Hu and Xue Journal of Inequalities and Applications (2015) 2015:98
DOI 10.1186/s13660-015-0622-7

Journal of Inequalities and Applications
a SpringerOpen Journal

RESEARCH **Open Access**

A note on reverses of Young type inequalities

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Abstract

In this paper, we obtain some improved reverses of Young type inequalities which were established by Burqan and Khandaqji (J. Math. Inequal. 9:113-120, 2015).

MSC: 15A45; 15A60

Keywords: unitarily invariant norms; Young type inequalities; positive semidefinite matrices; singular values

1 Introduction

Let M_n be the space of $n \times n$ complex matrices. Let $\|\cdot\|$ denote any unitarily invariant norm on M_n. So, $\|UAV\| = \|A\|$ for all $A \in M_n$ and for all unitary matrices $U, V \in M_n$. For $A = [a_{ij}] \in M_n$, the Hilbert-Schmidt norm and the trace norm of A are defined by $\|A\|_2 = \sqrt{\sum_{j=1}^n s_j^2(A)}$, $\|A\|_1 = \sum_{j=1}^n s_j(A)$, respectively, where $s_i(A)$ $(i = 1, \dots, n)$ are the singular values of A with $s_1(A) \geq \cdots \geq s_n(A)$, which are the eigenvalues of the positive semidefinite matrix $|A| = (AA^*)^{\frac{1}{2}}$, arranged in decreasing order and repeated according to multiplicity.

The classical Young inequality says that if $a, b \geq 0$ and $0 \leq \nu \leq 1$, then

$$a^\nu b^{1-\nu} \leq \nu a + (1-\nu)b \tag{1}$$

with equality if and only if $a = b$.

Kittaneh and Manasrah [1] obtained an improvement of inequality (1) which can be stated as follows:

$$a^\nu b^{1-\nu} + r_0(\sqrt{a} - \sqrt{b})^2 \leq \nu a + (1-\nu)b, \tag{2}$$

where $r_0 = \min\{\nu, 1-\nu\}$.

Recently, Burqan and Khandaqji [2] gave the following reverses of the scalar Young type inequalities:

$$\nu^2 a^2 + (1-\nu)^2 b^2 \leq (1-\nu)^2 (a-b)^2 + a^{2\nu}\big[(1-\nu)b\big]^{2-2\nu}, \quad 0 \leq \nu \leq \frac{1}{2}, \tag{3}$$

and

$$\nu^2 a^2 + (1-\nu)^2 b^2 \leq \nu^2 (a-b)^2 + (\nu a)^{2\nu} b^{2-2\nu}, \quad \frac{1}{2} \leq \nu \leq 1. \tag{4}$$

A matrix Young inequality, proved in [3], says that if $A, B \in M_n$ are positive semidefinite, then

$$s_j\left(A^\nu B^{1-\nu}\right) \le s_j\left(\nu A + (1-\nu)B\right)$$

for $j = 1, \ldots, n$.

Based on the reverses of the scalar Young type inequalities (3) and (4), Burqan and Khandaqji proved the following in [2] if $A, B, X \in M_n$ such that A and B are positive semidefinite. If $0 \le \nu \le \frac{1}{2}$, then

$$\begin{aligned}&\left\|\nu AX + (1-\nu)XB\right\|_2^2\\&\quad\le (1-\nu)^2\|AX - XB\|_2^2 + 2\nu(1-\nu)\left\|A^{\frac{1}{2}}XB^{\frac{1}{2}}\right\|_2^2 + (1-\nu)^{2(1-\nu)}\left\|A^\nu XB^{1-\nu}\right\|_2^2.\end{aligned} \tag{5}$$

If $\frac{1}{2} \le \nu \le 1$, then

$$\left\|\nu AX + (1-\nu)XB\right\|_2^2 \le \nu^2\|AX - XB\|_2^2 + 2\nu(1-\nu)\left\|A^{\frac{1}{2}}XB^{\frac{1}{2}}\right\|_2^2 + \nu^{2\nu}\left\|A^\nu XB^{1-\nu}\right\|_2^2. \tag{6}$$

