{1}{a_{k_0+1}} \frac{1}{a^{k-k_0}} = \frac{1}{a_{k_0+1}} \sum_{k=0}^{\infty} \frac{1}{a^k} = \frac{1}{a_{k_0+1}} \frac{a}{a-1}.
$$

Thus we have

$$
I \leq C \left(\frac{a}{a-1} \right)^{n+2}.
$$

If we define Φ_j^1 by $\hat{\Phi}_j^1(\xi) = \eta(\xi/a_j)$, we get $\Phi_j^1 = \Phi_{j-1}$. So, for Φ_j^1 we have the same estimate as for Φ_j. Therefore, we obtain

$$
\int_{|x|>2|y|} \left(\sum_{k \in \mathbb{Z}} \left|\Psi_k(x-y) - \Psi_k(x)\right|^2 \right)^{1/2} dx \leq C \left(\frac{a}{a-1} \right)^{n+2}. \tag{A.7}
$$

Now let $\mathcal{B}_1 = \mathbb{C}$, $\mathcal{B}_2 = \ell^2$, define an ℓ^2-valued function $\vec{K}(x)$ by $\vec{K}(x) = \{\Psi_k(x)\}_{k\in\mathbb{Z}}$, and the linear operator $\vec{T}$ by $\vec{T}(f) = \vec{K} * f$ for $f \in L^\infty(\mathbb{R}^n)$ with compact support. Then we have $\|\|\vec{T}(f)\|_{\mathcal{B}_2}\|_{L^r(\mathbb{R}^n)} = \|(\sum_{k\in\mathbb{Z}} |\Psi_k * f|^2)^{\frac{1}{2}}\|_{L^r(\mathbb{R}^n)}$, and so by Littlewood-Paley theory this is equivalent to $\|f\|_{L^r(\mathbb{R}^n)}$ for any $1 < r < \infty$. By (A.7), the kernel $\vec{K}(x)$ satisfies the Hörmander condition. Thus, we can apply Proposition 4.6.4 in Grafakos [21], and get the conclusion of Lemma 2.6.

Competing interests

The author declares that he has no competing interests.

Author's contributions

The author contributed to the writing of this paper. He read and approved the final manuscript.

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