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On block diagonal-Schur complements of the block strictly doubly diagonally dominant matrices

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Abstract

As is well known, the diagonal-Schur complements of strictly diagonally dominant matrices are strictly diagonally dominant. In this paper, we verify the block diagonal-Schur complements of I-block strictly doubly diagonally dominant matrices are I-block strictly doubly diagonally dominant matrices, the same is true for II-block strictly doubly diagonally dominant matrices. The theoretical analysis is supported by numerical experiments.

Keywords: Schur complements; block diagonal-Schur complements; I-(II-) block strictly doubly diagonally dominant matrices; I-(II-) block strictly doubly diagonally dominant matrices; I-(II-) block *H*-matrix; comparison matrices; preconditioner

1 Introduction

The Schur complements and the diagonal-Schur complements have appeared to be useful tools in the study of matrix theory (see *e.g.*, [1–3]), linear control theory (see *e.g.*, [4]), numerical analysis (see *e.g.*, [5–9]) and statistics (see *e.g.*, [10, 11]), and so on. At the same time, given a matrix family, it is always interesting to see whether some important properties or structures of the family of matrices are inherited by the submatrices or by the matrices associated with the original matrices. These heritable properties have been used for the convergence of iterations in numerical analysis (see *e.g.*, [12]).

A great deal of classic works on the relationships of the Schur complements and the diagonal-Schur complements with the original matrices have been contributed, for a complete survey of these works we refer to (see *e.g.*, [12]). As is shown in [1, 2], the Schur complements of positive semidefinite matrices are positive semidefinite and the Schur complements of strictly diagonally dominant matrices are strictly diagonally dominant, the same is true for M-matrices, H-matrices, inverse M-matrices, strictly doubly diagonally dominant matrices and generalized strictly diagonally dominant. In addition, Liu and Huang [3] proposed that the diagonal-Schur complements of strictly diagonally dominant matrices are strictly diagonally dominant, the same is true for strictly γ-diagonally dominant matrices and strictly product γ-diagonally dominant matrices.

As regards the block matrix, the concept of a diagonally dominant matrix is extended and two kinds of block diagonally dominant matrices are proposed, *i.e.*, I-block diagonally dominant matrices [13] and II-block diagonally dominant matrices [13]. Later, on the ba-

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sis of previous works, two kinds of generalized block strictly diagonally dominant matrices are also presented, in other words, I-block H-matrices [14] and II-block H-matrices [15]. Based on the above results, it is easy to see that a block diagonally dominant matrix is not always a diagonally dominant matrix; an example is given in [16], (2.6). Subsequently, Zhang *et al.* [17] showed that the Schur complement of I-(generalized) block diagonally dominant matrix is I-(generalized) block diagonally dominant, the same is true for II-(generalized) block diagonally dominant matrix.

Let $C^{n\times n}$ be the set of all $n \times n$ complex matrices. Suppose $A \in C^{n\times n}$, $N = \{1, 2, \dots, n\}$, and $|\alpha|$ equals the cardinality of α. For nonempty index sets $\alpha, \beta \subseteq N$, we denote by $A(\alpha, \beta)$ the submatrix of $A \in C^{n\times n}$ lying in the rows indicated by α and the columns indicated by β. The submatrix $A(\alpha, \alpha)$ is abbreviated to $A(\alpha)$. Let x^T be the transpose of the vector x and I_n be the $n \times n$ unit matrix.

Let $A = (a_{ij}) \in C^{n\times n}$. An $n \times n$ matrix A is strictly diagonally dominant (abbreviated SD_n), if

$$|a_{ii}| > P_i(A), \quad P_i(A) = \sum_{j=1, j\neq i} |a_{ij}|, \quad i = 1, 2, \dots, n.$$

An $n \times n$ matrix A is strictly generalized diagonally dominant (abbreviated SGD_n), if there exists $D = \operatorname{diag}(d_1, \dots, d_n) > 0$, such that AD is strictly diagonally dominant.

An $n \times n$ matrix A is strictly doubly diagonally dominant (abbreviated SDD_n), if

$$|a_{ii}a_{jj}| > P_i(A)P_j(A), \quad \forall 1 \le i < j \le n.$$

An $n \times n$ matrix A is generalized strictly doubly diagonally dominant (abbreviated $\mathrm{SGDD}_n^{N_1,N_2}$), with $N_1 \cup N_2 = N$, $N_1 \cap N_2 = \emptyset$, and $\emptyset$ denoting the empty set, for all $i \in N_1$ and $j \in N_2$, if

$$\bigl(|a_{ii}| - \gamma_i\bigr)\bigl(|a_{jj}| - \delta_j\bigr) > \gamma_j\delta_i,$$

where

$$\gamma_s = \sum_{t\in N_1, t\neq s} |a_{st}|, \qquad \delta_s = \sum_{t\in N_2, t\neq s} |a_{st}|, \quad \text{with } s = i \text{ or } j.$$

Let $R^{n\times n}$ be the set of all $n \times n$ real matrices. For $A = (a_{ij}) \in R^{m\times n}$ and $B = (b_{ij}) \in R^{m\times n}$, we write $A \ge B$, if $a_{ij} \ge b_{ij}$ for all i, j. A real $n \times n$ matrix A is called an M-matrix if $A = sI - B$, where $s \ge 0$, $B \ge 0$, and $s > \rho(B)$, $\rho(B)$ is the spectral radius of B. Let M_n denote the set of $n \times n$ M-matrices.

Suppose $A \in C^{n\times n}$, the comparison matrix $\mu(A) = (\mu_{ij})$, is defined by

$$\mu_{ij} = \begin{cases} -|a_{ij}|, & i \neq j, \\ |a_{ij}|, & i = j. \end{cases}$$

A complex $n \times n$ matrix A is called an H-matrix if $\mu(A) \in M_n$. By H_n is denoted the set of $n \times n$ H-matrices.

In this paper, we propose the concept of the block diagonal-Schur complement on block matrices and study the properties on the diagonal-Schur complement of two kinds of (generalized) block doubly diagonally dominant matrices.

2 Definitions and lemmas

Consider an $n \times n$ complex matrix A. Let s $(1 \leq s \leq n)$ be an arbitrary natural number and A be partitioned into the following form:

$$A = \begin{pmatrix} A(\alpha_1,\alpha_1) & A(\alpha_1,\alpha_2) & \cdots & A(\alpha_1,\alpha_s) \\ A(\alpha_2,\alpha_1) & A(\alpha_2,\alpha_2) & \cdots & A(\alpha_2,\alpha_s) \\ \vdots & \vdots & \ddots & \vdots \\ A(\alpha_s,\alpha_1) & A(\alpha_s,\alpha_2) & \cdots & A(\alpha_s,\alpha_s) \end{pmatrix}, \tag{1}$$

where $\alpha_0 = \emptyset$ and

$$\alpha_i = \left\{ \sum_{t=0}^{i-1} |\alpha_t| + 1, \ldots, \sum_{t=0}^{i} |\alpha_t| \right\}, \quad i = 1,2,\ldots,s, \sum_{t=0}^{s} |\alpha_t| = n,$$

with $A(\alpha_t,\alpha_t)$ being a $|\alpha_t| \times |\alpha_t|$ nonsingular principal submatrix of A, $t = 1,2,\ldots,s$.

