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Fixed points of α-admissible Meir-Keeler contraction mappings on quasi-metric spaces

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Abstract

We introduce α-admissible Meir-Keller and generalized α-admissible Meir-Keller contractions on quasi-metric spaces and discuss the existence of fixed points of such contractions. We apply our results to G-metric spaces and express some fixed point theorems in G-metric spaces as consequences of the results in quasi-metric spaces.
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1 Introduction and preliminaries

One of the generalizations of the metric spaces are the so-called quasi-metric spaces in which the commutativity condition does not hold in general. Recently, Jleli and Samet [1] obtained a relation between G-metric spaces introduced by Mustafa and Sims [2] and quasi-metric spaces. This work increased the interest to quasi-metric spaces (see [3, 4] for details).

In this paper, we investigate the existence of fixed points of Meir-Keeler type contractions defined on quasi-metric spaces and apply our results to G-metric spaces.

First, we recall the definition of a quasi-metric and quasi-metric space and some topological concepts on these spaces.

Definition 1 Let X be a nonempty set and $d : X \times X \to [0,\infty)$ be a function which satisfies:

(d1) $d(x,y) = 0$ if and only if $x = y$;

(d2) $d(x,y) \leq d(x,z) + d(z,y)$.

Then d called a quasi-metric and the pair (X,d) is called a quasi-metric space.

Remark 2 Any metric space is a quasi-metric space, but the converse is not true in general.

Definition 3 Let (X,d) be a quasi-metric space.

(1) A sequence $\{x_n\}$ in X is said to be convergent to x if $\lim_{n\to\infty} d(x_n,x) = \lim_{n\to\infty} d(x,x_n) = 0$.

(2) A sequence $\{x_n\}$ in X is called left-Cauchy if for every $\varepsilon > 0$ there exists a positive integer $N = N(\varepsilon)$ such that $d(x_n,x_m) < \varepsilon$ for all $n \geq m > N$.

(3) A sequence $\{x_n\}$ in X is called right-Cauchy if for every $\varepsilon > 0$ there exists a positive integer $N = N(\varepsilon)$ such that $d(x_n, x_m) < \varepsilon$ for all $m \geq n > N$.
(4) A sequence $\{x_n\}$ in X is called Cauchy sequence if for every $\varepsilon > 0$ there exists a positive integer $N = N(\varepsilon)$ such that $d(x_n, x_m) < \varepsilon$ for all $m, n > N$.

Remark 4 From Definition 3 it is obvious that a sequence $\{x_n\}$ in a quasi-metric space is Cauchy if and only if it is both left-Cauchy and right-Cauchy.

Definition 5 Let (X, d) be a quasi-metric space. Then:
(1) (X, d) is said to be left-complete if every left-Cauchy sequence in X is convergent.
(2) (X, d) is said to be right-complete if every right-Cauchy sequence in X is convergent.
(3) (X, d) is said to be complete if every Cauchy sequence in X is convergent.

In the sequel, we shall denote by $\mathbb{N}$ the set of nonnegative integers, that is, $\mathbb{N} = \{0, 1, 2, \dots\}$. We next define the concept of α-admissible mappings which have been recently introduced by Samet [5] and used by many authors to generalize contraction mappings of various types; see [6–8] for details.

Definition 6 A mapping $T : X \to X$ is called α-admissible if for all $x, y \in X$ we have

$$\alpha(x, y) \geq 1 \quad \Rightarrow \quad \alpha(Tx, Ty) \geq 1, \tag{1.1}$$

where $\alpha : X \times X \to [0, \infty)$ is a given function.

In the existence and uniqueness proofs of fixed points of α-admissible maps, an additional property is required. This property is given below.

Definition 7 A mapping $T : X \to X$ is called triangular α-admissible if it is α-admissible and satisfies

$$\left.\begin{aligned} \alpha(x, y) \geq 1, \\ \alpha(y, z) \geq 1 \end{aligned}\right\} \quad \Rightarrow \quad \alpha(x, z) \geq 1, \tag{1.2}$$

where $x, y, z \in X$ and $\alpha : X \times X \to [0, \infty)$ is a given function.

The following auxiliary result is going to be used in the proof of existence theorems.

Lemma 8 [7] *Let $T : X \to X$ be a triangular α-admissible mapping. Assume that there exists $x_0 \in X$ such that $\alpha(x_0, Tx_0) \geq 1$ and $\alpha(Tx_0, x_0) \geq 1$. If $x_n = T^n x_0$, then $\alpha(x_m, x_n) \geq 1$ for all $m, n \in \mathbb{N}$.*

Proof Let $x_0 \in X$ satisfies $\alpha(x_0, Tx_0) \geq 1$ and $\alpha(Tx_0, x_0) \geq 1$. Define the sequence $\{x_n\}$ in X as $x_{n+1} = Tx_n$ for $n \in \mathbb{N}$. Since T is α-admissible, we have

$$\alpha(x_0, Tx_0) = \alpha(x_0, x_1) \geq 1 \quad \Rightarrow \quad \alpha(x_1, x_2) \geq 1 \quad \Rightarrow \quad \cdots \quad \Rightarrow \quad \alpha(x_n, x_{n+1}) \geq 1,$$

$$\alpha(Tx_0, x_0) = \alpha(x_1, x_0) \geq 1 \quad \Rightarrow \quad \alpha(x_2, x_1) \geq 1 \quad \Rightarrow \quad \cdots \quad \Rightarrow \quad \alpha(x_{n+1}, x_n) \geq 1,$$

for all $n = 0, 1, \ldots$. On the other hand, since T is triangular α-admissible, we get

$$\left.\begin{aligned} \alpha(x_m, x_{m+1}) \geq 1, \\ \alpha(x_{m+1}, x_{m+2}) \geq 1 \end{aligned}\right\} \quad \Longrightarrow \quad \alpha(x_m, x_{m+2}) \geq 1,$$

and

$$\left.\begin{aligned} \alpha(x_m, x_{m-1}) \geq 1, \\ \alpha(x_{m-1}, x_{m-2}) \geq 1 \end{aligned}\right\} \quad \Longrightarrow \quad \alpha(x_m, x_{m-2}) \geq 1.$$

