Klaričić Bakula et al. *Journal of Inequalities and Applications* (2015) 2015:76
DOI 10.1186/s13660-015-0603-x

Journal of Inequalities and Applications
a SpringerOpen Journal

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Some Grüss type inequalities and corrected three-point quadrature formulae of Euler type

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Abstract
We obtain some new Grüss type inequalities for the general corrected three-point quadrature formulae of Euler type. As special cases, we derive some new bounds for the corrected Euler Simpson formula, the corrected dual Euler Simpson formula and the corrected Euler Maclaurin formula. Also, applications for the corrected Euler Bullen-Simpson formula are considered.
MSC: 26D15; 26D20; 26D99

Keywords: Chebyshev functional; Grüss inequality; corrected three-point quadrature formulae; corrected Euler Bullen-Simpson formula

1 Introduction

The corrected quadrature formulae are quadrature formulae where the integral is approximated not only by the values of the integrand in certain points but also by the values of its first derivative in the end points of the interval. These formulae have a degree of exactness higher than the adjoint original formulae. The term corrected quadrature formulae was first introduced in the inequalities area by Ujević and Roberts in [1].

The Chebyshev functional [2] is defined by

$$T(f,g)=\frac{1}{b-a}\int_a^b f(s)g(s)\,\mathrm{d}s-\frac{1}{b-a}\int_a^b f(s)\,\mathrm{d}s\cdot\frac{1}{b-a}\int_a^b g(s)\,\mathrm{d}s,$$

where $f,g:[a,b]\to\mathbb{R}$ are two real functions such that $f,g,f\cdot g\in L^1[a,b]$.

For two integrable functions $f,g:[a,b]\to\mathbb{R}$ such that $\gamma\le f(s)\le\Gamma$, and $\phi\le g(s)\le\Phi$, for all $s\in[a,b]$, where γ, Γ, ϕ, Φ are real constants, the following integral inequality is known as the Grüss inequality (see [2], p.296):

$$\left|\frac{1}{b-a}\int_a^b f(s)g(s)\,\mathrm{d}s-\frac{1}{b-a}\int_a^b f(s)\,\mathrm{d}s\cdot\frac{1}{b-a}\int_a^b g(s)\,\mathrm{d}s\right|\le\frac{1}{4}(\Gamma-\gamma)(\Phi-\phi).$$

Over the last decades some new inequalities of this type have been considered and applied in numerical analysis (see [2–8] and the references cited therein).

In [9], the authors proved the following inequalities for the Chebyshev functional:

Theorem 1 *Let $f, g : [a,b] \to \mathbb{R}$ be two absolutely continuous functions on $[a,b]$ with*

$$(\cdot - a)(b - \cdot)(f')^2, (\cdot - a)(b - \cdot)(g')^2 \in L^1[a,b],$$

then

$$\begin{aligned}\left|T(f,g)\right| &\le \frac{1}{\sqrt{2}}\left[T(f,f)\right]^{1/2}\frac{1}{\sqrt{b-a}}\left[\int_a^b (s-a)(b-s)\left(g'(s)\right)^2 \mathrm{d}s\right]^{1/2} \\ &\le \frac{1}{2(b-a)}\left[\int_a^b (s-a)(b-s)\left(f'(s)\right)^2 \mathrm{d}s\right]^{1/2} \\ &\quad \cdot \left[\int_a^b (s-a)(b-s)\left(g'(s)\right)^2 \mathrm{d}s\right]^{1/2}. \end{aligned} \tag{1}$$

The constants $1/\sqrt{2}$ and $1/2$ are the best possible.

Theorem 2 *Assume that $g : [a,b] \to \mathbb{R}$ is monotonic nondecreasing on $[a,b]$ and $f : [a,b] \to \mathbb{R}$ is absolutely continuous with $f' \in L^\infty[a,b]$, then*

$$\left|T(f,g)\right| \le \frac{1}{2(b-a)}\left\|f'\right\|_\infty \cdot \int_a^b (s-a)(b-s)\,\mathrm{d}g(s). \tag{2}$$

The constant $1/2$ is the best possible.

The aim of this note is to consider some new Grüss type inequalities for the general corrected three-point quadrature formulae of Euler type. This will be done by using the above theorems and the corrected three-point quadrature formulae recently introduced in [10]. Also, we use the obtained results to get the error estimates for the corrected Euler Simpson formula, the corrected dual Euler Simpson formula and the corrected Euler Maclaurin formula. Finally, the corresponding error estimates for the corrected Euler Bullen-Simpson formula are derived.

More about quadrature formulae and error estimations (from the point of view of inequality theory) can be found in the monographs [11] and [12].

Since we deal with quadrature formulae of Euler type, let us recall a few features of the Bernoulli polynomials. The symbol $B_k(s)$ denotes the Bernoulli polynomials, $B_k = B_k(0)$ the Bernoulli numbers, and $B_k^*(s)$, $k \ge 0$, periodic functions of period 1 defined by the condition

$$B_k^*(s+1) = B_k^*(s), \quad s \in \mathbb{R},$$

and related to the Bernoulli polynomials as follows:

$$B_k^*(s) = B_k(s), \quad 0 \le s < 1.$$

The Bernoulli polynomials $B_k(s)$, $k \ge 0$, are uniquely determined by the identities

$$B_k'(s) = kB_{k-1}(s), \quad k \ge 1; \qquad B_0(s) = 1, \qquad B_k(s+1) - B_k(s) = ks^{k-1}, \quad k \ge 0.$$

Further, $B_0^*(s) = 1$, $B_1^*(s)$ is a discontinuous function with a jump of -1 at each integer and for $k \geq 2$, $B_k^*(s)$ are continuous functions. We get

$$B_k^{*\prime}(s) = kB_{k-1}^*(s), \quad k \geq 1, \tag{3}$$

for every $s \in \mathbb{R}$ when $k \geq 3$, and for every $s \in \mathbb{R} \setminus Z$ when $k = 1, 2$.