Let $C_s^{n\times n}$ be the set of all $s \times s$ block matrices in $C^{n\times n}$ partitioned as (1). Suppose $A = (A(\alpha_l,\alpha_m))_{s\times s} \in C_s^{n\times n}$ and let $N(A) = (\|A(\alpha_l,\alpha_m)\|)_{s\times s}$ denote the norm matrix of block matrix A.

Let $\alpha \subset N$, $\alpha^c = N - \alpha$, and $A(\alpha)$ be nonsingular. The Schur complement of $A(\alpha)$ in A is defined by

$$A/A(\alpha) = A/\alpha = A(\alpha^c) - A(\alpha^c,\alpha)[A(\alpha)]^{-1}A(\alpha,\alpha^c), \tag{2}$$

and the block diagonal-Schur complement of $A(\alpha)$ in A is defined by

$$A/_\circ A(\alpha) = A/_\circ\alpha = A(\alpha^c) - \{A(\alpha^c,\alpha)[A(\alpha)]^{-1}A(\alpha,\alpha^c)\} \circ E(\alpha), \tag{3}$$

where '$\circ$' denotes the Kronecker product symbol and

$$\alpha = \alpha_{i_1} \cup \alpha_{i_2} \cup \cdots \cup \alpha_{i_k}, \qquad \alpha^c = \alpha_{j_1} \cup \alpha_{j_2} \cup \cdots \cup \alpha_{j_l},$$
$$i_1 < i_2 < \cdots < i_k, j_1 < j_2 < \cdots < j_l, k + l = s,$$
$$E_{|\alpha|} = \begin{pmatrix} E_{|\alpha_1|} & & & \\ & E_{|\alpha_2|} & & \\ & & \ddots & \\ & & & |\alpha_s| \end{pmatrix}, \qquad E_{|\alpha_i|} = \begin{pmatrix} 1 & 1 & \ldots & 1 \\ 1 & 1 & \ldots & 1 \\ \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & \ldots & 1 \end{pmatrix}_{|\alpha_i|\times|\alpha_i|},$$

with $i = 1,2,\ldots,s$.

Definition 2.1 [13] Let $A = (A(\alpha_l,\alpha_m))_{s\times s} \in C_s^{n\times n}$ and $A(\alpha_l,\alpha_l)$ $(l = 1,2,\ldots,s)$ be nonsingular. If

$$\left\| [A(\alpha_l,\alpha_l)]^{-1} \right\|^{-1} > \sum_{\substack{m=1 \\ m\neq l}}^{s} \left\| A(\alpha_l,\alpha_m) \right\|, \quad \forall l \in s, \tag{4}$$

then A is an I-block strictly diagonally dominant matrix (abbreviated I-BSD$_s$).

Definition 2.2 [13] Let $A = (A(\alpha_l,\alpha_m))_{s\times s} \in C_s^{n\times n}$ and $A(\alpha_l,\alpha_l)$ $(l = 1,2,\dots,s)$ be nonsingular. If

$$\sum_{\substack{m=1\\ m\neq l}}^{s} \big\| \big[A(\alpha_l,\alpha_l)\big]^{-1} A(\alpha_l,\alpha_m)\big\| < 1, \quad \forall l \in s, \tag{5}$$

then A is an II-block strictly diagonally dominant matrix (abbreviated II-BSD$_s$).

Definition 2.3 [5] Let $A = (A(\alpha_l,\alpha_m))_{s\times s} \in C_s^{n\times n}$ and $A(\alpha_l,\alpha_l)$ $(l = 1,2,\dots,s)$ be nonsingular. For $1 \le i < j \le s$, if and only if

$$\big\| \big[A(\alpha_i,\alpha_i)\big]^{-1}\big\|^{-1} \big\| \big[A(\alpha_j,\alpha_j)\big]^{-1}\big\|^{-1} > \sum_{\substack{m=1\\ m\neq i}}^{s} \big\|A(\alpha_i,\alpha_m)\big\| \sum_{\substack{m=1\\ m\neq j}}^{s} \big\|A(\alpha_j,\alpha_m)\big\|, \tag{6}$$

then A is an I-block strictly doubly diagonally dominant matrix (abbreviated I-BSDD$_s$).

Definition 2.4 [5] Let $A = (A(\alpha_l,\alpha_m))_{s\times s} \in C_s^{n\times n}$ and $A(\alpha_l,\alpha_l)$ $(l = 1,2,\dots,s)$ be nonsingular. For $1 \le i < j \le s$, if and only if

$$\sum_{\substack{m=1\\ m\neq i}}^{s} \big\| \big[A(\alpha_i,\alpha_i)\big]^{-1} A(\alpha_i,\alpha_m)\big\| \sum_{\substack{m=1\\ m\neq j}}^{s} \big\| \big[A(\alpha_j,\alpha_j)\big]^{-1} A(\alpha_j,\alpha_m)\big\| < 1, \tag{7}$$

then A is an II-block strictly doubly diagonally dominant matrix (abbreviated II-BSDD$_s$).

Lemma 2.1 [5] *Let* $A = (A(\alpha_l,\alpha_m))_{s\times s} \in C_s^{(n\times n)}$ *be an* II-BSDD$_s$. *Then* $D^{-1}A$ *is an* I-BSDD$_s$, *where* $D = \operatorname{diag}(A(\alpha_1,\alpha_1), A(\alpha_2,\alpha_2),\dots,A(\alpha_s,\alpha_s))$.

Remark 2.1 [5] If A is an I-BSD$_s$ (or I-BSDD$_s$), according to the following inequality:

$$\big\|A(\alpha_l,\alpha_l)A(\alpha_l,\alpha_m)\big\| \le \big\|A(\alpha_l,\alpha_l)\big\| \big\|A(\alpha_l,\alpha_m)\big\|, \tag{8}$$

then A is an II-BSD$_s$ (or II-BSDD$_s$).

Definition 2.5 [14, 15] Let $A = (A(\alpha_l,\alpha_m))_{s\times s} \in C_s^{n\times n}$ and $A(\alpha_l,\alpha_l)$ $(l = 1,2,\dots,s)$ be nonsingular. If the comparison matrices of block matrix A, $\mu_I(A) = (\omega_{l,m}) \in R^{s\times s}$ or $\mu_{II}(A) = (\varpi_{l,m}) \in R^{s\times s}$ is an M-matrix, respectively, where

$$\omega_{lm} = \begin{cases} \|[A(\alpha_l,\alpha_l)]^{-1}\|^{-1}, & \text{if } l = m, \\ -\|A(\alpha_l,\alpha_m)\|, & \text{if } l \neq m, \end{cases}$$

$$\varpi_{lm} = \begin{cases} 1, & \text{if} l = m, \\ -\|[A(\alpha_l,\alpha_l)]^{-1}A(\alpha_l,\alpha_m)\|, & \text{if } l \neq m, \end{cases}$$

then A is called an I-block H-matrix or an II-block H-matrix.

Lemma 2.2 [13] *Let* $A = (A(\alpha_l,\alpha_m))_{s\times s} \in$ I-(II-) block H-matrix. *Then* A *is nonsingular.*

Remark 2.2 If A is an I-(II-)BSD$_s$ or an I-(II-)BSDD$_s$, by (4)-(7), then $\mu_I(A)$ (or $\mu_{II}(A)$) is an M-matrix. Further, by Definition 2.5 and Lemma 2.2, then A is a nonsingular I-(II-) block H-matrix.