Similarly,

$$\left.\begin{aligned} \alpha(x_m, x_{m+2}) \geq 1, \\ \alpha(x_{m+2}, x_{m+3}) \geq 1 \end{aligned}\right\} \quad \Longrightarrow \quad \alpha(x_m, x_{m+3}) \geq 1,$$

and

$$\left.\begin{aligned} \alpha(x_m, x_{m-2}) \geq 1, \\ \alpha(x_{m-2}, x_{m-3}) \geq 1 \end{aligned}\right\} \quad \Longrightarrow \quad \alpha(x_m, x_{m-3}) \geq 1.$$

Continuing in this way, we obtain, for all $m, n \in \mathbb{N}$,

$$\alpha(x_m, x_n) \geq 1. \qquad \square$$

In this paper we study α-admissible Meir-Keeler (or shortly α-Meir-Keeler) contractions which can be regarded as generalizations of the Meir-Keeler contractions defined in [9]. In fact, we insert α-admissibility into the definition of the original Meir-Keeler contraction.

Definition 9 Let (X, d) be a quasi-metric space. Let $T : X \to X$ be a triangular α-admissible mapping. Suppose that for every $\varepsilon > 0$ there exists $\delta > 0$ such that

$$\varepsilon \leq d(x, y) < \varepsilon + \delta \quad \text{implies} \quad \alpha(x, y) d(Tx, Ty) < \varepsilon, \tag{1.3}$$

for all $x, y \in X$. Then T is called α-Meir-Keeler contraction.

Remark 10 Let T be an α-Meir-Keeler contractive mapping. Then

$$\alpha(x, y) d(Tx, Ty) < d(x, y),$$

for all $x, y \in X$ when $x \neq y$. Also, if $x = y$ then $d(Tx, Ty) = 0$, *i.e.*,

$$\alpha(x, y) d(Tx, Ty) \leq d(x, y),$$

for all $x, y \in X$.

We also generalize the α-Meir-Keeler contraction by using a more general expression in the contractive condition. Specifically, we define two types of generalized α-Meir-Keeler contraction, say type (I) and type (II) as follows.

Definition 11 Let (X,d) be a quasi-metric space. Let $T: X \to X$ be a triangular α-admissible mapping. Assume that for every $\varepsilon > 0$ there exists $\delta > 0$ such that

$$\varepsilon \leq M(x,y) < \varepsilon + \delta \quad \text{implies} \quad \alpha(x,y)d(Tx,Ty) < \varepsilon, \tag{1.4}$$

where

$$M(x,y) = \max\{d(x,y), d(Tx,x), d(Ty,y)\}, \tag{1.5}$$

for all $x,y \in X$. Then T is called a generalized α-Meir-Keeler contraction of type (I).

Definition 12 Let (X,d) be a quasi-metric space. Let $T: X \to X$ be a triangular α-admissible mapping. Assume that for every $\varepsilon > 0$ there exists $\delta > 0$ such that

$$\varepsilon \leq N(x,y) < \varepsilon + \delta \quad \text{implies} \quad \alpha(x,y)d(Tx,Ty) < \varepsilon, \tag{1.6}$$

where

$$N(x,y) = \max\left\{d(x,y), \frac{1}{2}\left[d(Tx,x) + d(Ty,y)\right]\right\}, \tag{1.7}$$

for all $x,y \in X$. Then T is called a generalized α-Meir-Keeler contraction of type (II).

Remark 13 Let $T: X \to X$ be a generalized α-Meir-Keeler contraction of type (I) (respectively, type (II)). Then

$$\alpha(x,y)d(Tx,Ty) < M(x,y) \quad (\text{respectively}, N(x,y)),$$

for all $x,y \in X$ when $M(x,y) > 0$ (respectively, $N(x,y) > 0$). Also, if $M(x,y) = 0$ (respectively, $N(x,y) = 0$), then $x = y$, which implies $d(x,y) = 0$, *i.e.*,

$$\alpha(x,y)d(Tx,Ty) \leq M(x,y) \quad (\text{respectively}, N(x,y)),$$

for all $x,y \in X$.

Remark 14 It is obvious that $N(x,y) \leq M(x,y)$ for all $x,y \in X$, where $M(x,y)$ and $N(x,y)$ are defined in (1.5) and (1.7), respectively.

2 Main results

Our first result is a fixed point theorem for generalized α-Meir-Keeler contractions of type (I) on quasi-metric spaces.

Theorem 15 *Let (X,d) be a complete quasi-metric space and $T: X \to X$ be a continuous generalized α-Meir-Keeler contraction of type* (I). *If $\alpha(x_0, Tx_0) \geq 1$ and $\alpha(Tx_0, x_0) \geq 1$ for some $x_0 \in X$, then T has a fixed point in X.*