At the same time, Burqan and Khandaqji proved the following in [2] if $A, B \in M_n$ such that A and B are positive semidefinite. If $0 \le \nu \le \frac{1}{2}$, then

$$\begin{aligned}&(1-\nu)^{1-\nu}\left\|A^\nu\right\|_2\left\|B^{1-\nu}\right\|_2\\&\quad\ge \sqrt{\nu^2\|A\|_2^2 + (1-\nu)^2\|B\|_2^2 - (1-\nu)^2\left(\|A\|_2^2 + \|B\|_2^2 - 2\|AB\|_1\right)}.\end{aligned} \tag{7}$$

If $\frac{1}{2} \le \nu \le 1$, then

$$\nu^\nu\left\|A^\nu\right\|_2\left\|B^{1-\nu}\right\|_2 \ge \sqrt{\nu^2\|A\|_2^2 + (1-\nu)^2\|B\|_2^2 - \nu^2\left(\|A\|_2^2 + \|B\|_2^2 - 2\|AB\|_1\right)}. \tag{8}$$

For more information on matrix versions of the Young inequality (1) the reader is referred to [4–9].

The main purpose of this paper is to give improved reverses of Young type inequalities (3) and (4). Then we use these inequalities to establish corresponding inequalities for matrices. To achieve our goal we need the following reverses of Young type inequalities for scalars.

2 Reverses of Young type inequalities for scalars

We begin this section with the reverses of Young type inequalities for scalars.

Theorem 1 *Let $a, b \ge 0$. If $0 \le \nu \le \frac{1}{2}$, then*

$$\nu^2a^2 + (1-\nu)^2b^2 + r_0a\left(\sqrt{(1-\nu)b} - \sqrt{a}\right)^2 \le (1-\nu)^2(a-b)^2 + a^{2\nu}\left[(1-\nu)b\right]^{2-2\nu}, \tag{9}$$

where $r_0 = \min\{2\nu, 1-2\nu\}$.

If $\frac{1}{2} \le \nu \le 1$, then

$$\nu^2a^2 + (1-\nu)^2b^2 + r_0b(\sqrt{b} - \sqrt{\nu a})^2 \le \nu^2(a-b)^2 + (\nu a)^{2\nu}b^{2-2\nu}, \tag{10}$$

where $r_0 = \min\{2\nu - 1, 2 - 2\nu\}$.

Proof If $0 \le \nu \le \frac{1}{2}$, then, by inequality (2), we have

$$\begin{aligned}
&(1-\nu)^2(a-b)^2 - \nu^2 a^2 - (1-\nu)^2 b^2 - r_0 a\bigl(\sqrt{(1-\nu)b} - \sqrt{a}\bigr)^2 + a^{2\nu}\bigl[(1-\nu)b\bigr]^{2-2\nu} \\
&\quad = a\bigl[(1-2\nu)a + 2\nu(1-\nu)b\bigr] - r_0 a\bigl(\sqrt{(1-\nu)b} - \sqrt{a}\bigr)^2 - 2(1-\nu)ab + a^{2\nu}\bigl[(1-\nu)b\bigr]^{2-2\nu} \\
&\quad \ge a\bigl\{a^{1-2\nu}\bigl[(1-\nu)b\bigr]^{2\nu}\bigr\} - 2(1-\nu)ab + a^{2\nu}\bigl[(1-\nu)b\bigr]^{2-2\nu} \\
&\quad = a^{2-2\nu}\bigl[(1-\nu)b\bigr]^{2\nu} + a^{2\nu}\bigl[(1-\nu)b\bigr]^{2-2\nu} - 2(1-\nu)ab \\
&\quad = \bigl[a^{1-\nu}(1-\nu)^{\nu} b^{\nu} - a^{\nu}(1-\nu)^{1-\nu} b^{1-\nu}\bigr]^2 \ge 0,
\end{aligned}$$

and so

$$\nu^2 a^2 + (1-\nu)^2 b^2 + r_0 a\bigl(\sqrt{(1-\nu)b} - \sqrt{a}\bigr)^2 \le (1-\nu)^2(a-b)^2 + a^{2\nu}\bigl[(1-\nu)b\bigr]^{2-2\nu}.$$