Lemma 2.3 [15] *If* $A \in \mathrm{SD}_n, \mathrm{SDD}_n, \mathrm{SGD}_n$ *or* $\mathrm{SGDD}_n^{N_1,N_2}$. *Then* $\mu(A)$ *is an* M*-matrix, i.e.,* A *is an* H*-matrix.*

Lemma 2.4 [1] *Let* $A \in C^{n\times n}$. *If* $\|A\| < 1$, *then* $I_n - A$ *is nonsingular and*

$$\left\|(I_n - A)^{-1}\right\| \le \frac{1}{1-\|A\|},$$

where I_n *denotes the* $n \times n$ *identity matrix.*

Lemma 2.5 [18] *Let* $A = (A(\alpha_l,\alpha_m))_{s\times s} \in$ I-(II-)BSD$_s$. *For all* $t = 1,2,\ldots,l$, *then*

$$\psi_t = 1 - \left\| \left[A(\alpha_{j_t},\alpha_{j_t})\right]^{-1}\left(A(\alpha_{j_t},\alpha_{i_1}),\ldots,A(\alpha_{j_t},\alpha_{i_k})\right)\left[A(\alpha)\right]^{-1}\begin{pmatrix} A(\alpha_{i_1},\alpha_{j_t}) \\ \vdots \\ A(\alpha_{i_k},\alpha_{j_t})\end{pmatrix}\right\| > 0.$$

3 On block diagonal-Schur complement of I-(II-)BSDD$_s$

In this section, to verify the heritable properties of the block diagonal-Schur complements from the original matrix I-BSDD$_s$ and I-BSDD$_s$, we only need to consider two cases as follows:

(1) If A is an I-BSDD$_s$ but is not an I-BSD$_s$, by Definition 2.3, there exists one and only one index i_0, $A(\alpha_{i_0},\alpha_{i_0})$ being nonsingular and such that

$$\left\|\left[A(\alpha_{i_0},\alpha_{i_0})\right]^{-1}\right\|^{-1} \le P_{i_0}(A). \tag{9}$$

(2) If A is an II-BSDD$_s$ but not an II-BSD$_s$, by Definition 2.4, there exists one and only one index i_0, $A(\alpha_{i_0},\alpha_{i_0})$ being nonsingular and satisfying

$$\sum_{r=1}^{s}\left\|\left[A(\alpha_{i_0},\alpha_{i_0})\right]^{-1}A(\alpha_{i_0},\alpha_r)\right\| \ge 1. \tag{10}$$

Theorem 3.1 *Let* A *be an* $n \times n$ I-BSDD$_s$ *but be not an* $n\times n$ I-BSD$_s$, *and* i_0 $(1 \le i_0 \le s)$ *satisfy the condition in* (9). *For any index set* $\alpha \subseteq N$, *writing* $\alpha = \alpha_{i_1}\cup\alpha_{i_2}\cup\cdots\cup\alpha_{i_k}$ *and* $\alpha^c = \alpha_{j_1}\cup\alpha_{j_2}\cup\cdots\cup\alpha_{j_l}$, *with* $k + l = s$, *then*:

(i) *If* $\alpha_{i_0} \subseteq \alpha$, *then* $A/_\circ\alpha \in$ I-BSD$_l$.

(ii) *If* $\alpha_{i_0} \subseteq \alpha^c$, *then* $A/_\circ\alpha \in$ I-BSDD$_l$.

Proof Without loss of generality, we can assume $A/_\circ\alpha = (\tilde{A}(\alpha_t,\alpha_r))$ and denote

$$\Phi_\omega = \left(A(\alpha_{j_\omega},\alpha_{i_1}),\ldots,A(\alpha_{j_\omega},\alpha_{i_k})\right)\left[A(\alpha)\right]^{-1}\begin{pmatrix} A(\alpha_{i_1},\alpha_{j_\omega}) \\ \vdots \\ A(\alpha_{i_k},\alpha_{j_\omega})\end{pmatrix},$$

$$K_\omega = \left(\left\|A(\alpha_{j_\omega},\alpha_{i_1})\right\|,\ldots,\left\|A(\alpha_{j_\omega},\alpha_{i_k})\right\|\right), \qquad H_\upsilon = \left(\left\|A(\alpha_{i_1},\alpha_{j_\upsilon})\right\|,\ldots,\left\|A(\alpha_{i_k},\alpha_{j_\upsilon})\right\|\right)^T,$$

$$\Psi_\omega = K_\omega\cdot\mu_I\left\{\left[A(\alpha)\right]^{-1}\right\}\cdot H_\omega, \qquad \Upsilon_{\omega,\upsilon} = K_\omega\cdot\left\{\mu_I\left[A(\alpha)\right]\right\}^{-1}\cdot H_\upsilon, \quad \text{with } \omega,\upsilon = t,u.$$

By the definition of the block diagonal-Schur complement (3), denote $|\alpha_{j_t}| = J_t$. According to Remark 2.2, we obtain $\|[A(\alpha_{j_\omega},\alpha_{j_\omega})]^{-1}\Phi_\omega\| < 1$, $\omega = t, u$. Further, consider the following two cases:

(i) If $\alpha_{i_0} \subseteq \alpha$, then

$$\begin{aligned}
&\big\|\big[\tilde{A}(\alpha_t,\alpha_t)\big]^{-1}\big\|^{-1} - \sum_{\substack{r=1\\ r\neq t}}^{l}\big\|\tilde{A}(\alpha_t,\alpha_r)\big\| \\
&= \big\|\big\{A(\alpha_{j_t},\alpha_{j_t}) - \Phi_t\big\}^{-1}\big\|^{-1} - \sum_{\substack{r=1\\ r\neq t}}^{l}\big\|A(\alpha_{j_t},\alpha_{j_r})\big\| \\
&= \big\|\big\{I_{J_t} - \big[A(\alpha_{j_t},\alpha_{j_t})\big]^{-1}\Phi_t\big\}^{-1}\big[A(\alpha_{j_t},\alpha_{j_t})\big]^{-1}\big\|^{-1} - \sum_{\substack{r=1\\ r\neq t}}^{l}\big\|A(\alpha_{j_t},\alpha_{j_r})\big\| \\
&\geq \big\|\big\{I_{J_t} - \big[A(\alpha_{j_t},\alpha_{j_t})\big]^{-1}\Phi_t\big\}^{-1}\big\|^{-1}\big\|\big[A(\alpha_{j_t},\alpha_{j_t})\big]^{-1}\big\|^{-1} - \sum_{\substack{r=1\\ r\neq t}}^{l}\big\|A(\alpha_{j_t},\alpha_{j_r})\big\| \\
&\geq \big\{1 - \big\|\big[A(\alpha_{j_t},\alpha_{j_t})\big]^{-1}\Phi_t\big\|\big\}\big\|\big[A(\alpha_{j_t},\alpha_{j_t})\big]^{-1}\big\|^{-1} - \sum_{\substack{r=1\\ r\neq t}}^{l}\big\|A(\alpha_{j_t},\alpha_{j_r})\big\| \\
&= \big\|\big[A(\alpha_{j_t},\alpha_{j_t})\big]^{-1}\big\|^{-1} - \big\|\big[A(\alpha_{j_t},\alpha_{j_t})\big]^{-1}\big\|^{-1}\big\|\big[A(\alpha_{j_t},\alpha_{j_t})\big]^{-1}\Phi_t\big\| - \sum_{\substack{r=1\\ r\neq t}}^{l}\big\|A(\alpha_{j_t},\alpha_{j_r})\big\| \\
&\geq \big\|\big[A(\alpha_{j_t},\alpha_{j_t})\big]^{-1}\big\|^{-1} - \big\|\big[A(\alpha_{j_t},\alpha_{j_t})\big]^{-1}\big\|^{-1}\big\|\big[A(\alpha_{j_t},\alpha_{j_t})\big]^{-1}\big\|\|\Phi_t\| - \sum_{\substack{r=1\\ r\neq t}}^{l}\big\|A(\alpha_{j_t},\alpha_{j_r})\big\| \\
&\geq \big\|\big[A(\alpha_{j_t},\alpha_{j_t})\big]^{-1}\big\|^{-1} - \sum_{\substack{r=1\\ r\neq t}}^{l}\big\|A(\alpha_{j_t},\alpha_{j_r})\big\| - \Psi_\omega \\
&\geq \big\|\big[A(\alpha_{j_t},\alpha_{j_t})\big]^{-1}\big\|^{-1} - \sum_{\substack{r=1\\ r\neq t}}^{l}\big\|A(\alpha_{j_t},\alpha_{j_r})\big\| - \Upsilon_{\omega,\omega} \\
&\triangleq \frac{\det B_1}{\det[\mu_I(A(\alpha))]},
\end{aligned}$$