Proof Let $x_0 \in X$ satisfy $\alpha(x_0, Tx_0) \geq 1$ and $\alpha(Tx_0, x_0) \geq 1$. Define the sequence $\{x_n\}$ in X as

$$x_{n+1} = Tx_n \quad \text{for } n \in \mathbb{N}.$$

Notice that if $x_{n_0} = x_{n_0+1}$ for some $n_0 > 0$, then x_{n_0} is a fixed point of T and the proof is done. Assume that $x_n \neq x_{n+1}$ for all $n \geq 0$. Since T is α-admissible,

$$\alpha(x_0, Tx_0) = \alpha(x_0, x_1) \geq 1 \quad \Rightarrow \quad \alpha(Tx_0, Tx_1) = \alpha(x_1, x_2) \geq 1, \tag{2.1}$$

and continuing we obtain

$$\alpha(x_n, x_{n+1}) \geq 1 \quad \forall n \in \mathbb{N}. \tag{2.2}$$

Upon substituting $x = x_n$ and $y = x_{n+1}$ in (1.4) we find that for every $\varepsilon > 0$ there exists $\delta > 0$ such that

$$\varepsilon \leq M(x_n, x_{n+1}) < \varepsilon + \delta \quad \Longrightarrow \quad \alpha(x_n, x_{n+1}) d(Tx_n, Tx_{n+1}) < \varepsilon, \tag{2.3}$$

where

$$M(x_n, x_{n+1}) = \max\bigl\{d(x_n, x_{n+1}), d(x_{n+1}, x_n), d(x_{n+2}, x_{n+1})\bigr\}.$$

In what follows, we examine three cases.

Case 1. Assume that $M(x_n, x_{n+1}) = d(x_n, x_{n+1})$. Then (2.3) becomes

$$\varepsilon \leq d(x_n, x_{n+1}) < \varepsilon + \delta \quad \Longrightarrow \quad \alpha(x_n, x_{n+1}) d(Tx_n, Tx_{n+1}) < \varepsilon.$$

Therefore, we deduce that

$$d(x_{n+1}, x_{n+2}) \leq \alpha(x_n, x_{n+1}) d(Tx_n, Tx_{n+1}) < \varepsilon \leq d(x_n, x_{n+1}),$$

for all n. That is, $\{d(x_n, x_{n+1})\}$ is a decreasing positive sequence in $\mathbb{R}_+$ and it converges to some $r \geq 0$. To show that $r = 0$, we assume the contrary, that is, $r > 0$. Then we must have

$$0 < r \leq d(x_n, x_{n+1}) \quad \text{for all } n \in \mathbb{N}. \tag{2.4}$$

Since the condition (2.3) holds for every $\varepsilon > 0$, we may choose $\varepsilon = r$. For this ε, there exists $\delta(\varepsilon) > 0$ satisfying (2.3). In other words,

$$r \leq d_n = d(x_n, x_{n+1}) < r + \delta \quad \Longrightarrow \quad \alpha(x_n, x_{n+1}) d(Tx_n, Tx_{n+1}) < r.$$

However, this implies

$$\begin{aligned} & r \leq M(x_n, x_{n+1}) = d(x_n, x_{n+1}) < r + \delta \\ & \quad \Longrightarrow \quad d(x_{n+1}, x_{n+2}) \leq \alpha(x_n, x_{n+1}) d(Tx_n, Tx_{n+1}) < r, \end{aligned} \tag{2.5}$$

which contradicts (2.4). Thus, $r = 0$, that is,

$$\lim_{n\to\infty} d(x_n, x_{n+1}) = 0. \tag{2.6}$$

Case 2. Assume that $M(x_n, x_{n+1}) = d(x_{n+1}, x_n)$. In this case (2.3) becomes

$$\varepsilon \leq d(x_{n+1}, x_n) < \varepsilon + \delta \quad \Longrightarrow \quad \alpha(x_n, x_{n+1}) d(Tx_n, Tx_{n+1}) < \varepsilon,$$

from which it follows that

$$d(x_{n+1}, x_{n+2}) \leq \alpha(x_n, x_{n+1}) d(Tx_n, Tx_{n+1}) < \varepsilon \leq d(x_{n+1}, x_n).$$

Therefore, we obtain

$$d(x_{n+1}, x_{n+2}) < d(x_{n+1}, x_n), \tag{2.7}$$

for all $n \in \mathbb{N}$. Note that by Remark 13, since

$$M(x_n, x_{n-1}) = \max\{d(x_n, x_{n-1}), d(x_{n+1}, x_n), d(x_n, x_{n-1})\} > 0,$$

we get

$$d(x_{n+1}, x_n) = d(Tx_n, Tx_{n-1}) \leq \alpha(x_n, x_{n-1}) d(Tx_n, Tx_{n-1}) < M(x_n, x_{n-1}), \tag{2.8}$$

where

$$\begin{aligned} M(x_n, x_{n-1}) &= \max\{d(x_n, x_{n-1}), d(x_{n+1}, x_n), d(x_n, x_{n-1})\} \\ &= \max\{d(x_n, x_{n-1}), d(x_{n+1}, x_n)\}. \end{aligned}$$

Then (2.8) becomes

$$d(x_{n+1}, x_n) < \max\{d(x_n, x_{n-1}), d(x_{n+1}, x_n)\}, \tag{2.9}$$

for all $n \in \mathbb{N}$.

Clearly, the case $\max\{d(x_n, x_{n-1}), d(x_{n+1}, x_n)\} = d(x_{n+1}, x_n)$ is not possible. Indeed, in this case we would get

$$0 < d(x_{n+1}, x_n) < d(x_{n+1}, x_n).$$

Therefore, we should have $\max\{d(x_n, x_{n-1}), d(x_{n+1}, x_n)\} = d(x_n, x_{n-1})$, which implies

$$0 < d(x_{n+1}, x_n) < d(x_n, x_{n-1}), \tag{2.10}$$

for all $n \in \mathbb{N}$. That is, the sequence $\{d(x_{n+1}, x_n)\}$ is decreasing and positive sequence and hence it converges to $L \geq 0$. In fact, the limit L of this sequence is 0, which can be shown by mimicking the proof of (2.6) done above. In other words, we get

$$\lim_{n\to\infty} d(x_{n+1}, x_n) = 0. \tag{2.11}$$

Finally, taking the limit as $n \to \infty$ in (2.7) and using (2.11) we obtain

$$\lim_{n\to\infty} d(x_{n+1}, x_{n+2}) = 0. \tag{2.12}$$

Case 3. Assume that $M(x_n, x_{n+1}) = d(x_{n+2}, x_{n+1})$. In this case (2.3) becomes

$$\varepsilon \le d(x_{n+2}, x_{n+1}) < \varepsilon + \delta \quad \Longrightarrow \quad \alpha(x_n, x_{n+1}) d(Tx_n, Tx_{n+1}) < \varepsilon,$$

or

$$d(x_{n+1}, x_{n+2}) \le \alpha(x_n, x_{n+1}) d(Tx_n, Tx_{n+1}) < \varepsilon \le d(x_{n+2}, x_{n+1}).$$