For some further details as regards Bernoulli polynomials, Bernoulli numbers and periodic functions B_k^*, see [13].

2 Main results

Let $x \in [0, 1/2)$ and $f : [0, 1] \to \mathbb{R}$ be such that $f^{(2n+1)}$ is a continuous function of bounded variation on $[0, 1]$ for some $n \geq 0$. In [10], the authors proved the following general three-point quadrature formula of Euler type:

$$\begin{aligned} &\int_0^1 f(s)\,\mathrm{d}s - w(x)f(x) - \bigl(1 - 2w(x)\bigr)f\left(\frac{1}{2}\right) - w(x)f(1-x) + T_{2n}(x) \\ &\quad = \frac{1}{(2n+2)!}\int_0^1 F_{2n+2}(x,s)\,\mathrm{d}f^{(2n+1)}(s), \end{aligned} \tag{4}$$

where

$$T_{2n}(x) = \sum_{k=2}^{2n} \frac{1}{k!} G_k(x,0)\bigl[f^{(k-1)}(1) - f^{(k-1)}(0)\bigr], \tag{5}$$

$$G_k(x,s) = w(x)\bigl[B_k^*(x-s) + B_k^*(1-x-s)\bigr] + \bigl(1-2w(x)\bigr)B_k^*\left(\frac{1}{2} - s\right), \quad k \geq 1, \tag{6}$$

$$F_k(x,s) = G_k(x,s) - G_k(x,0), \quad k \geq 2 \tag{7}$$

and $s \in \mathbb{R}$.

From the properties of the Bernoulli polynomials it easily follows that

$$\begin{aligned} &G_k(x, 1-s) = (-1)^k G_k(x,s), \quad s \in [0,1], \\ &\frac{\partial G_k(x,s)}{\partial s} = -kG_{k-1}(x,s) \end{aligned}$$

and $G_{2k-1}(x, 0) = 0$, for $k \geq 2$ and for any choice of the weight w. In general $G_{2k}(x, 0) \neq 0$.

If we impose the condition $G_4(x, 0) = 0$ the obtained formula will include the value of the first derivative at the end points of the interval and is known in the literature as a corrected quadrature formula. So, condition $G_4(x, 0) = 0$ gives

$$w(x) = \frac{7}{30(2x-1)^2(-4x^2+4x+1)}, \quad x \in \left[0, \frac{1}{2}\right). \tag{8}$$

Now, for $f : [0, 1] \to \mathbb{R}$ such that $f^{(2n+1)}$ is a continuous function of bounded variation on $[0, 1]$ for some $n \geq 0$, $x \in [0, 1/2)$, (4) becomes

$$\int_0^1 f(s)\,\mathrm{d}s - Q_C\left(x, \frac{1}{2}, 1-x\right) + T_{2n}^{CQ3}(x) = \frac{1}{(2n+2)!}\int_0^1 F_{2n+2}^{CQ3}(x,s)\,\mathrm{d}f^{(2n+1)}(s), \tag{9}$$

where

$$
\begin{aligned}
&Q_C\left(x,\frac{1}{2},1-x\right)\\
&\quad=\frac{1}{30(2x-1)^2(-4x^2+4x+1)}\left[7f(x)-480B_4(x)f\left(\frac{1}{2}\right)+7f(1-x)\right],\\
&T_{2n}^{CQ3}(x)=\sum_{k=1}^{n}\frac{1}{(2k)!}G_{2k}^{CQ3}(x,0)\left[f^{(2k-1)}(1)-f^{(2k-1)}(0)\right]\\
&\qquad=\frac{10x^2-10x+1}{60(-4x^2+4x+1)}\left[f'(1)-f'(0)\right]\\
&\qquad\quad+\sum_{k=3}^{n}\frac{1}{(2k)!}G_{2k}^{CQ3}(x,0)\left[f^{(2k-1)}(1)-f^{(2k-1)}(0)\right],\\
&G_k^{CQ3}(x,s)=\frac{1}{30(2x-1)^2(-4x^2+4x+1)}\\
&\qquad\cdot\left[7B_k^*(x-s)-480B_4(x)\cdot B_k^*\left(\frac{1}{2}-s\right)+7B_k^*(1-x-s)\right],\quad k\geq 1,\\
&F_k^{CQ3}(x,s)=G_k^{CQ3}(x,s)-G_k^{CQ3}(x,0),\quad k\geq 2
\end{aligned}
\tag{10}
$$

and $s\in\mathbb{R}$.

Assuming $f^{(2n-1)}$ is a continuous function of bounded variation on $[0,1]$ for some $n\geq 1$, then the following identity holds:

$$
\int_0^1 f(s)\,\mathrm{d}s-Q_C\left(x,\frac{1}{2},1-x\right)+T_{2n}^{CQ3}(x)=\frac{1}{(2n)!}\int_0^1 G_{2n}^{CQ3}(x,s)\,\mathrm{d}f^{(2n-1)}(s),\tag{11}
$$

while assuming $f^{(2n)}$ is a continuous function of bounded variation on $[0,1]$ for some $n\geq 0$ it follows that

$$
\int_0^1 f(s)\,\mathrm{d}s-Q_C\left(x,\frac{1}{2},1-x\right)+T_{2n}^{CQ3}(x)=\frac{1}{(2n+1)!}\int_0^1 G_{2n+1}^{CQ3}(x,s)\,\mathrm{d}f^{(2n)}(s).\tag{12}
$$

The identities (9), (11), (12) and the following lemma were proved in [12], p.99.