where

$$B_1 = \begin{pmatrix} \|[A(\alpha_{j_t},\alpha_{j_t})]^{-1}\|^{-1} - \sum_{\substack{r=1\\ r\neq t}}^{l}\|A(\alpha_{j_t},\alpha_{j_r})\| & -K_t \\ -H_t & \mu_I[A(\alpha)] \end{pmatrix}.$$

Since A is an I-BSDD$_s$, $\alpha_{i_0} \subseteq \alpha$, and $\forall \alpha_{j_t} \subseteq \alpha^c$,

$$\big\|\big[A(\alpha_{j_t},\alpha_{j_t})\big]^{-1}\big\|^{-1} - \sum_{\substack{r=1\\ r\neq t}}^{l}\big\|A(\alpha_{j_t},\alpha_{j_r})\big\| > \sum_{r=1}^{k}\big\|A(\alpha_{j_t},\alpha_{i_r})\big\|.$$

For $\forall\alpha_{i_x} \subseteq \alpha$, $x = 1, 2, \ldots, k$, if $i_x \neq i_0$, then

$$\big\|\big[A(\alpha_{i_x}, \alpha_{i_x})\big]^{-1}\big\|^{-1} > \sum_{\substack{r=1\\ r\neq i_x}}^{s} \big\|A(\alpha_{i_x}, \alpha_r)\big\| \geq \sum_{\substack{r=1\\ r\neq x}}^{k} \big\|A(\alpha_{i_x}, \alpha_{i_r})\big\| + \big\|A(\alpha_{i_x}, \alpha_{j_t})\big\|,$$

if $i_x = i_0$, by Definition 2.3 and the inequality (9), we have

$$\begin{aligned}
&\left\{\big\|\big[A(\alpha_{j_t}, \alpha_{j_t})\big]^{-1}\big\|^{-1} - \sum_{\substack{r=1\\ r\neq t}}^{l} \big\|A(\alpha_{j_t}, \alpha_{j_r})\big\|\right\} \big\|\big[A(\alpha_{i_0}, \alpha_{i_0})\big]^{-1}\big\|^{-1} \\
&= \big\|\big[A(\alpha_{j_t}, \alpha_{j_t})\big]^{-1}\big\|^{-1} \big\|\big[A(\alpha_{i_0}, \alpha_{i_0})\big]^{-1}\big\|^{-1} - \big\|\big[A(\alpha_{i_0}, \alpha_{i_0})\big]^{-1}\big\|^{-1} \sum_{\substack{r=1\\ r\neq t}}^{l} \big\|A(\alpha_{j_t}, \alpha_{j_r})\big\| \\
&> \sum_{\substack{r=1\\ r\neq j_t}}^{s} \big\|A(\alpha_{j_t}, \alpha_r)\big\| \sum_{\substack{r=1\\ r\neq i_0}}^{s} \big\|A(\alpha_{i_0}, \alpha_r)\big\| - \sum_{\substack{r=1\\ r\neq i_0}}^{s} \big\|A(\alpha_{i_0}, \alpha_r)\big\| \sum_{\substack{r=1\\ r\neq t}}^{l} \big\|A(\alpha_{j_t}, \alpha_{j_r})\big\| \\
&= \left\{\sum_{\substack{r=1\\ r\neq j_t}}^{s} \big\|A(\alpha_{j_t}, \alpha_r)\big\| - \sum_{\substack{r=1\\ r\neq t}}^{l} \big\|A(\alpha_{j_t}, \alpha_{j_r})\big\|\right\} \sum_{\substack{r=1\\ r\neq i_0}}^{s} \big\|A(\alpha_{i_0}, \alpha_r)\big\| \\
&\geq \sum_{r=1}^{k} \big\|A(\alpha_{j_t}, \alpha_{i_r})\big\| \left\{\sum_{\substack{r=1\\ i_r\neq i_0}}^{k} \big\|A(\alpha_{i_0}, \alpha_{i_r})\big\| + \big\|A(\alpha_{i_0}, \alpha_{j_t})\big\|\right\}.
\end{aligned}$$

Thus, $B_1 \in \mathrm{SDD}_{k+1}$. Further, by Lemma 2.3, we have $B_1 = \mu(B_1) \in M_{k+1}$ and $\mu_I[A(\alpha)] \in M_k$. Therefore, $\det(B_1) > 0$ and $\det[\mu_I(A(\alpha))] > 0$, *i.e.*,

$$A/_\circ\alpha \in \text{I-BSD}_l.$$

(ii) If $\alpha_{i_0} \subseteq \alpha^c$, for $\forall t, u = 1, 2, \ldots, l$ and $t \neq u$, we have

$$\begin{aligned}
&\big\|\big[\tilde{A}(\alpha_t, \alpha_t)\big]^{-1}\big\|^{-1} \big\|\big[\tilde{A}(\alpha_u, \alpha_u)\big]^{-1}\big\|^{-1} - \sum_{\substack{r=1\\ r\neq t}}^{l} \big\|\tilde{A}(\alpha_t, \alpha_r)\big\| \sum_{\substack{r=1\\ r\neq u}}^{l} \big\|\tilde{A}(\alpha_u, \alpha_r)\big\| \\
&= \big\|\big\{A(\alpha_{j_t}, \alpha_{j_t}) - \Phi_t\big\}^{-1}\big\|^{-1} \big\|\big\{A(\alpha_{j_u}, \alpha_{j_u}) - \Phi_u\big\}^{-1}\big\|^{-1} \\
&\quad - \sum_{\substack{r=1\\ r\neq t}}^{l} \big\|A(\alpha_{j_t}, \alpha_{j_r})\big\| \sum_{\substack{r=1\\ r\neq u}}^{l} \big\|A(\alpha_{j_u}, \alpha_{j_r})\big\| \\
&= \big\|\big\{I_{J_t} - \big[A(\alpha_{j_t}, \alpha_{j_t})\big]^{-1}\Phi_t\big\}^{-1}\big[A(\alpha_{j_t}, \alpha_{j_t})\big]^{-1}\big\|^{-1} \\
&\quad \times \big\|\big\{I_{J_u} - \big[A(\alpha_{j_u}, \alpha_{j_u})\big]^{-1}\Phi_u\big\}^{-1}\big[A(\alpha_{j_u}, \alpha_{j_u})\big]^{-1}\big\|^{-1} \\
&\quad - \sum_{\substack{r=1\\ r\neq t}}^{l} \big\|A(\alpha_{j_t}, \alpha_{j_r})\big\| \sum_{\substack{r=1\\ r\neq u}}^{l} \big\|A(\alpha_{j_u}, \alpha_{j_r})\big\| \\
&\geq \big\|\big\{I_{J_t} - \big[A(\alpha_{j_t}, \alpha_{j_t})\big]^{-1}\Phi_t\big\}^{-1}\big\|^{-1} \big\|\big\{I_{J_u} - \big[A(\alpha_{j_u}, \alpha_{j_u})\big]^{-1}\Phi_u\big\}^{-1}\big\|^{-1}
\end{aligned}$$