Therefore, we deduce

$$d(x_{n+1}, x_{n+2}) < d(x_{n+2}, x_{n+1}), \tag{2.13}$$

for all $n \in \mathbb{N}$. By Remark 13, we have

$$d(x_{n+2}, x_{n+1}) = d(Tx_{n+1}, Tx_n) \le \alpha(x_{n+1}, x_n) d(Tx_{n+1}, Tx_n) < M(x_{n+1}, x_n), \tag{2.14}$$

where

$$\begin{aligned} M(x_{n+1}, x_n) &= \max\bigl\{d(x_{n+1}, x_n), d(x_{n+2}, x_{n+1}), d(x_{n+1}, x_n)\bigr\} \\ &= \max\bigl\{d(x_{n+1}, x_n), d(x_{n+2}, x_{n+1})\bigr\} \end{aligned}$$

is clearly positive. Then (2.14) becomes

$$d(x_{n+2}, x_{n+1}) < \max\bigl\{d(x_{n+1}, x_n), d(x_{n+2}, x_{n+1})\bigr\}, \tag{2.15}$$

for all $n \in \mathbb{N}$.

The case $\max\{d(x_{n+1}, x_n), d(x_{n+2}, x_{n+1})\} = d(x_{n+2}, x_{n+1})$ is impossible, since it yields

$$0 < d(x_{n+2}, x_{n+1}) < d(x_{n+2}, x_{n+1}).$$

The other case, that is, $\max\{d(x_{n+1}, x_n), d(x_{n+2}, x_{n+1})\} = d(x_{n+1}, x_n)$ implies

$$0 < d(x_{n+2}, x_{n+1}) < d(x_{n+1}, x_n), \tag{2.16}$$

for all $n \in \mathbb{N}$. As in Case 2, the sequence $\{d(x_{n+1}, x_n)\}$ is decreasing and positive sequence and hence it converges to $L = 0$. Finally, taking the limit as $n \to \infty$ in (2.13) we end up with

$$\lim_{n\to\infty} d(x_{n+1}, x_{n+2}) = 0. \tag{2.17}$$

As a result, we see that in all three cases, the sequence $\{d_n\}$ defined by $d_n := d(x_n, x_{n+1})$ converges to 0 as $n \to \infty$. Using similar arguments, it can be shown that the sequence $\{f_n\}$ where $f_n := d(x_{n+1}, x_n)$ also converges to 0. We first note that

$$\alpha(Tx_0, x_0) = \alpha(x_1, x_0) \ge 1 \quad \Rightarrow \quad \alpha(Tx_1, Tx_0) = \alpha(x_2, x_1) \ge 1, \tag{2.18}$$

and continuing in this way, we obtain

$$\alpha(x_{n+1}, x_n) \geq 1 \quad \forall n \in \mathbb{N}. \tag{2.19}$$

Substituting $x = x_{n+1}$ and $y = x_n$ in (1.4) we find that for every $\varepsilon > 0$ there exists $\delta > 0$ such that

$$\varepsilon \leq M(x_{n+1}, x_n) < \varepsilon + \delta \quad \Longrightarrow \quad \alpha(x_{n+1}, x_n)d(Tx_{n+1}, Tx_n) < \varepsilon, \tag{2.20}$$

where

$$\begin{aligned} M(x_{n+1}, x_n) &= \max\{d(x_{n+1}, x_n), d(x_{n+2}, x_{n+1}), d(x_{n+1}, x_n)\} \\ &= \max\{d(x_{n+1}, x_n), d(x_{n+2}, x_{n+1})\}. \end{aligned}$$

We need to examine two cases.

Case 1. Assume that $M(x_{n+1}, x_n) = d(x_{n+1}, x_n)$. Then (2.20) becomes

$$\varepsilon \leq d(x_{n+1}, x_n) < \varepsilon + \delta \quad \Longrightarrow \quad \alpha(x_{n+1}, x_n)d(Tx_{n+1}, Tx_n) < \varepsilon.$$

Then we have

$$d(x_{n+2}, x_{n+1}) \leq \alpha(x_{n+1}, x_n)d(Tx_{n+1}, Tx_n) < \varepsilon \leq d(x_{n+1}, x_n),$$

for all n. That is, $\{d(x_{n+1}, x_n)\}$ is a decreasing positive sequence in $\mathbb{R}_+$ and it converges to $L \geq 0$. As above, it can be shown that $L = 0$.

Case 2. Assume that $M(x_{n+1}, x_n) = d(x_{n+2}, x_{n+1})$. In this case (2.20) becomes

$$\varepsilon \leq d(x_{n+2}, x_{n+1}) < \varepsilon + \delta \quad \Longrightarrow \quad \alpha(x_{n+1}, x_n)d(Tx_{n+1}, Tx_n) < \varepsilon,$$

or

$$d(x_{n+2}, x_{n+1}) \leq \alpha(x_{n+1}, x_n)d(Tx_{n+1}, Tx_n) < \varepsilon \leq d(x_{n+2}, x_{n+1}),$$

which results in

$$0 < d(x_{n+2}, x_{n+1}) < d(x_{n+2}, x_{n+1}), \tag{2.21}$$

for all $n \in \mathbb{N}$ and is not possible.