Lemma 1 *For $x\in[0,\frac{1}{2}-\frac{\sqrt{15}}{10})\cup[\frac{1}{6},\frac{1}{2})$ and $k\geq 2$, $G_{2k+1}^{CQ3}(x,s)$ has no zeros in variable s on the interval $(0,\frac{1}{2})$. The sign of this function is determined by*

$$
(-1)^{k+1}G_{2k+1}^{CQ3}(x,s)>0,\quad \textit{for } x\in\left[0,\frac{1}{2}-\frac{\sqrt{15}}{10}\right)
$$

and

$$
(-1)^{k}G_{2k+1}^{CQ3}(x,s)>0,\quad \textit{for } x\in\left[\frac{1}{6},\frac{1}{2}\right).
$$

Now, we can state some new Grüss type inequalities for the general corrected three-point quadrature formulae of Euler type.

Theorem 3 *Let $f : [0,1] \to \mathbb{R}$ be such that $f^{(2n)}$ is an absolutely continuous function for some $n \geq 1$ and $x \in [0,1/2)$. Then the following equality holds*:

$$\int_0^1 f(s)\,\mathrm{d}s - Q_C\left(x, \frac{1}{2}, 1-x\right) + T_{2n}^{CQ3}(x) = K_{2n}^{CQ3}(f) \tag{13}$$

and the remainder $K_{2n}^{CQ3}(f)$ satisfies the inequality

$$\begin{aligned} &\left|K_{2n}^{CQ3}(f)\right| \\ &\quad\leq \frac{1}{30(2x-1)^2(-4x^2+4x+1)}\left[\frac{-1}{2(4n)!}\left(98B_{4n} + 98B_{4n}(1-2x)\right.\right. \\ &\qquad\left.\left. - 13{,}440B_4(x)B_{4n}\left(x+\frac{1}{2}\right) + 480^2B_4^2(x)B_{4n}\right)\right]^{1/2} \\ &\qquad\cdot\left[\int_0^1 s(1-s)\left(f^{(2n+1)}(s)\right)^2\,\mathrm{d}s\right]^{1/2}. \end{aligned} \tag{14}$$

For $f : [0,1] \to \mathbb{R}$ such that $f^{(2n+1)}$ is an absolutely continuous function for some $n \geq 0$ and $x \in [0,1/2)$, the following representation holds:

$$\int_0^1 f(s)\,\mathrm{d}s - Q_C\left(x, \frac{1}{2}, 1-x\right) + T_{2n}^{CQ3}(x) = K_{2n+1}^{CQ3}(f) \tag{15}$$

and the remainder $K_{2n+1}^{CQ3}(f)$ satisfies the inequality

$$\begin{aligned} &\left|K_{2n+1}^{CQ3}(f)\right| \\ &\quad\leq \frac{1}{30(2x-1)^2(-4x^2+4x+1)}\left[\frac{1}{2(4n+2)!}\left(98B_{4n+2} + 98B_{4n+2}(1-2x)\right.\right. \\ &\qquad\left.\left. - 13{,}440B_4(x)B_{4n+2}\left(x+\frac{1}{2}\right) + 480^2B_4^2(x)B_{4n+2}\right)\right]^{1/2} \\ &\qquad\cdot\left[\int_0^1 s(1-s)\left(f^{(2n+2)}(s)\right)^2\,\mathrm{d}s\right]^{1/2}. \end{aligned} \tag{16}$$

Proof Applying Theorem 1 with G_k^{CQ3} in place of f and $f^{(k)}$ in place of g we obtain

$$\begin{aligned} &\left|\int_0^1 G_k^{CQ3}(x,s)f^{(k)}(s)\,\mathrm{d}s - \int_0^1 G_k^{CQ3}(x,s)\,\mathrm{d}s\cdot\int_0^1 f^{(k)}(s)\,\mathrm{d}s\right| \\ &\quad\leq \frac{1}{\sqrt{2}}\left[T\left(G_k^{CQ3}(x,\cdot), G_k^{CQ3}(x,\cdot)\right)\right]^{1/2}\left[\int_0^1 s(1-s)\left(f^{(k+1)}(s)\right)^2\,\mathrm{d}s\right]^{1/2}, \end{aligned} \tag{17}$$

where

$$T\left(G_k^{CQ3}(x,\cdot), G_k^{CQ3}(x,\cdot)\right) = \int_0^1 \left(G_k^{CQ3}(x,s)\right)^2\,\mathrm{d}s - \left(\int_0^1 G_k^{CQ3}(x,s)\,\mathrm{d}s\right)^2.$$