If $\frac{1}{2} \le \nu \le 1$, then

$$\begin{aligned}
&\nu^2(a-b)^2 - \nu^2 a^2 - (1-\nu)^2 b^2 - r_0 b(\sqrt{b} - \sqrt{\nu a})^2 + (\nu a)^{2\nu} b^{2-2\nu} \\
&\quad = (2\nu-1)b^2 + (2-2\nu)\nu ab - r_0 b(\sqrt{b} - \sqrt{\nu a})^2 - 2\nu ab + (\nu a)^{2\nu} b^{2-2\nu} \\
&\quad = b\bigl[(2\nu-1)b + (2-2\nu)\nu a - r_0(\sqrt{b} - \sqrt{\nu a})^2\bigr] - 2\nu ab + (\nu a)^{2\nu} b^{2-2\nu} \\
&\quad \ge b\bigl[b^{2\nu-1}(\nu a)^{2-2\nu}\bigr] - 2\nu ab + (\nu a)^{2\nu} b^{2-2\nu} \\
&\quad = \bigl[b^{\nu}(\nu a)^{1-\nu} - (\nu a)^{\nu} b^{1-\nu}\bigr]^2 \ge 0,
\end{aligned}$$

and so

$$\nu^2 a^2 + (1-\nu)^2 b^2 + r_0 b(\sqrt{b} - \sqrt{\nu a})^2 \le \nu^2(a-b)^2 + (\nu a)^{2\nu} b^{2-2\nu}.$$

This completes the proof. □

Remark 1 Obviously, (9) and (10) are improvement reverses of the scalar Young type inequalities (3) and (4).

3 Reverses of Young type inequalities for matrices

Based on the reverses of the scalar Young type inequalities (9) and (10), we obtain matrix versions of these inequalities.

Theorem 2 *Let $A, B, X \in M_n$ such that A and B are positive semidefinite. If $0 \le \nu \le \frac{1}{2}$, then*

$$\begin{aligned}
&\bigl\|\nu AX + (1-\nu)XB\bigr\|_2^2 + r_0\bigl[(1-\nu)\bigl\|A^{\frac{1}{2}}XB^{\frac{1}{2}}\bigr\|_2^2 + \|AX\|_2^2 - 2\sqrt{1-\nu}\bigl\|A^{\frac{3}{4}}XB^{\frac{1}{4}}\bigr\|_2^2\bigr] \\
&\quad \le (1-\nu)^2\|AX - XB\|_2^2 + 2\nu(1-\nu)\bigl\|A^{\frac{1}{2}}XB^{\frac{1}{2}}\bigr\|_2^2 + (1-\nu)^{2(1-\nu)}\bigl\|A^{\nu}XB^{1-\nu}\bigr\|_2^2, \qquad (11)
\end{aligned}$$

where $r_0 = \min\{2\nu, 1-2\nu\}$.

If $\frac{1}{2} \le \nu \le 1$, then

$$\begin{aligned}
&\bigl\|\nu AX + (1-\nu)XB\bigr\|_2^2 + r_0\bigl[\nu\bigl\|A^{\frac{1}{2}}XB^{\frac{1}{2}}\bigr\|_2^2 + \|XB\|_2^2 - 2\sqrt{\nu}\bigl\|A^{\frac{1}{4}}XB^{\frac{3}{4}}\bigr\|_2^2\bigr] \\
&\quad \le \nu^2\|AX - XB\|_2^2 + 2\nu(1-\nu)\bigl\|A^{\frac{1}{2}}XB^{\frac{1}{2}}\bigr\|_2^2 + \nu^{2\nu}\bigl\|A^{\nu}XB^{1-\nu}\bigr\|_2^2, \qquad (12)
\end{aligned}$$

where $r_0 = \min\{2\nu-1, 2-2\nu\}$.