Thus, we have only one possibility, $M(x_{n+1}, x_n) = d(x_{n+1}, x_n)$, which leads to the fact that the sequence $\{f_n\} = \{d(x_{n+1}, n_n)\}$ converges to 0.

We next show that the sequence $\{x_n\}$ is both right and left Cauchy. First, we show that $\{x_n\}$ is a right-Cauchy sequence in (X, d). We will prove that for every $\varepsilon > 0$ there exists $N \in \mathbb{N}$ such that

$$d(x_l, x_{l+k}) < \varepsilon, \tag{2.22}$$

for all $l \geq N$ and $k \in \mathbb{N}$. Since the sequences $\{d_n\}$ and $\{f_n\}$ both converge to 0 as $n \to \infty$, for every $\delta > 0$ there exist $N_1, N_2 \in \mathbb{N}$ such that

$$d(x_n, x_{n+1}) < \delta \quad \text{for all } n \geq N_1 \in \mathbb{N} \quad \text{and} \quad d(x_{n+1}, x_n) < \delta \quad \text{for all } n \geq N_2 \in \mathbb{N}. \quad (2.23)$$

Choose δ such as $\delta < \varepsilon$. We will prove (2.22) by using induction on k. For $k = 1$, (2.22) becomes

$$d(x_l, x_{l+1}) < \varepsilon, \quad (2.24)$$

and clearly holds for all $l \geq N = \max\{N_1, N_2\}$ due to (2.23) and the choice of δ. Assume that the inequality (2.22) holds for some $k = m$, that is,

$$d(x_l, x_{l+m}) < \varepsilon, \quad \text{for all } l \geq N. \quad (2.25)$$

For $k = m + 1$ we have to show that $d(x_l, x_{l+m+1}) < \varepsilon$ for all $l \geq N$. From the triangle inequality, we have

$$d(x_{l-1}, x_{l+m}) < d(x_{l-1}, x_l) + d(x_l, x_{l+m}) < \delta + \varepsilon, \quad (2.26)$$

for all $l \geq N$. If $d(x_{l-1}, x_{l+m}) \geq \varepsilon$, then for

$$M(x_{l-1}, x_{l+m}) = \max\bigl\{d(x_{l-1}, x_{l+m}), d(x_l, x_{l-1}), d(x_{l+m+1}, x_{l+m})\bigr\},$$

we have

$$\begin{aligned} \varepsilon &\leq d(x_{l-1}, x_{l+m}) \leq M(x_{l-1}, x_{l+m}) \\ &= \max\bigl\{d(x_{l-1}, x_{l+m}), d(x_l, x_{l-1}), d(x_{l+m+1}, x_{l+m})\bigr\} < \{\varepsilon + \delta, \delta, \delta\} \leq \varepsilon + \delta, \end{aligned}$$

and because of Lemma 8, the contractive condition (1.4) with $x = x_{l-1}$ and $y = x_{l+m}$ yields

$$\begin{aligned} &\varepsilon \leq M(x_{l-1}, x_{l+m}) < \delta + \varepsilon \\ &\quad \Longrightarrow \quad d(x_l, x_{l+m+1}) \leq \alpha(x_{l-1}, x_{l+m}) d(x_l, x_{l+m+1}) = \alpha(x_{l-1}, x_{l+m}) d(Tx_{l-1}, Tx_{l+m}) < \varepsilon, \end{aligned}$$

and hence (2.22) holds for $k = m + 1$.

If $d(x_{l-1}, x_{l+m}) < \varepsilon$, then

$$\begin{aligned} M(x_{l-1}, x_{l+m}) &= \max\bigl\{d(x_{l-1}, x_{l+m}), d(x_l, x_{l-1}), d(x_{l+m+1}, x_{l+m})\bigr\} \\ &< \{\varepsilon, \delta, \delta\} \leq \varepsilon. \end{aligned}$$

Regarding Remark 13, we deduce

$$\begin{aligned} d(x_l, x_{l+m+1}) &\leq \alpha(x_{l-1}, x_{l+m}) d(x_l, x_{l+m+1}) \\ &\leq M(x_{l-1}, x_{l+m}) < \varepsilon, \end{aligned}$$

that is, inequality (2.22) holds for $k = m + 1$. Hence $d(x_l, x_{l+k}) < \varepsilon$ for all $l \geq N$ and $k \geq 1$, which means

$$d(x_n, x_m) < \varepsilon, \quad \text{for all } m \geq n \geq N. \tag{2.27}$$

Consequently, $\{x_n\}$ is a right-Cauchy sequence in (X, d). Due to the similarity, the proof that $\{x_n\}$ is a left-Cauchy sequence in (X, d) is omitted. By Remark 4, we deduce that $\{x_n\}$ is a Cauchy sequence in complete quasi-metric space (X, d). Therefore, there exists $z \in X$ such that

$$\lim_{n\to\infty} d(x_n, z) = \lim_{n\to\infty} d(z, x_n) = 0. \tag{2.28}$$

Employing the property (d1) and the continuity of T we get

$$\lim_{n\to\infty} d(x_n, Tz) = \lim_{n\to\infty} d(Tx_{n-1}, Tz) = 0 \tag{2.29}$$

and

$$\lim_{n\to\infty} d(Tz, x_n) = \lim_{n\to\infty} d(Tz, Tx_{n-1}) = 0. \tag{2.30}$$

Combining (2.29) and (2.30), we deduce

$$\lim_{n\to\infty} d(x_n, Tz) = \lim_{n\to\infty} d(Tz, x_n) = 0. \tag{2.31}$$

From (2.28) and (2.31), due to the uniqueness of the limit, we conclude that $z = Tz$, that is, z is a fixed point of T. □

Below we state an existence theorem for fixed point of generalized α-Meir-Keeler contraction of type (II). Taking Remark 14 into account, we observe that the proof of this theorem is similar to the proof of Theorem 15.