By elementary calculations we obtain

$$\int_0^1 G_k^{CQ3}(x,s)\,\mathrm{d}s = 0. \tag{18}$$

Using integration by parts we get

$$\begin{aligned}
&\int_0^1 \left(G_k^{CQ3}(x,s)\right)^2 \mathrm{d}s \\
&= (-1)^{k-1} \frac{k(k-1)\cdots 2}{(k+1)(k+2)\cdots(2k-1)} \\
&\quad \cdot \left[-\frac{1}{2k} G_{2k}^{CQ3}(x,s) G_1(x,s)\Big|_0^1 + \frac{1}{2k} \int_0^1 G_{2k}^{CQ3}(x,s)\, \mathrm{d}G_1(x,s) \right] \\
&= \frac{(-1)^{k-1}(k!)^2}{30(2x-1)^2(-4x^2+4x+1)(2k)!} \\
&\quad \cdot \left[30(2x-1)^2\left(4x^2-4x-1\right) \int_0^1 G_{2k}^{CQ3}(x,s)\, \mathrm{d}s \right. \\
&\quad \left. + 7G_{2k}^{CQ3}(x,x) - 480B_4(x) G_{2k}^{CQ3}\left(x, \frac{1}{2}\right) + 7G_{2k}^{CQ3}(x,1-x) \right] \\
&= \frac{(-1)^{k-1}(k!)^2}{900(2x-1)^4(-4x^2+4x+1)^2(2k)!} \\
&\quad \cdot \left[98B_{2k} + 98B_{2k}(1-2x) - 13{,}440B_4(x)B_{2k}\left(x+\frac{1}{2}\right) + 480^2 B_4^2(x) B_{2k} \right].
\end{aligned}$$

Finally, if we put $k = 2n$ using (11) and (17), we obtain representation (13) and inequality (14). Since, for $k = 2n + 1$ by (12) and (17), representation (15) and estimate (16) follow. □

Remark 1 From (10) and (18) we get

$$\int_0^1 F_k^{CQ3}(x,s)\, \mathrm{d}s = \int_0^1 G_k^{CQ3}(x,s)\, \mathrm{d}s - \int_0^1 G_k^{CQ3}(x,0)\, \mathrm{d}s = -G_k^{CQ3}(x,0)$$

and

$$\begin{aligned}
\int_0^1 \left(F_k^{CQ3}(x,s)\right)^2 \mathrm{d}s &= \int_0^1 \left(G_k^{CQ3}(x,s)\right)^2 \mathrm{d}s - 2G_k^{CQ3}(x,0) \int_0^1 G_k^{CQ3}(x,s)\, \mathrm{d}s \\
&\quad + \left(G_k^{CQ3}(x,0)\right)^2.
\end{aligned}$$

Further, if we put $k = 2n + 2$ in the proof of Theorem 3, using (9) similar to (17) (with $n \leftrightarrow n + 1$), we deduce equality (13) and bound (14).

Corollary 1 *Let $f : [0,1] \to \mathbb{R}$ be such that $f^{(2n+1)}$ is absolutely continuous for some $n \geq 2$ and $f^{(2n+1)} \geq 0$ on $[0,1]$. Then for $x \in [\frac{1}{6}, \frac{1}{2})$,*

$$\begin{aligned}
0 &\leq (-1)^n \left\{ \int_0^1 f(s)\, \mathrm{d}s - Q_C\left(x, \frac{1}{2}, 1-x\right) + T_{2n}^{CQ3}(x) \right\} \\
&\leq \frac{1}{30(2x-1)^2(-4x^2+4x+1)} \left[\frac{1}{2(4n+2)!} \left(98B_{4n+2} + 98B_{4n+2}(1-2x) \right. \right. \\
&\quad \left. \left. - 13{,}440B_4(x)B_{4n+2}\left(x+\frac{1}{2}\right) + 480^2 B_4^2(x) B_{4n+2} \right) \right]^{1/2} \\
&\quad \cdot \left[\int_0^1 s(1-s)\left(f^{(2n+2)}(s)\right)^2 \mathrm{d}s \right]^{1/2}, \qquad (19)
\end{aligned}$$

and for $x \in [0, \frac{1}{2} - \frac{\sqrt{15}}{10})$,

$$\begin{aligned} 0 &\leq (-1)^{n+1}\left\{\int_0^1 f(s)\,\mathrm{d}s - Q_C\left(x, \frac{1}{2}, 1-x\right) + T_{2n}^{CQ3}(x)\right\} \\ &\leq \frac{1}{30(2x-1)^2(-4x^2+4x+1)}\left[\frac{1}{2(4n+2)!}\left(98B_{4n+2} + 98B_{4n+2}(1-2x)\right.\right. \\ &\quad \left.\left. - 13{,}440B_4(x)B_{4n+2}\left(x+\frac{1}{2}\right) + 480^2B_4^2(x)B_{4n+2}\right)\right]^{1/2} \\ &\quad \cdot \left[\int_0^1 s(1-s)\left(f^{(2n+2)}(s)\right)^2\,\mathrm{d}s\right]^{1/2}. \end{aligned} \tag{20}$$

Proof We use Lemma 1, representation (15) and inequality (16) to obtain inequalities (19) and (20). □

As special cases of Theorem 3 for $x = 0$, $x = 1/4$ and $x = 1/6$ we derive inequalities related to the corrected Euler Simpson formula, the corrected dual Euler Simpson formula and the corrected Euler Maclaurin formula, respectively.