$$\begin{aligned}
&\times \big\|\big[A(\alpha_{j_t},\alpha_{j_t})\big]^{-1}\big\|^{-1}\big\|\big[A(\alpha_{j_u},\alpha_{j_u})\big]^{-1}\big\|^{-1} - \sum_{\substack{r=1\\ r\neq t}}^{l}\big\|A(\alpha_{j_t},\alpha_{j_r})\big\|\sum_{\substack{r=1\\ r\neq u}}^{l}\big\|A(\alpha_{j_u},\alpha_{j_r})\big\| \\
&\geq \big\{1-\big\|\big[A(\alpha_{j_t},\alpha_{j_t})\big]^{-1}\Phi_t\big\|\big\}\big\{1-\big\|\big[A(\alpha_{j_u},\alpha_{j_u})\big]^{-1}\Phi_u\big\|\big\} \\
&\quad\times \big\|\big[A(\alpha_{j_t},\alpha_{j_t})\big]^{-1}\big\|^{-1}\big\|\big[A(\alpha_{j_u},\alpha_{j_u})\big]^{-1}\big\|^{-1} - \sum_{\substack{r=1\\ r\neq t}}^{l}\big\|A(\alpha_{j_t},\alpha_{j_r})\big\|\sum_{\substack{r=1\\ r\neq u}}^{l}\big\|A(\alpha_{j_u},\alpha_{j_r})\big\| \\
&\geq \big\{1-\big\|\big[A(\alpha_{j_t},\alpha_{j_t})\big]^{-1}\big\|\|\Phi_t\|\big\}\big\{1-\big\|\big[A(\alpha_{j_u},\alpha_{j_u})\big]^{-1}\big\|\|\Phi_u\|\big\} \\
&\quad\times \big\|\big[A(\alpha_{j_t},\alpha_{j_t})\big]^{-1}\big\|^{-1}\big\|\big[A(\alpha_{j_u},\alpha_{j_u})\big]^{-1}\big\|^{-1} - \sum_{\substack{r=1\\ r\neq t}}^{l}\big\|A(\alpha_{j_t},\alpha_{j_r})\big\|\sum_{\substack{r=1\\ r\neq u}}^{l}\big\|A(\alpha_{j_u},\alpha_{j_r})\big\| \\
&= \big\{\big\|\big[A(\alpha_{j_t},\alpha_{j_t})\big]^{-1}\big\|^{-1}-\|\Phi_t\|\big\}\big\{\big\|\big[A(\alpha_{j_u},\alpha_{j_u})\big]^{-1}\big\|^{-1}-\|\Phi_u\|\big\} \\
&\quad - \sum_{\substack{r=1\\ r\neq t}}^{l}\big\|A(\alpha_{j_t},\alpha_{j_r})\big\|\sum_{\substack{r=1\\ r\neq u}}^{l}\big\|A(\alpha_{j_u},\alpha_{j_r})\big\| \\
&\geq \big\{\big\|\big[A(\alpha_{j_t},\alpha_{j_t})\big]^{-1}\big\|^{-1}-\Psi_t\big\}\big\{\big\|\big[A(\alpha_{j_u},\alpha_{j_u})\big]^{-1}\big\|^{-1}-\Psi_u\big\} \\
&\quad - \sum_{\substack{r=1\\ r\neq t}}^{l}\big\|A(\alpha_{j_t},\alpha_{j_r})\big\|\sum_{\substack{r=1\\ r\neq u}}^{l}\big\|A(\alpha_{j_u},\alpha_{j_r})\big\| \\
&\geq \big\{\big\|\big[A(\alpha_{j_t},\alpha_{j_t})\big]^{-1}\big\|^{-1}-\Upsilon_{tt}\big\}\big\{\big\|\big[A(\alpha_{j_u},\alpha_{j_u})\big]^{-1}\big\|^{-1}-\Upsilon_{uu}\big\} \\
&\quad - \sum_{\substack{r=1\\ r\neq t}}^{l}\big\|A(\alpha_{j_t},\alpha_{j_r})\big\|\sum_{\substack{r=1\\ r\neq u}}^{l}\big\|A(\alpha_{j_u},\alpha_{j_r})\big\| \\
&\geq \big\{\big\|\big[A(\alpha_{j_t},\alpha_{j_t})\big]^{-1}\big\|^{-1}-\Upsilon_{tt}\big\}\big\{\big\|\big[A(\alpha_{j_u},\alpha_{j_u})\big]^{-1}\big\|^{-1}-\Upsilon_{uu}\big\} \\
&\quad - \left\{\sum_{\substack{r=1\\ r\neq t}}^{l}\big\|A(\alpha_{j_t},\alpha_{j_r})\big\|+\Upsilon_{tu}\right\}\left\{\sum_{\substack{r=1\\ r\neq u}}^{l}\big\|A(\alpha_{j_u},\alpha_{j_r})\big\|+\Upsilon_{ut}\right\} \\
&\triangleq \frac{\det B_2}{\det[\mu_I(A)(\alpha)]},
\end{aligned}$$

where

$$B_2 = \begin{pmatrix} \|[A(\alpha_{j_t},\alpha_{j_t})]^{-1}\|^{-1} & -\sum_{\substack{r=1\\ r\neq t}}^{l}\|A(\alpha_{j_t},\alpha_{j_r})\| & -K_t \\ -\sum_{\substack{r=1\\ r\neq u}}^{l}\|A(\alpha_{j_u},\alpha_{j_r})\| & \|[A(\alpha_{j_u},\alpha_{j_u})]^{-1}\|^{-1} & -K_u \\ -H_t & -H_u & \mu_I(A)(\alpha) \end{pmatrix}.$$

Since A is an I-BSDD$_s$, $\alpha_{i_0} \subseteq \alpha^c$ and $\alpha_{j_\omega} \subseteq \alpha^c$, $\omega = t, u$,

$$\big\|\big[A(\alpha_{j_t},\alpha_{j_t})\big]^{-1}\big\|^{-1}\big\|\big[A(\alpha_{j_u},\alpha_{j_u})\big]^{-1}\big\|^{-1} > \sum_{\substack{r=1\\ r\neq j_t}}^{s}\big\|A(\alpha_{j_t},\alpha_r)\big\|\sum_{\substack{r=1\\ r\neq j_u}}^{s}\big\|A(\alpha_{j_u},\alpha_r)\big\|.$$