Proof Since every positive semidefinite matrix is unitarily diagonalizable, it follows that there are unitary matrices $U, V \in M_n$ such that $A = UDU^*$ and $B = VEV^*$, where

$$D = \operatorname{diag}(\lambda_1, \ldots, \lambda_n), \qquad E = \operatorname{diag}(\mu_1, \ldots, \mu_n), \quad \text{and } \lambda_i, \mu_i \geq 0, i = 1, \ldots, n.$$

Let $Y = U^*XV = [y_{ij}]$. Then

$$\nu AX + (1-\nu)XB = U\bigl(\nu DY + (1-\nu)YE\bigr)V^* = U\bigl[\bigl(\nu\lambda_i + (1-\nu)\mu_j\bigr)y_{ij}\bigr]V^*,$$

$$AX - XB = U\bigl[(\lambda_i - \mu_j)y_{ij}\bigr]V^*, \qquad A^{\frac{1}{2}}XB^{\frac{1}{2}} = U\bigl[\lambda_i^{\frac{1}{2}}\mu_j^{\frac{1}{2}}y_{ij}\bigr]V^*,$$

and

$$A^\nu XB^{1-\nu} = U\bigl[\lambda_i^\nu \mu_j^{1-\nu} y_{ij}\bigr]V^*.$$

If $0 \leq \nu \leq \frac{1}{2}$, by inequality (9), we have

$$\begin{aligned}
&\bigl\|\nu AX + (1-\nu)XB\bigr\|_2^2 \\
&\quad= \sum_{i,j=1}^n \bigl(\nu\lambda_i + (1-\nu)\mu_j\bigr)^2 |y_{ij}|^2 \\
&\quad\leq (1-\nu)^2 \sum_{i,j=1}^n (\lambda_i - \mu_j)^2 |y_{ij}|^2 + (1-\nu)^{2(1-\nu)} \sum_{i,j=1}^n \bigl(\lambda_i^\nu \mu_j^{1-\nu}\bigr)^2 |y_{ij}|^2 \\
&\qquad - r_0 \sum_{i,j=1}^n \lambda_i \Bigl(\sqrt{(1-\nu)\mu_j} - \sqrt{\lambda_i}\Bigr)^2 |y_{ij}|^2 + 2\nu(1-\nu) \sum_{i,j=1}^n \bigl(\lambda_i^{\frac{1}{2}}\mu_j^{\frac{1}{2}}\bigr)^2 |y_{ij}|^2 \\
&\quad= (1-\nu)^2 \sum_{i,j=1}^n (\lambda_i - \mu_j)^2 |y_{ij}|^2 + (1-\nu)^{2(1-\nu)} \sum_{i,j=1}^n \bigl(\lambda_i^\nu \mu_j^{1-\nu}\bigr)^2 |y_{ij}|^2 \\
&\qquad + (2\nu - r_0)(1-\nu) \sum_{i,j=1}^n \bigl(\lambda_i^{\frac{1}{2}}\mu_j^{\frac{1}{2}}\bigr)^2 |y_{ij}|^2 \\
&\qquad - r_0 \sum_{i,j=1}^n \lambda_i^2 |y_{ij}|^2 + 2r_0\sqrt{(1-\nu)} \sum_{i,j=1}^n \bigl(\lambda_i^{\frac{3}{4}}\mu_j^{\frac{1}{4}}\bigr)^2 |y_{ij}|^2 \\
&\quad= (1-\nu)^2 \|AX - XB\|_2^2 + (1-\nu)^{2(1-\nu)} \bigl\|A^\nu XB^{1-\nu}\bigr\|_2^2 \\
&\qquad + (2\nu - r_0)(1-\nu)\bigl\|A^{\frac{1}{2}}XB^{\frac{1}{2}}\bigr\|_2^2 - r_0\|AX\|_2^2 + 2r_0\sqrt{1-\nu}\bigl\|A^{\frac{3}{4}}XB^{\frac{1}{4}}\bigr\|_2^2,
\end{aligned}$$