Corollary 2 *Let* $f : [0,1] \to \mathbb{R}$ *be such that* $f^{(2n)}$ *is absolutely continuous for some* $n \geq 1$. *Then*

$$\begin{aligned} &\left|\int_0^1 f(s)\,\mathrm{d}s - \frac{1}{30}\left[7f(0) + 16f\left(\frac{1}{2}\right) + 7f(1)\right] + T_{2n}^{CQ3}(0)\right| \\ &\quad \leq \frac{1}{15}\left[-\frac{1 + 7\cdot 2^{5-4n}}{2(4n)!}B_{4n}\right]^{1/2} \cdot \left[\int_0^1 s(1-s)\left(f^{(2n+1)}(s)\right)^2\,\mathrm{d}s\right]^{1/2}. \end{aligned} \tag{21}$$

If $f^{(2n+1)}$ *is absolutely continuous for some* $n \geq 0$ *then*

$$\begin{aligned} &\left|\int_0^1 f(s)\,\mathrm{d}s - \frac{1}{30}\left[7f(0) + 16f\left(\frac{1}{2}\right) + 7f(1)\right] + T_{2n}^{CQ3}(0)\right| \\ &\quad \leq \frac{1}{15}\left[\frac{1 + 7\cdot 2^{3-4n}}{2(4n+2)!}B_{4n+2}\right]^{1/2} \cdot \left[\int_0^1 s(1-s)\left(f^{(2n+2)}(s)\right)^2\,\mathrm{d}s\right]^{1/2}, \end{aligned} \tag{22}$$

where $T_0^{CQ3}(0) = 0$, $T_2^{CQ3}(0) = T_4^{CQ3}(0) = \frac{1}{60}[f'(1) - f'(0)]$ *and for* $n \geq 3$,

$$\begin{aligned} T_{2n}^{CQ3}(0) &= \frac{1}{60}\left[f'(1) - f'(0)\right] \\ &\quad + \sum_{k=3}^{n} \frac{1}{15(2k)!}\left(-1 + 2^{4-2k}\right)B_{2k}\left[f^{(2k-1)}(1) - f^{(2k-1)}(0)\right]. \end{aligned}$$

Remark 2 Specially, if f' is absolutely continuous then for $n = 0$ in Corollary 2, we derive

$$\begin{aligned} &\left|\int_0^1 f(s)\,\mathrm{d}s - \frac{1}{30}\left[7f(0) + 16f\left(\frac{1}{2}\right) + 7f(1)\right]\right| \\ &\quad \leq \frac{1}{30}\sqrt{\frac{19}{2}} \cdot \left[\int_0^1 s(1-s)\left(f''(s)\right)^2\,\mathrm{d}s\right]^{1/2}. \end{aligned}$$

Further, if f'' is absolutely continuous then for $n = 1$ in Corollary 2 we obtain

$$\left|\int_0^1 f(s)\,\mathrm{d}s - \frac{1}{30}\left[7f(0) + 16f\left(\frac{1}{2}\right) + 7f(1)\right] + \frac{1}{60}\left[f'(1) - f'(0)\right]\right| \le \frac{1}{60\sqrt{6}} \cdot \left[\int_0^1 s(1-s)\left(f'''(s)\right)^2\,\mathrm{d}s\right]^{1/2}.$$

Corollary 3 *Let $f : [0,1] \to \mathbb{R}$ be such that $f^{(2n)}$ is absolutely continuous for some $n \ge 1$. Then*

$$\left|\int_0^1 f(s)\,\mathrm{d}s - \frac{1}{15}\left[8f\left(\frac{1}{4}\right) - f\left(\frac{1}{2}\right) + 8f\left(\frac{3}{4}\right)\right] + T_{2n}^{CQ3}\left(\frac{1}{4}\right)\right| \le \frac{1}{15}\left[-\frac{1 + 9 \cdot 2^{5-4n} - 2^{6-8n}}{2(4n)!}B_{4n}\right]^{1/2} \cdot \left[\int_0^1 s(1-s)\left(f^{(2n+1)}(s)\right)^2\,\mathrm{d}s\right]^{1/2}. \tag{23}$$

If $f^{(2n+1)}$ is absolutely continuous for some $n \ge 0$ then

$$\left|\int_0^1 f(s)\,\mathrm{d}s - \frac{1}{15}\left[8f\left(\frac{1}{4}\right) - f\left(\frac{1}{2}\right) + 8f\left(\frac{3}{4}\right)\right] + T_{2n}^{CQ3}\left(\frac{1}{4}\right)\right| \le \frac{1}{15}\left[\frac{1 + 9 \cdot 2^{3-4n} - 2^{2-8n}}{2(4n+2)!}B_{4n+2}\right]^{1/2} \cdot \left[\int_0^1 s(1-s)\left(f^{(2n+2)}(s)\right)^2\,\mathrm{d}s\right]^{1/2}, \tag{24}$$

where $T_0^{CQ3}(\frac{1}{4}) = 0$, $T_2^{CQ3}(\frac{1}{4}) = T_4^{CQ3}(\frac{1}{4}) = -\frac{1}{120}[f'(1) - f'(0)]$ and for $n \ge 3$,

$$T_{2n}^{CQ3}\left(\frac{1}{4}\right) = -\frac{1}{120}\left[f'(1) - f'(0)\right] + \sum_{k=3}^{n} \frac{1}{15(2k)!}\left(2^{5-4k} - 9 \cdot 2^{1-2k} + 1\right)B_{2k}\left[f^{(2k-1)}(1) - f^{(2k-1)}(0)\right].$$

Remark 3 If f' is absolutely continuous then for $n = 0$ in Corollary 3 we obtain

$$\left|\int_0^1 f(s)\,\mathrm{d}s - \frac{1}{15}\left[8f\left(\frac{1}{4}\right) - f\left(\frac{1}{2}\right) + 8f\left(\frac{3}{4}\right)\right]\right| \le \frac{1}{30}\sqrt{\frac{23}{2}} \cdot \left[\int_0^1 s(1-s)\left(f''(s)\right)^2\,\mathrm{d}s\right]^{1/2}.$$