For $\forall\alpha_{i_x} \subseteq \alpha$, $x = 1, 2, \ldots, k$, and $\alpha_{j_w} \subseteq \alpha^c$, $w = t, u$,

$$\begin{aligned}&\left\|\left[A(\alpha_{i_x},\alpha_{i_x})\right]^{-1}\right\|^{-1}\left\|\left[A(\alpha_{j_\omega},\alpha_{j_\omega})\right]^{-1}\right\|^{-1}\\&\quad>\left\{\sum_{\substack{r=1\\r\neq x}}^{k}\left\|A(\alpha_{i_x},\alpha_{i_r})\right\|+\left\|A(\alpha_{i_x},\alpha_{j_t})\right\|+\left\|A(\alpha_{i_x},\alpha_{j_u})\right\|\right\}\sum_{\substack{r=1\\r\neq j_\omega}}^{s}\left\|A(\alpha_{j_\omega},\alpha_r)\right\|\end{aligned}$$

and

$$\left\|\left[A(\alpha_{i_x},\alpha_{i_x})\right]^{-1}\right\|^{-1}>\left\{\sum_{\substack{r=1\\r\neq x}}^{k}\left\|A(\alpha_{i_x},\alpha_{i_r})\right\|+\left\|A(\alpha_{i_x},\alpha_{j_u})\right\|+\left\|A(\alpha_{i_x},\alpha_{j_t})\right\|\right\}.$$

Thus, $B_2 \in \mathrm{SDD}_{k+2}$. Further, by Lemma 2.3, we have $B_2 = \mu(B_2) \in M_{k+2}$ and $\mu_I[A(\alpha)] \in M_k$. Therefore, $\det(B_2) > 0$ and $\det[\mu_I(A)(\alpha)] > 0$, *i.e.*,

$$A/_\circ\alpha \in \text{I-BSDD}_l.$$

Combining the proof of (i) and (ii), we complete the proof of Theorem 3.1. □

Theorem 3.2 *Let A be an $n \times n$ II-BSDD$_s$ but be not an $n \times n$ II-BSD$_s$, and i_0 $(1 \le i_0 \le s)$ satisfy the condition in* (10). *For any index set $\alpha \subseteq N$, writing $\alpha = \alpha_{i_1} \cup \alpha_{i_2} \cup \cdots \cup \alpha_{i_k}$ and $\alpha^c = \alpha_{j_1} \cup \alpha_{j_2} \cup \cdots \cup \alpha_{j_l}$, with $k + l = s$, then*:

(i) *If $\alpha_{i_0} \subseteq \alpha$, then $A/_\circ\alpha \in$ II-BSD$_l$,*

(ii) *If $\alpha_{i_0} \subseteq \alpha^c$, then $A/_\circ\alpha \in$ II-BSDD$_l$.*

Proof Without loss of generality, we can assume $A/_\circ\alpha = (\tilde{A}(\alpha_t, \alpha_r))$, and we denote

$$\begin{aligned}&\Phi_\omega=\left(A(\alpha_{j_\omega},\alpha_{i_1}),\ldots,A(\alpha_{j_\omega},\alpha_{i_k})\right)\left[A(\alpha)\right]^{-1}\begin{pmatrix}A(\alpha_{i_1},\alpha_{j_\omega})\\\vdots\\A(\alpha_{i_k},\alpha_{j_\omega})\end{pmatrix},\\&K_\omega=\left(\left\|\left[A(\alpha_{j_\omega},\alpha_{i_1})\right]^{-1}A(\alpha_{j_\omega},\alpha_{i_1})\right\|,\ldots,\left\|\left[A(\alpha_{j_\omega},\alpha_{i_1})\right]^{-1}A(\alpha_{j_\omega},\alpha_{i_k})\right\|\right),\\&H_\omega=\left(\left\|\left[A(\alpha_{i_1},\alpha_{i_1})\right]^{-1}A(\alpha_{i_1},\alpha_{j_\omega})\right\|,\ldots,\left\|\left[A(\alpha_{i_1},\alpha_{i_1})\right]^{-1}A(\alpha_{i_k},\alpha_{j_\omega})\right\|\right)^T,\\&D_1=\operatorname{diag}\left(A(\alpha_{i_1},\alpha_{i_1}),\ldots,A(\alpha_{i_k},\alpha_{i_k})\right),\\&\Psi_\omega=K_\omega\cdot\mu_{II}\left\{\left[A(\alpha)\right]^{-1}D_1\right\}\cdot H_\omega,\qquad\Upsilon_{\omega,\upsilon}=K_\omega\cdot\left\{\mu_{II}\left[A(\alpha)\right]D_1^{-1}\right\}^{-1}\cdot H_\upsilon,\\&\quad\text{with }\omega,\upsilon=t,u.\end{aligned}$$

(i) If $\alpha_{i_0} \subseteq \alpha$, according to the proof of Theorem 3.1(i), for $\forall t \in \alpha^c$, we obtain

$$\begin{aligned}&1-\sum_{\substack{r=1\\r\neq t}}^{l}\left\|\left[\tilde{A}(\alpha_t,\alpha_t)\right]^{-1}\tilde{A}(\alpha_t,\alpha_r)\right\|\\&\quad=1-\sum_{\substack{r=1\\r\neq t}}^{l}\left\|\left[A(\alpha_{j_t},\alpha_{j_t})-\Phi_t\right]^{-1}A(\alpha_{j_t},\alpha_{j_r})\right\|\end{aligned}$$

$$= 1 - \sum_{\substack{r=1 \\ r\neq t}}^{l} \left\| \left\{ I_{J_t} - \left[A(\alpha_{j_t}, \alpha_{j_t}) \right]^{-1} \Phi_t \right\}^{-1} \left[A(\alpha_{j_t}, \alpha_{j_t}) \right]^{-1} A(\alpha_{j_t}, \alpha_{j_r}) \right\|$$

$$\geq 1 - \sum_{\substack{r=1 \\ r\neq t}}^{l} \left\| \left\{ I_{J_t} - \left[A(\alpha_{j_t}, \alpha_{j_t}) \right]^{-1} \Phi_t \right\}^{-1} \right\| \left\| \left[A(\alpha_{j_t}, \alpha_{j_t}) \right]^{-1} A(\alpha_{j_t}, \alpha_{j_r}) \right\|$$

$$\geq 1 - \sum_{\substack{r=1 \\ r\neq t}}^{l} \left\{ 1 - \left\| \left[A(\alpha_{j_t}, \alpha_{j_t}) \right]^{-1} \Phi_t \right\| \right\}^{-1} \left\| \left[A(\alpha_{j_t}, \alpha_{j_t}) \right]^{-1} A(\alpha_{j_t}, \alpha_{j_r}) \right\|$$

$$= \left\{ 1 - \left\| \left[A(\alpha_{j_t}, \alpha_{j_t}) \right]^{-1} \Phi_t \right\| \right\}^{-1} \times \left\{ 1 - \left\| \left[A(\alpha_{j_t}, \alpha_{j_t}) \right]^{-1} \Phi_t \right\| - \sum_{\substack{r=1 \\ r\neq t}}^{l} \left\| \left[A(\alpha_{j_t}, \alpha_{j_t}) \right]^{-1} A(\alpha_{j_t}, \alpha_{j_r}) \right\| \right\}$$