Theorem 16 *Let (X, d) be a complete quasi-metric space and $T : X \to X$ be a continuous generalized α-Meir-Keeler contraction of type* (II). *If $\alpha(x_0, Tx_0) \geq 1$ and $\alpha(Tx_0, x_0) \geq 1$ for some $x_0 \in X$, then T has a fixed point in X.*

One advantage of α-admissibility is that the continuity of the contraction is not required whenever the following condition is satisfied.

(A) If $\{x_n\}$ is a sequence in X which converges to x and satisfies $\alpha(x_{n+1}, x_n) \geq 1$ and $\alpha(x_n, x_{n+1}) \geq 1$ for all n then there exists a subsequence $\{x_{n(k)}\}$ of $\{x_n\}$ such that $\alpha(x, x_{n(k)}) \geq 1$ and $\alpha(x_{n(k)}, x) \geq 1$ for all k.

Replacing the continuity of the contraction in Theorem 16 by the condition (A) on the space (X, d) we deduce another existence theorem.

Theorem 17 *Let (X, d) be a complete quasi-metric space and $T : X \to X$ be a generalized α-Meir-Keeler contraction of type* (II) *and let (X, d) satisfies the condition* (A). *If $\alpha(x_0, Tx_0) \geq 1$ and $\alpha(Tx_0, x_0) \geq 1$ for some $x_0 \in X$, then T has a fixed point in X.*

Proof Following the lines of the proof of Theorem 15, we know that the sequence $\{x_n\}$ defined by $x_{n+1} = Tx_n$, where $x_0 \in X$ satisfies $\alpha(x_0, Tx_0) \geq 1$ and $\alpha(Tx_0, x_0) \geq 1$, converges to some $z \in X$. From (2.2) and condition (A), there exists a subsequence $\{x_{n(k)}\}$ of $\{x_n\}$ such that $\alpha(z, x_{n(k)}) \geq 1$ and $\alpha(x_{n(k)}, z) \geq 1$ for all k. Regarding Remark 13 we have for all $k \in \mathbb{N}$

$$\begin{aligned} d(Tz, x_{n(k)+1}) &= d(Tz, Tx_{n(k)}) \leq \alpha(z, x_{n(k)}) d(Tz, Tx_{n(k)}) \leq N(z, x_{n(k)}), \\ d(x_{n(k)+1}, Tz) &= d(Tx_{n(k)}, Tz) \leq \alpha(x_{n(k)}, z) d(Tx_{n(k)}, Tz) \leq N(x_{n(k)}, z), \end{aligned} \tag{2.32}$$

where

$$\begin{aligned} N(z, x_{n(k)}) &= \max\left\{ d(z, x_{n(k)}), \frac{1}{2}\left[d(Tz, z) + d(Tx_{n(k)}, x_{n(k)})\right]\right\}, \\ N(x_{n(k)}, z) &= \max\left\{ d(x_{n(k)}, z), \frac{1}{2}\left[d(Tx_{n(k)}, x_{n(k)}) + d(Tz, z)\right]\right\}. \end{aligned} \tag{2.33}$$

Letting $k \to \infty$ in (2.33) we obtain

$$\begin{aligned} \lim_{k\to\infty} N(z, x_{n(k)}) &= \max\left\{0, \frac{d(Tz, z)}{2}\right\} = \frac{d(Tz, z)}{2}, \\ \lim_{k\to\infty} N(x_{n(k)}, z) &= \max\left\{0, \frac{d(Tz, z)}{2}\right\} = \frac{d(Tz, z)}{2}. \end{aligned}$$

Thus, upon taking the limit in (2.32) as $k \to \infty$, we conclude

$$\begin{aligned} 0 &\leq d(Tz, z) \leq \frac{d(Tz, z)}{2}, \\ 0 &\leq d(z, Tz) \leq \frac{d(Tz, z)}{2}. \end{aligned} \tag{2.34}$$

The first inequality implies $d(Tz, z) = 0$, and hence $Tz = z$, which completes the proof. □

We next consider a particular case of the main theorems in which the mapping is an α-Meir-Keeler contractive mapping, that is, it satisfies Definition 9.

Corollary 18 *Let (X, d) be a complete quasi-metric space and $T : X \to X$ be a continuous, α-Meir-Keeler contraction, that is, for every $\varepsilon > 0$ there exists $\delta > 0$ such that*

$$\varepsilon \leq d(x, y) < \varepsilon + \delta \quad \Longrightarrow \quad \alpha(x, y) d(Tx, Ty) < \varepsilon, \tag{2.35}$$

holds for all $x, y \in X$. If $\alpha(x_0, Tx_0) \geq 1$ and $\alpha(Tx_0, x_0) \geq 1$ for some $x_0 \in X$, then T has a fixed point in X.

Proof It is obvious that if (2.35) holds, then using the fact that

$$d(x, y) \leq M(x, y) = \max\{d(x, y), d(Tx, x), d(Ty, y)\},$$

we conclude

$$\varepsilon \leq M(x, y) < \varepsilon + \delta \quad \Longrightarrow \quad \varepsilon \leq d(x, y) < \varepsilon + \delta \quad \Longrightarrow \quad \alpha(x, y) d(Tx, Ty) < \varepsilon, \tag{2.36}$$

for all $x, y \in X$. In other words, T satisfies the conditions in the statement of Theorem 15 and hence has a fixed point. □

Finally, we replace the continuity of the contraction in Corollary 18 by the condition (A) on the space (X, d), which results in the following existence theorem the proof of which is identical to the proof of Theorem 17.