If f'' is absolutely continuous then for $n = 1$ in Corollary 3 we get

$$\left|\int_0^1 f(s)\,\mathrm{d}s - \frac{1}{15}\left[8f\left(\frac{1}{4}\right) - f\left(\frac{1}{2}\right) + 8f\left(\frac{3}{4}\right)\right] - \frac{1}{120}\left[f'(1) - f'(0)\right]\right| \le \frac{1}{360}\sqrt{\frac{15}{2}} \cdot \left[\int_0^1 s(1-s)\left(f'''(s)\right)^2\,\mathrm{d}s\right]^{1/2}.$$

Corollary 4 *Let $f:[0,1]\to\mathbb{R}$ be such that $f^{(2n)}$ is absolutely continuous for some $n\geq 1$. Then*

$$\left|\int_0^1 f(s)\,\mathrm{d}s-\frac{1}{80}\left[27f\left(\frac{1}{6}\right)+26f\left(\frac{1}{2}\right)+27f\left(\frac{5}{6}\right)\right]+T_{2n}^{CQ3}\left(\frac{1}{6}\right)\right| \leq \frac{1}{80}\left[-\frac{1+79\cdot 3^{4-4n}}{2(4n)!}B_{4n}\right]^{1/2}\cdot\left[\int_0^1 s(1-s)\left(f^{(2n+1)}(s)\right)^2\mathrm{d}s\right]^{1/2}. \tag{25}$$

If $f^{(2n+1)}$ is absolutely continuous for some $n\geq 0$ then

$$\left|\int_0^1 f(s)\,\mathrm{d}s-\frac{1}{80}\left[27f\left(\frac{1}{6}\right)+26f\left(\frac{1}{2}\right)+27f\left(\frac{5}{6}\right)\right]+T_{2n}^{CQ3}\left(\frac{1}{6}\right)\right| \leq \frac{1}{80}\left[\frac{1+79\cdot 3^{2-4n}}{2(4n+2)!}B_{4n+2}\right]^{1/2}\cdot\left[\int_0^1 s(1-s)\left(f^{(2n+2)}(s)\right)^2\mathrm{d}s\right]^{1/2}, \tag{26}$$

where $T_0^{CQ3}(\frac{1}{6})=0$, $T_2^{CQ3}(\frac{1}{6})=T_4^{CQ3}(\frac{1}{6})=-\frac{1}{240}[f'(1)-f'(0)]$ and for $n\geq 3$,

$$T_{2n}^{CQ3}\left(\frac{1}{6}\right)=-\frac{1}{240}\left[f'(1)-f'(0)\right]+\sum_{k=3}^{n}\frac{1}{80(2k)!}\left(1-2^{1-2k}\right)\left(1-3^{4-2k}\right)B_{2k}\left[f^{(2k-1)}(1)-f^{(2k-1)}(0)\right].$$

Remark 4 Specially, if f' is absolutely continuous then for $n=0$ in Corollary 4 we obtain

$$\left|\int_0^1 f(s)\,\mathrm{d}s-\frac{1}{80}\left[27f\left(\frac{1}{6}\right)+26f\left(\frac{1}{2}\right)+27f\left(\frac{5}{6}\right)\right]\right| \leq \frac{1}{80}\sqrt{\frac{89}{3}}\cdot\left[\int_0^1 s(1-s)\left(f''(s)\right)^2\mathrm{d}s\right]^{1/2}.$$

If f'' is absolutely continuous then for $n=1$ in Corollary 4 we get

$$\left|\int_0^1 f(s)\,\mathrm{d}s-\frac{1}{80}\left[27f\left(\frac{1}{6}\right)+26f\left(\frac{1}{2}\right)+27f\left(\frac{5}{6}\right)\right]-\frac{1}{240}\left[f'(1)-f'(0)\right]\right| \leq \frac{1}{240\sqrt{2}}\cdot\left[\int_0^1 s(1-s)\left(f'''(s)\right)^2\mathrm{d}s\right]^{1/2}.$$

Here, as in the rest of the paper, the symbol $[f^{(k)};0,1]$ denotes the divided difference of the function $f^{(k)}$,

$$\left[f^{(k)};0,1\right]=f^{(k)}(1)-f^{(k)}(0).$$

Theorem 4 *Let $f:[0,1]\to\mathbb{R}$ be such that $f^{(2n)}$ is an absolutely continuous function and $f^{(2n+1)}\geq 0$ on $[0,1]$. Then representation* (13) *holds and the remainder $K_{2n}^{CQ3}(f)$ satisfies the following inequality*:

$$\left|K_{2n}^{CQ3}(f)\right|\leq\frac{1}{(2n-1)!}\left\|G_{2n-1}^{CQ3}(x,s)\right\|_\infty\left\{\frac{f^{(2n-1)}(0)+f^{(2n-1)}(1)}{2}-\left[f^{(2n-2)};0,1\right]\right\}. \tag{27}$$

If $f^{(2n+1)}$ *is an absolutely continuous function and* $f^{(2n+2)} \geq 0$ *on* $[0,1]$, *then equality* (15) *holds and the remainder* $K^{Q3}_{2n+1}(f)$ *satisfies the inequality*

$$\bigl|K^{CQ3}_{2n+1}(f)\bigr| \leq \frac{1}{(2n)!}\bigl\|G^{CQ3}_{2n}(x,s)\bigr\|_{\infty}\left\{\frac{f^{(2n)}(0)+f^{(2n)}(1)}{2} - \bigl[f^{(2n-1)};0,1\bigr]\right\}. \tag{28}$$