$$\geq \left\{ 1 - \left\| \left[A(\alpha_{j_t}, \alpha_{j_t}) \right]^{-1} \Phi_t \right\| \right\}^{-1} \left\{ 1 - \Upsilon_{tt} - \sum_{\substack{r=1 \\ r\neq t}}^{l} \left\| \left[A(\alpha_{j_t}, \alpha_{j_t}) \right]^{-1} A(\alpha_{j_t}, \alpha_{j_r}) \right\| \right\}$$

$$\triangleq \frac{\det B_3}{(1 - \| [A(\alpha_{j_t}, \alpha_{j_t})]^{-1} \Phi_t \|) \det[\mu_{II}(A)(\alpha)]},$$

where

$$B_3 = \begin{pmatrix} 1 - \sum_{\substack{r=1 \\ r\neq t}}^{l} \| [A(\alpha_{j_t}, \alpha_{j_t})]^{-1} A(\alpha_{j_t}, \alpha_{j_r}) \| & -K_t \\ -H_t & \mu_{II}(A)(\alpha) \end{pmatrix}.$$

Since A is an II-BSDD$_s$, $\alpha_{i_0} \subseteq \alpha$, and $\alpha_{j_t} \subseteq \alpha^c$,

$$1 - \sum_{\substack{r=1 \\ r\neq t}}^{l} \left\| \left[A(\alpha_{j_t}, \alpha_{j_t}) \right]^{-1} A(\alpha_{j_t}, \alpha_{j_r}) \right\| > \sum_{r=1}^{k} \left\| \left[A(\alpha_{j_t}, \alpha_{j_t}) \right]^{-1} A(\alpha_{j_t}, \alpha_{i_r}) \right\|.$$

For $\forall \alpha_{i_x} \subseteq \alpha$, $x = 1, 2, \ldots, k$, if $i_x \neq i_0$, then

$$\left\| \left[A(\alpha_{i_x}, \alpha_{i_x}) \right]^{-1} A(\alpha_{i_x}, \alpha_{j_t}) \right\| + \sum_{\substack{r=1 \\ r\neq x}}^{k} \left\| \left[A(\alpha_{i_x}, \alpha_{i_x}) \right]^{-1} A(\alpha_{i_x}, \alpha_{i_r}) \right\|$$

$$\leq \sum_{\substack{r=1 \\ r\neq i_x}}^{s} \left\| \left[A(\alpha_{i_x}, \alpha_{i_x}) \right]^{-1} A(\alpha_{i_x}, \alpha_r) \right\| < 1,$$

if $i_x = i_0$, by Definition 2.4 and the inequality (10), we have

$$1 - \sum_{\substack{r=1 \\ r\neq t}}^{l} \left\| \left[A(\alpha_{j_t}, \alpha_{j_t}) \right]^{-1} A(\alpha_{j_t}, \alpha_{j_r}) \right\|$$

$$\geq 1 - \sum_{\substack{r=1 \\ r\neq t}}^{l} \left\| \left[A(\alpha_{j_t}, \alpha_{j_t}) \right]^{-1} A(\alpha_{j_t}, \alpha_{j_r}) \right\| \sum_{\substack{r=1 \\ r\neq i_0}}^{s} \left\| \left[A(\alpha_{i_0}, \alpha_{i_0}) \right]^{-1} A(\alpha_{i_0}, \alpha_r) \right\|$$

$$> \sum_{r=1}^{k} \| [A(\alpha_{j_t}, \alpha_{j_t})]^{-1} A(\alpha_{j_t}, \alpha_{j_r}) \| \sum_{\substack{r=1 \\ r \neq i_0}}^{s} \| [A(\alpha_{i_0}, \alpha_{i_0})]^{-1} A(\alpha_{i_0}, \alpha_r) \|$$

$$\geq \sum_{r=1}^{k} \| [A(\alpha_{j_t}, \alpha_{j_t})]^{-1} A(\alpha_{j_t}, \alpha_{j_r}) \|$$

$$\times \left\{ \| [A(\alpha_{i_0}, \alpha_{i_0})]^{-1} A(\alpha_{i_0}, \alpha_{j_t}) \| + \sum_{\substack{r=1 \\ i_r \neq i_0}}^{k} \| [A(\alpha_{i_0}, \alpha_{i_0})]^{-1} A(\alpha_{i_0}, \alpha_{i_r}) \| \right\}.$$

Thus, $B_3 \in \mathrm{SDD}_{k+1}$. Further, by Lemma 2.3, we obtain $B_3 = \mu(B_1) \in M_{k+1}$ and $\mu_{II}[A(\alpha)] \in M_k$. Therefore, $\det(B_3) > 0$ and $\det[\mu_{II}(A(\alpha))] > 0$, *i.e.*,

$$A/_\circ\alpha \in \text{II-BSD}_l.$$

(ii) If $\alpha_{i_0} \subseteq \alpha^c$, for $\forall t, u \in \alpha^c$, with $t, u = 1, 2, \ldots, l$, $t \neq u$, we obtain

$$1 - \sum_{\substack{r=1 \\ r \neq t}}^{l} \| [\tilde{A}(\alpha_t, \alpha_t)]^{-1} \tilde{A}(\alpha_t, \alpha_r) \| \sum_{\substack{r=1 \\ r \neq u}}^{l} \| [\tilde{A}(\alpha_u, \alpha_u)]^{-1} \tilde{A}(\alpha_u, \alpha_r) \|$$

$$= 1 - \sum_{\substack{r=1 \\ r \neq t}}^{l} \| [A(\alpha_{j_t}, \alpha_{j_t}) - \Phi_t]^{-1} A(\alpha_{j_t}, \alpha_{j_r}) \| \sum_{\substack{r=1 \\ r \neq u}}^{l} \| [A(\alpha_{j_u}, \alpha_{j_u}) - \Phi_u]^{-1} A(\alpha_{j_u}, \alpha_{j_r}) \|$$

$$= 1 - \sum_{\substack{r=1 \\ r \neq t}}^{l} \| \{ I_{J_t} - [A(\alpha_{j_t}, \alpha_{j_t})]^{-1} \Phi_t \}^{-1} [A(\alpha_{j_t}, \alpha_{j_t})]^{-1} A(\alpha_{j_t}, \alpha_{j_r}) \|$$

$$\times \sum_{\substack{r=1 \\ r \neq u}}^{l} \| \{ I_{J_u} - [A(\alpha_{j_u}, \alpha_{j_u})]^{-1} \Phi_u \}^{-1} [A(\alpha_{j_u}, \alpha_{j_u})]^{-1} A(\alpha_{j_u}, \alpha_{j_r}) \|$$

$$\geq 1 - \sum_{\substack{r=1 \\ r \neq t}}^{l} \| \{ I_{J_t} - [A(\alpha_{j_t}, \alpha_{j_t})]^{-1} \Phi_t \}^{-1} \| \| [A(\alpha_{j_t}, \alpha_{j_t})]^{-1} A(\alpha_{j_t}, \alpha_{j_r}) \|$$