Corollary 19 *Let* (X, d) *be a complete quasi-metric space satisfying the condition* (A) *and let* $T : X \to X$ *be an* α*-Meir-Keeler contractive mapping. If there exists* $x_0 \in X$ *such that* $\alpha(x_0, Tx_0) \geq 1$ *and* $\alpha(Tx_0, x_0) \geq 1$, *then* T *has a fixed point.*

We end this section with an example of an α-Meir-Keeler contraction defined on a quasi-metric space.

Example 20 Let $X = [0, \infty)$. Define

$$d(x,y) = \begin{cases} x - y & \text{if } x \geq y, \\ \frac{y-x}{2} & \text{if } x < y. \end{cases}$$

The function $d(x, y)$ is a quasi-metric but not a metric on X. Indeed, note that $d(1, 3) = 1 \neq d(3, 1) = 2$. The space (X, d) is a complete quasi-metric space. Define the mappings $T : X \to X$ and $\alpha : X \times X \to [0, \infty)$ as follows:

$$Tx = \begin{cases} \frac{x^2}{8} & \text{if } x \in [0,1], \\ 3x & \text{if } x \in (1,\infty), \end{cases} \qquad \alpha(x,y) = \begin{cases} 1 & \text{if } x,y \in [0,1], \\ -1 & \text{otherwise.} \end{cases}$$

It is easy to see that T is triangular α-admissible. Note that if $\alpha(x, y) \geq 1$, then $x, y \in [0, 1]$ and hence both Tx and Ty are also in $[0, 1]$. Thus, $\alpha(Tx, Ty) = \alpha(\frac{x^2}{8}, \frac{y^2}{8}) = 1 \geq 1$. Also, if $\alpha(x, z) \geq 1$ and $\alpha(z, y) \geq 1$, then $x, y, z \in [0, 1]$ and thus, $\alpha(x, y) = 1 \geq 1$. The map T is not continuous, however, the condition (A) holds on X. More precisely, if the sequence $\{x_n\} \subset X$ satisfies $\alpha(x_n, x_{n+1}) \geq 1$ and $\alpha(x_{n+1}, x_n) \geq 1$, and if $\lim_{n\to\infty} x_n = x$, then $\{x_n\} \subset [0, 1]$, and hence $x \in [0, 1]$. Then $\alpha(x_n, x) \geq 1$.

Note that for $x, y \in [0, 1]$, we have $x + y \leq 2$.

If $x \geq y$ then for $\varepsilon > 0$ we choose $\delta = 3\varepsilon$ so that $\varepsilon \leq d(x, y) = x - y < \varepsilon + \delta$ implies $\alpha(x, y)d(Tx, Ty) = \frac{x^2-y^2}{8} = \frac{(x-y)(x+y)}{8} < \frac{2(\varepsilon+\delta)}{8} = \varepsilon$.

If $x < y$ then for $\varepsilon > 0$ we choose again $\delta = 3\varepsilon$ so that $\varepsilon \leq d(x, y) = \frac{y-x}{2} < \varepsilon + \delta$ implies $\alpha(x, y)d(Tx, Ty) = \frac{x^2/8-y^2/8}{2} = \frac{(x-y)(x+y)}{16} < \frac{2(\varepsilon+\delta)}{16} = \frac{\varepsilon}{2}$.

In other words, for every $\varepsilon > 0$, there exists δ which is actually $\delta = 3\varepsilon$. Therefore, the map T is an α-Meir-Keeler contraction. Finally, note that $\alpha(0, T0) \geq 1$ and $\alpha(T0, 0) \geq 1$. All conditions of Corollary 19 are satisfied and T has a fixed point $x = 0$.

3 Consequences: *G*-metric spaces

In this section, we present some results which show that several fixed point theorems on G-metric spaces are in fact direct consequences of the existence theorems given in the previous section.

First, we briefly recollect some basic notions of G-metric and G-metric space [2].

Definition 21 Let X be a nonempty set, $G : X \times X \times X \to [0, \infty)$ be a function satisfying the following conditions:

(G1) $G(x, y, z) = 0$ if $x = y = z$,

(G2) $0 < G(x, x, y)$ for all $x, y \in X$ with $x \neq y$,

(G3) $G(x, x, y) \leq G(x, y, z)$ for all $x, y, z \in X$ with $z \neq y$,

(G4) $G(x, y, z) = G(x, z, y) = G(y, z, x) = \cdots$ (symmetry in all variables),

(G5) $G(x, y, z) \leq G(x, a, a) + G(a, y, z)$ for all $x, y, z, a \in X$ (rectangle inequality).

Then the function G is called a G-metric on X and the pair (X, G) is called a G-metric space.

It is obvious that for every G-metric on the set X, the expression

$$d_G(x, y) = G(x, x, y) + G(x, y, y)$$

is a standard metric on X.

Definition 22 (see [2]) Let (X, G) be a G-metric space and let $\{x_n\}$ be a sequence in X.

(1) A point $x \in X$ is said to be the limit of the sequence $\{x_n\}$ if

$$\lim_{n,m\to\infty} G(x, x_n, x_m) = 0$$

and the sequence $\{x_n\}$ is said to be G-convergent to x.

(2) A sequence $\{x_n\}$ is called a G-Cauchy sequence if for every $\varepsilon > 0$, there is a positive integer N such that $G(x_n, x_m, x_l) < \varepsilon$ for all $n, m, l \geq N$; that is, if $G(x_n, x_m, x_l) \to 0$ as $n, m, l \to \infty$.

(3) (X, G) is said to be G-complete (or a complete G-metric space) if every G-Cauchy sequence in (X, G) is G-convergent in X.