Proof Applying Theorem 2 with G^{CQ3}_{2n} in place of f and $f^{(2n)}$ in place of g we deduce

$$\begin{aligned}&\left|\int_0^1 G^{CQ3}_{2n}(x,s)f^{(2n)}(s)\,\mathrm{d}s - \int_0^1 G^{CQ3}_{2n}(x,s)\,\mathrm{d}s\cdot\int_0^1 f^{(2n)}(s)\,\mathrm{d}s\right| \\ &\quad\leq \frac{2n}{2}\bigl\|G^{CQ3}_{2n-1}(x,s)\bigr\|_{\infty}\int_0^1 s(1-s)f^{(2n+1)}(s)\,\mathrm{d}s.\end{aligned} \tag{29}$$

Further,

$$\begin{aligned}\int_0^1 s(1-s)f^{(2n+1)}(s)\,\mathrm{d}s &= \int_0^1 (2s-1)f^{(2n)}(s)\,\mathrm{d}s \\ &= f^{(2n-1)}(1)+f^{(2n-1)}(0) - 2\bigl[f^{(2n-2)}(1)-f^{(2n-2)}(0)\bigr].\end{aligned}$$

Finally, using equality (13) and inequality (29), we obtain estimate (27). Similarly, from identity (15) we get inequality (28). □

3 Applications for the corrected Euler Bullen-Simpson formula

In [14], the author proved that if $f:[0,1]\to\mathbb{R}$ is a 4-convex function then the following Bullen-Simpson inequality holds:

$$\begin{aligned}0 &\leq \int_0^1 f(s)\,\mathrm{d}s - \frac{1}{3}\left[2f\left(\frac{1}{4}\right) - f\left(\frac{1}{2}\right) + 2f\left(\frac{3}{4}\right)\right] \\ &\leq \frac{1}{6}\left[f(0)+4f\left(\frac{1}{2}\right)+f(1)\right] - \int_0^1 f(s)\,\mathrm{d}s.\end{aligned} \tag{30}$$

In [15] a generalisation of inequality (30) for a class of $(2k)$-convex functions was established.

Franjić and Pečarić in [16] derived similar type inequalities by using the corrected Simpson formula and the corrected dual Simpson formula. They proved that the corrected dual Simpson quadrature rule is more accurate than the corrected Simpson quadrature rule, that is,

$$\begin{aligned}0 &\leq \int_0^1 f(s)\,\mathrm{d}s - \frac{1}{15}\left[8f\left(\frac{1}{4}\right) - f\left(\frac{1}{2}\right) + 8f\left(\frac{3}{4}\right)\right] - \frac{1}{120}\bigl[f'(1)-f'(0)\bigr] \\ &\leq \frac{1}{30}\left[7f(0)+16f\left(\frac{1}{2}\right)+7f(1)\right] - \frac{1}{60}\bigl[f'(1)-f'(0)\bigr] - \int_0^1 f(s)\,\mathrm{d}s.\end{aligned} \tag{31}$$

Also, they obtained a generalisation of inequality (31) for a class of $(2k)$-convex functions.

Let us define

$$D^C(0,1) = \frac{1}{60}\left[7f(0)+16f\left(\frac{1}{4}\right)+14f\left(\frac{1}{2}\right)+16f\left(\frac{3}{4}\right)+7f(1)\right].$$

We consider the sequences of functions $(G_k^C(s))_{k\geq 1}$ and $(F_k^C(s))_{k\geq 1}$ defined by

$$G_k^C(s) = 7B_k^*(1-s) + 8B_k^*\left(\frac{1}{4}-s\right) + 7B_k^*\left(\frac{1}{2}-s\right) + 8B_k^*\left(\frac{3}{4}-s\right), \quad s\in\mathbb{R}$$

and

$$F_k^C(s) = G_k^C(s) - \tilde{B}_k, \quad s\in\mathbb{R}, \tag{32}$$

where

$$\tilde{B}_k = 7B_k + 8B_k\left(\frac{1}{4}\right) + 7B_k\left(\frac{1}{2}\right) + 8B_k\left(\frac{3}{4}\right).$$

By direct calculation we get $\tilde{B}_2 = 1/4$ and $\tilde{B}_3 = \tilde{B}_4 = \tilde{B}_5 = 0$. Further, it is easy to see that $\tilde{B}_{2k-1} = 0$, $k\geq 2$.

For any function $f:[0,1]\to\mathbb{R}$ such that $f^{(n-1)}$ exists on $[0,1]$ for some $n\geq 1$ we define $T_0^C(f) = T_1^C(f) = 0$,

$$T_2^C(f) = T_3^C(f) = T_4^C(f) = T_5^C(f) = -\frac{1}{240}\left[f'(1) - f'(0)\right]$$

and for $m\geq 6$,

$$T_m^C(f) = -\frac{1}{240}\left[f'(1) - f'(0)\right] + \frac{1}{15}\sum_{k=3}^{\lfloor m/2\rfloor}\frac{1}{(2k)!}2^{-2k}\left(1-2^{4-2k}\right)B_{2k}\left[f^{(2k-1)}(1) - f^{(2k-1)}(0)\right]. \tag{33}$$

In [16], the authors established the following corrected Euler Bullen-Simpson formulae.

Lemma 2 *Let $f:[0,1]\to\mathbb{R}$ be such that $f^{(n-1)}$ is a continuous function of bounded variation on $[0,1]$, for some $n\geq 1$. Then*

$$\int_0^1 f(s)\,\mathrm{d}s = D^C(0,1) + T_n^C(f) + R_n^C(f) \tag{34}$$

and

$$\int_0^1 f(s)\,\mathrm{d}s = D^C(0,1) + T_{n-1}^C(f) + \hat{R}_n^C(f), \tag{35}$$

where

$$R_n^C(f) = \frac{1}{30(n!)}\int_0^1 G_n^C(s)\,\mathrm{d}f^{(n-1)}(s)$$

and

$$\hat{R}_n^C(f) = \frac{1}{30(n!)} \int_0^1 F_n^C(s)\,\mathrm{d}f^{(n-1)}(s).$$

Using Theorem 1 for identity (34) we get the following Grüss type inequality.