and so

$$\begin{aligned}
&\bigl\|\nu AX + (1-\nu)XB\bigr\|_2^2 + r_0\Bigl[(1-\nu)\bigl\|A^{\frac{1}{2}}XB^{\frac{1}{2}}\bigr\|_2^2 + \|AX\|_2^2 - 2\sqrt{1-\nu}\bigl\|A^{\frac{3}{4}}XB^{\frac{1}{4}}\bigr\|_2^2\Bigr] \\
&\quad\leq (1-\nu)^2\|AX - XB\|_2^2 + 2\nu(1-\nu)\bigl\|A^{\frac{1}{2}}XB^{\frac{1}{2}}\bigr\|_2^2 + (1-\nu)^{2(1-\nu)}\bigl\|A^\nu XB^{1-\nu}\bigr\|_2^2.
\end{aligned}$$

If $\frac{1}{2} \leq \nu \leq 1$, then by inequality (10) and the same method above, we have inequality (12). This completes the proof. □

Remark 2 Obviously, (11) and (12) are improvement reverses of the matrix Young type inequalities (5) and (6).

In the end, we present two new inequalities, by means of inequalities (9) and (10). To do this, we need the following lemmas.

Lemma 1 (Cauchy-Schwarz inequality) [10] *Let $a_i \geq 0$, $b_i \geq 0$, for $i = 1, 2, \dots, n$, then*

$$\sum_{i=1}^{n} a_i b_i \leq \left(\sum_{i=1}^{n} a_i^2\right)^{\frac{1}{2}} \left(\sum_{i=1}^{n} b_i^2\right)^{\frac{1}{2}}.$$

Lemma 2 [10] *Let $A, B \in M_n$, then*

$$\sum_{j=1}^{n} s_j(AB) \leq \sum_{j=1}^{n} s_j(A) s_j(B).$$

Theorem 3 *Let $A, B \in M_n$ such that A and B are positive semidefinite. If $0 \leq \nu \leq \frac{1}{2}$, then*

$$\begin{aligned}(1-\nu)^{1-\nu} \|A^\nu\|_2 \|B^{1-\nu}\|_2 \\ \geq \sqrt{\nu^2 \|A\|_2^2 + (1-\nu)^2 \|B\|_2^2 - (1-\nu)^2 \left(\|A\|_2^2 + \|B\|_2^2 - 2\|AB\|_1\right) + M_1},\end{aligned} \tag{13}$$

where $r_0 = \min\{2\nu, 1-2\nu\}$, $M_1 = r_0[(1-\nu)\|AB\|_1 + \|A\|_2^2 - 2\sqrt{1-\nu}\|A^{\frac{3}{2}}\|_1 \|B^{\frac{1}{2}}\|_1]$.

If $\frac{1}{2} \leq \nu \leq 1$, then

$$\nu^\nu \|A^\nu\|_2 \|B^{1-\nu}\|_2 \geq \sqrt{\nu^2 \|A\|_2^2 + (1-\nu)^2 \|B\|_2^2 - \nu^2 \left(\|A\|_2^2 + \|B\|_2^2 - 2\|AB\|_1\right) + M_2}, \tag{14}$$

where $r_0 = \min\{2\nu - 1, 2 - 2\nu\}$, $M_2 = r_0[\nu\|AB\|_1 + \|B\|_2^2 - 2\sqrt{\nu}\|A^{\frac{1}{2}}\|_1 \|B^{\frac{3}{2}}\|_1]$.