Theorem 23 [1] *Let (X, G) be a G-metric space. Let $d : X \times X \to [0, \infty)$ be the function defined by $d(x, y) = G(x, y, y)$. Then*

(1) *(X, d) is a quasi-metric space*;

(2) *$\{x_n\} \subset X$ is G-convergent to $x \in X$ if and only if $\{x_n\}$ is convergent to x in (X, d)*;

(3) *$\{x_n\} \subset X$ is G-Cauchy if and only if $\{x_n\}$ is Cauchy in (X, d)*;

(4) *(X, G) is G-complete if and only if (X, d) is complete.*

Admissible mappings in the context of G-metric spaces can be defined as follows [10].

Definition 24 A mapping $T : X \to X$ is called β-admissible if for all $x, y \in X$ we have

$$\beta(x, y, y) \geq 1 \quad \Rightarrow \quad \beta(Tx, Ty, Ty) \geq 1, \tag{3.1}$$

where $\beta : X \times X \times X \to [0, \infty)$ is a given function. If in addition,

$$\left.\begin{aligned} \beta(x, y, y) \geq 1, \\ \beta(y, z, z) \geq 1 \end{aligned}\right\} \quad \Rightarrow \quad \beta(x, z, z) \geq 1, \tag{3.2}$$

for all $x, y, z \in X$, then T is called triangular β-admissible.

Definition 25 Let (X, G) be a G-metric space. Let $T : X \to X$ be a triangular β-admissible mapping. Suppose that for every $\varepsilon > 0$ there exists $\delta > 0$ such that

$$\varepsilon \leq G(x,y,y) < \varepsilon + \delta \quad \text{implies} \quad \beta(x,y,y)G(Tx,Ty,Ty) < \varepsilon, \tag{3.3}$$

for all $x, y \in X$. Then T is called β-Meir-Keeler contraction.

For more details on β-admissible maps on G-metric spaces we refer the reader to [10].

Definition 26 (I) Let (X, G) be a G-metric space. Let $T : X \to X$ be a triangular β-admissible mapping. Suppose that for every $\varepsilon > 0$ there exists $\delta > 0$ such that

$$\varepsilon \leq M(x,y,y) < \varepsilon + \delta \quad \text{implies} \quad \beta(x,y,y)G(Tx,Ty,Ty) < \varepsilon, \tag{3.4}$$

for all $x, y \in X$, where

$$M(x,y,y) = \max\bigl\{G(x,y,y), G(Tx,x,x), G(Ty,y,y)\bigr\}. \tag{3.5}$$

Then T is called a generalized β-Meir-Keeler contraction of type (I).

(II) Let (X, G) be a G-metric space. Let $T : X \to X$ be a triangular β-admissible mapping. Suppose that for every $\varepsilon > 0$ there exists $\delta > 0$ such that

$$\varepsilon \leq N(x,y,y) < \varepsilon + \delta \quad \text{implies} \quad \beta(x,y,y)G(Tx,Ty,Ty) < \varepsilon, \tag{3.6}$$

for all $x, y \in X$, where

$$M(x,y,y) = \max\left\{G(x,y,y), \frac{G(Tx,x,x) + G(Ty,y,y)}{2}\right\}. \tag{3.7}$$

Then T is called a generalized β-Meir-Keeler contraction of type (II).

Lemma 27 *Let $T : X \to X$ where X is nonempty set. Then T is β-admissible on (X, G) if and only if T is α-admissible on (X, d).*

Proof The proof is obvious by taking $\alpha(x,y) = \beta(x,y,y)$. □

The next theorem is a consequence of Corollary 18.

Theorem 28 *Let (X, G) be a complete G-metric space and $T : X \to X$ be a continuous, β-Meir-Keeler contraction. If $\beta(x_0, Tx_0, Tx_0) \geq 1$ and $\beta(Tx_0, Tx_0, x_0) \geq 1$ for some $x_0 \in X$, then T has a fixed point in X.*

Proof Consider the quasi-metric $d(x,y) = G(x,y,y)$ for all $x, y \in X$. Due to Lemma 27 and (3.3), we find that for every $\varepsilon > 0$ there exists $\delta > 0$ such that

$$\varepsilon \leq d(x,y) < \varepsilon + \delta \quad \text{implies} \quad \alpha(x,y)d(Tx,Ty) < \varepsilon, \tag{3.8}$$

for all $x, y \in X$. Then the proof follows from Corollary 18. □

Our last theorem is a consequence of Theorems 15 and 16.

Theorem 29 *Let* (X,G) *be a complete G-metric space and* $T : X \to X$ *be a continuous, generalized* β*-Meir-Keeler contraction of type* (I) *or* (II). *If* $\beta(x_0, Tx_0, Tx_0) \geq 1$ *and* $\alpha(Tx_0, Tx_0, x_0) \geq 1$ *for some* $x_0 \in X$, *then* T *has a fixed point in* X.

Proof Since the function $d(x,y) = G(x,y,y)$ is a quasi-metric on X, employing Lemma 27 and (3.4) (respectively (3.6)), we see that for every $\varepsilon > 0$ there exists $\delta > 0$ such that

$$\begin{aligned} &\varepsilon \leq M(x,y) < \varepsilon + \delta \quad \text{implies} \quad \alpha(x,y)d(Tx,Ty) < \varepsilon, \quad \text{or} \\ &\varepsilon \leq N(x,y) < \varepsilon + \delta \quad \text{implies} \quad \alpha(x,y)d(Tx,Ty) < \varepsilon, \end{aligned} \tag{3.9}$$

for all $x,y \in X$ for the generalized α-Meir-Keeler mappings of types (I) or (II), respectively. Then the proof follows from Theorem 15 (respectively, Theorem 16). □

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
All authors contributed equally and significantly in writing this article. All authors read and approved the final manuscript.

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