Theorem 5 *Let $f : [0,1] \to \mathbb{R}$ be such that $f^{(n)}$ is absolutely continuous for some $n \geq 1$. Then*

$$\int_0^1 f(s)\,\mathrm{d}s - D^C(0,1) - T_n^C(f) = K_n^C(f), \tag{36}$$

and the remainder $K_n^C(f)$ satisfies the inequality

$$\begin{aligned} \left|K_n^C(f)\right| \leq{}& \frac{1}{15}\left[\frac{(-1)^{n-1}}{2(2n)!}\left(2^{-2n} + 7 \cdot 2^{5-4n}\right)B_{2n}\right]^{1/2} \\ &\cdot \left[\int_0^1 s(1-s)\left(f^{(n+1)}(s)\right)^2\,\mathrm{d}s\right]^{1/2}. \end{aligned} \tag{37}$$

Proof Applying Theorem 1 for $f \to G_n^C$, $g \to f^{(n)}$, we obtain

$$\begin{aligned} &\left|\int_0^1 G_n^C(s) f^{(n)}(s)\,\mathrm{d}s - \int_0^1 G_n^C(s)\,\mathrm{d}s \cdot \int_0^1 f^{(n)}(s)\,\mathrm{d}s\right| \\ &\quad \leq \frac{1}{\sqrt{2}}\left[T\left(G_n^C(\cdot), G_n^C(\cdot)\right)\right]^{1/2} \cdot \left[\int_0^1 s(1-s)\left(f^{(n+1)}(s)\right)^2\,\mathrm{d}s\right]^{1/2}, \end{aligned} \tag{38}$$

where

$$T\left(G_n^C(\cdot), G_n^C(\cdot)\right) = \int_0^1 \left(G_n^C(s)\right)^2\,\mathrm{d}s - \left[\int_0^1 G_n^C(s)\,\mathrm{d}s\right]^2.$$

Easily we get $\int_0^1 G_n^C(s)\,\mathrm{d}s = 0$ and using integration by parts we have

$$\begin{aligned} &\int_0^1 \left(G_n^C(s)\right)^2\,\mathrm{d}s \\ &\quad = (-1)^{n-1}\frac{n(n-1)\cdots 2}{(n+1)(n+2)\cdots(2n-1)}\left[\int_0^1 G_1^C(s) G_{2n-1}^C(s)\,\mathrm{d}s\right] \\ &\quad = (-1)^{n-1}\frac{(n!)^2}{(2n)!}\left[-30\int_0^1 G_{2n}^C(s)\,\mathrm{d}s + 14G_{2n}^C(0) + 16G_{2n}^C\left(\frac{1}{4}\right)\right] \\ &\quad = (-1)^{n-1}\frac{(n!)^2}{(2n)!}\left[226B_{2n} + 448B_{2n}\left(\frac{1}{4}\right) + 226B_{2n}\left(\frac{1}{2}\right)\right]. \end{aligned}$$

Using (34) and (38), we deduce representation (36) and bound (37). □

Remark 5 Because of (32) we get

$$\int_0^1 F_k^C(s)\,\mathrm{d}s = \int_0^1 G_k^C(s)\,\mathrm{d}s - \int_0^1 \tilde{B}_k\,\mathrm{d}s = -\tilde{B}_k,$$

and also

$$\int_0^1 \left(F_k^C(s)\right)^2 \mathrm{d}s = \int_0^1 \left(G_k^C(s)\right)^2 \mathrm{d}s - 2\tilde{B}_k \int_0^1 G_k^C(s)\,\mathrm{d}s + \tilde{B}_k^2.$$

So, using (35), similar to (38), we obtain equality (36) and inequality (37), too.

The following Grüss type inequality also holds.

Theorem 6 *Let $f : [0,1] \to \mathbb{R}$ be such that $f^{(n)}$ is absolutely continuous and $f^{(n+1)} \geq 0$ on $[0,1]$. Then representation* (36) *holds and the remainder $K_n^C(f)$ satisfies the bound*

$$\left|K_n^C(f)\right| \leq \frac{1}{30(n-1)!}\left\|G_{n-1}^C(s)\right\|_\infty \left\{\frac{f^{(n-1)}(0)+f^{(n-1)}(1)}{2} - \left[f^{(n-2)};0,1\right]\right\}. \tag{39}$$

Proof Applying Theorem 2 for $f \to G_n^C$, $g \to f^{(n)}$, we obtain

$$\begin{aligned} &\left|\int_0^1 G_n^C(s) f^{(n)}(s)\,\mathrm{d}s - \int_0^1 G_n^C(s)\,\mathrm{d}s \cdot \int_0^1 f^{(n)}(s)\,\mathrm{d}s\right| \\ &\quad \leq \frac{n}{2}\left\|G_{n-1}^C(s)\right\|_\infty \left(\int_0^1 s(1-s) f^{(n+1)}(s)\,\mathrm{d}s\right). \end{aligned} \tag{40}$$

So, similar to Theorem 4, using representation (36) and inequality (40), we deduce (39). □

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
All authors jointly worked on the results and they read and approved the final manuscript.

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Acknowledgements
This work has been fully supported by Croatian Science Foundation under the project 5435.

Received: 14 August 2014 Accepted: 18 February 2015 Published online: 03 March 2015

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