Proof If $0 \leq \nu \leq \frac{1}{2}$, then using Lemma 1, Lemma 2, and inequality (9), we have

$$\begin{aligned}
&\operatorname{tr}\left(\nu^2 A^2 + (1-\nu)^2 B^2\right) \\
&\quad = \nu^2 \operatorname{tr} A^2 + (1-\nu)^2 \operatorname{tr} B^2 \\
&\quad = \sum_{j=1}^{n} \left(\nu^2 s_j^2(A) + (1-\nu)^2 s_j^2(B)\right) \\
&\quad \leq (1-\nu)^2 \left[\sum_{j=1}^{n} s_j^2(A) + \sum_{j=1}^{n} s_j^2(B) - 2\sum_{j=1}^{n} s_j(A) s_j(B)\right] \\
&\qquad + \sum_{j=1}^{n} (1-\nu)^{2(1-\nu)} \left[s_j\left(A^\nu\right) s_j\left(B^{1-\nu}\right)\right]^2 - r_0 \sum_{j=1}^{n} s_j(A) \left[\sqrt{(1-\nu) s_j(B)} - \sqrt{s_j(A)}\right]^2 \\
&\quad \leq (1-\nu)^2 \left[\|A\|_2^2 + \|B\|_2^2 - 2\sum_{j=1}^{n} s_j(AB)\right] + (1-\nu)^{2(1-\nu)} \left[\sum_{j=1}^{n} s_j\left(A^\nu\right) s_j\left(B^{1-\nu}\right)\right]^2 \\
&\qquad - r_0 \left[(1-\nu)\sum_{j=1}^{n} s_j(A) s_j(B) + \sum_{j=1}^{n} s_j^2(A) - 2\sqrt{1-\nu}\left(\sum_{j=1}^{n} s_j^{\frac{3}{2}}(A) s_j^{\frac{1}{2}}(B)\right)\right]
\end{aligned}$$

$$
\begin{aligned}
&\le (1-\nu)^2\big[\|A\|_2^2+\|B\|_2^2-2\|AB\|_1\big]+(1-\nu)^{2(1-\nu)}\left[\sum_{j=1}^n s_j^2\big(A^\nu\big)\sum_{j=1}^n s_j^2\big(B^{1-\nu}\big)\right]\\
&\quad - r_0\left[(1-\nu)\sum_{j=1}^n s_j(AB)+\|A\|_2^2-2\sqrt{1-\nu}\left(\sum_{j=1}^n s_j^{\frac{3}{4}}(A)s_j^{\frac{1}{4}}(B)\right)^2\right]\\
&\le (1-\nu)^2\big[\|A\|_2^2+\|B\|_2^2-2\|AB\|_1\big]+(1-\nu)^{2(1-\nu)}\big\|A^\nu\big\|_2^2\big\|B^{1-\nu}\big\|_2^2\\
&\quad - r_0\left[(1-\nu)\|AB\|_1+\|A\|_2^2-2\sqrt{1-\nu}\left(\sum_{j=1}^n s_j^{\frac{3}{2}}(A)\sum_{j=1}^n s_j^{\frac{1}{2}}(B)\right)\right].
\end{aligned}
$$

Thus

$$
\begin{aligned}
\nu^2\|A\|_2^2+(1-\nu)^2\|B\|_2^2 &\le (1-\nu)^2\big(\|A\|_2^2+\|B\|_2^2-2\|AB\|_1\big)+(1-\nu)^{2(1-\nu)}\big\|A^\nu\big\|_2^2\big\|B^{1-\nu}\big\|_2^2\\
&\quad - r_0\big[(1-\nu)\|AB\|_1+\|A\|_2^2-2\sqrt{1-\nu}\big\|A^{\frac{3}{2}}\big\|_1\big\|B^{\frac{1}{2}}\big\|_1\big].
\end{aligned}
$$

If $\frac{1}{2}\le\nu\le 1$, then by inequality (10) and the same method above, we have inequality (14). This completes the proof. □

Remark 3 It should be noticed that neither (7) nor (13) is uniformly better than the other. At the same time, neither (8) nor (14) is uniformly better than the other.

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
All authors contributed equally to the writing of this paper. All authors read and approved the final manuscript.

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Acknowledgements
The authors wish to express their heartfelt thanks to the referees for their detailed and helpful suggestions for revising the manuscript.

Received: 30 November 2014 Accepted: 6 March 2015 Published online: 17 March 2015

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