$$\times \sum_{\substack{r=1 \\ r \neq t}}^{l} \| \{ I_{J_u} - [A(\alpha_{j_u}, \alpha_{j_u})]^{-1} \Phi_u \}^{-1} \| \| [A(\alpha_{j_u}, \alpha_{j_u})]^{-1} A(\alpha_{j_u}, \alpha_{j_r}) \|$$

$$\geq 1 - \sum_{\substack{r=1 \\ r \neq t}}^{l} \{ 1 - \| [A(\alpha_{j_t}, \alpha_{j_t})]^{-1} \Phi_t \| \}^{-1} \| [A(\alpha_{j_t}, \alpha_{j_t})]^{-1} A(\alpha_{j_t}, \alpha_{j_r}) \|$$

$$\times \sum_{\substack{r=1 \\ r \neq u}}^{l} \{ 1 - \| [A(\alpha_{j_u}, \alpha_{j_u})]^{-1} \Phi_u \| \}^{-1} \| [A(\alpha_{j_u}, \alpha_{j_u})]^{-1} A(\alpha_{j_u}, \alpha_{j_r}) \|$$

$$= \{ 1 - \| [A(\alpha_{j_t}, \alpha_{j_t})]^{-1} \Phi_t \| \}^{-1} \{ 1 - \| [A(\alpha_{j_u}, \alpha_{j_u})]^{-1} \Phi_u \| \}^{-1}$$

$$\times \Big\{ [1 - \| [A(\alpha_{j_t}, \alpha_{j_t})]^{-1} \Phi_t \|][1 - \| [A(\alpha_{j_u}, \alpha_{j_u})]^{-1} \Phi_u \|]$$

$$- \sum_{\substack{r=1 \\ r \neq t}}^{l} \| [A(\alpha_{j_t}, \alpha_{j_t})]^{-1} A(\alpha_{j_t}, \alpha_{j_r}) \| \sum_{\substack{r=1 \\ r \neq u}}^{l} \| [A(\alpha_{j_u}, \alpha_{j_u})]^{-1} A(\alpha_{j_u}, \alpha_{j_r}) \| \Bigg\}$$

$$\geq \{1 - \| [A(\alpha_{j_t}, \alpha_{j_t})]^{-1} \Phi_t \| \}^{-1} \{1 - \| [A(\alpha_{j_u}, \alpha_{j_u})]^{-1} \Phi_u \| \}^{-1} \Bigg\{ (1 - \Upsilon_{tt})(1 - \Upsilon_{uu})$$

$$- \Bigg[\sum_{\substack{r=1 \\ r \neq t}}^{l} \| [A(\alpha_{j_t}, \alpha_{j_t})]^{-1} A(\alpha_{j_t}, \alpha_{j_r}) \| + \Upsilon_{tu} \Bigg] \Bigg[\sum_{\substack{r=1 \\ r \neq u}}^{l} \| [A(\alpha_{j_u}, \alpha_{j_u})]^{-1} A(\alpha_{j_u}, \alpha_{j_r}) \| + \Upsilon_{ut} \Bigg] \Bigg\}$$

$$= \{1 - \| [A(\alpha_{j_t}, \alpha_{j_t})]^{-1} \Phi_t \| \}^{-1} \{1 - \| [A(\alpha_{j_u}, \alpha_{j_u})]^{-1} \Phi_u \| \}^{-1}$$

$$\triangleq \frac{\det B_4}{\det[\mu_{II}(A)(\alpha)]},$$

where

$$B_4 = \begin{pmatrix} 1 & -\xi_t & -K_t \\ -\xi_u & 1 & -K_u \\ -H_t & -H_u & \mu_{II}(A)(\alpha) \end{pmatrix}, \qquad \xi_\omega = \sum_{\substack{r=1 \\ r \neq \omega}}^{l} \| [A(\alpha_{j_\omega}, \alpha_{j_\omega})]^{-1} A(\alpha_{j_\omega}, \alpha_{j_r}) \|,$$

with $\omega = t, u$.

Since A is an II-BSDD$_s$, $\alpha_{i_0} \subseteq \alpha^c$, and $\alpha_{j_\omega} \subseteq \alpha^c$, $\omega = t, u$,

$$\sum_{\substack{r=1 \\ r \neq j_t}}^{s} \| [A(\alpha_{j_t}, \alpha_{j_t})]^{-1} A(\alpha_{j_t}, \alpha_r) \| \sum_{\substack{r=1 \\ r \neq j_u}}^{s} \| [A(\alpha_{j_u}, \alpha_{j_u})]^{-1} A(\alpha_{j_u}, \alpha_r) \| < 1.$$

For $\forall \alpha_{i_x} \subseteq \alpha$, $x = 1, 2, \ldots, k$, and $\alpha_{j_w} \subseteq \alpha^c$, $w = t, u$.

$$\Bigg\{ \sum_{\substack{r=1 \\ r \neq x}}^{s} \| [A(\alpha_{i_x}, \alpha_{i_x})]^{-1} A(\alpha_{i_x}, \alpha_{i_r}) \| + \| [A(\alpha_{i_x}, \alpha_{i_x})]^{-1} A(\alpha_{i_x}, \alpha_{j_t}) \|$$

$$+ \| [A(\alpha_{i_x}, \alpha_{i_x})]^{-1} A(\alpha_{i_x}, \alpha_{j_u}) \| \Bigg\} \sum_{\substack{r=1 \\ r \neq j_\omega}}^{s} \| [A(\alpha_{j_\omega}, \alpha_{j_\omega})]^{-1} A(\alpha_{j_\omega}, \alpha_r) \| < 1.$$

For $\forall \alpha_{i_x} \subseteq \alpha$, with $x = 1, 2, \ldots, k$,

$$\Bigg\{ \sum_{\substack{r=1 \\ r \neq x}}^{k} \| [A(\alpha_{i_x}, \alpha_{i_x})]^{-1} A(\alpha_{i_x}, \alpha_{i_r}) \| + \| [A(\alpha_{i_x}, \alpha_{i_x})]^{-1} A(\alpha_{i_x}, \alpha_{j_u}) \|$$

$$+ \| [A(\alpha_{i_x}, \alpha_{i_x})]^{-1} A(\alpha_{i_x}, \alpha_{j_t}) \| \Bigg\} < 1.$$

Thus, $B_4 \in \mathrm{SDD}_{k+2}$. Further, by Lemma 2.3, we have $B_4 = \mu(B_4) \in M_{k+2}$, $\mu_{II}[A(\alpha)] \in M_k$. Therefore, $\det(B_4) > 0$ and $\det[\mu_{II}(A)(\alpha)] > 0$, *i.e.*,

$$A/_\circ \alpha \in \text{II-BSDD}_l.$$

Combining the proof of (i) and (ii), we complete this proof. □

4 Numerical examples

In this section, we use two numerical examples to verify the accuracy of the theoretical analysis, from two aspects of the iteration number (denoted by IT) and the solution time in seconds (denoted by CPU), and then we further illustrate the feasibility and effectiveness of PGMSS(l) [19] iteration method, and verify the superiority of PGMSS iteration method is more efficient than that of the ordinary GMRES(l) iteration method. Here, we use the integer l in GMRES(l) to denote the number of restarting steps.

In our numerical experiments, we choose the zero vector as the initial guess and take the right-hand-side vector b so that the exact solutions x and y are the unity vectors with all entries equal to one. In addition, all runs are initiated with the initial vector $x^{(0)} = 0$. We use

$$\mathrm{RES} = \frac{\|b - Ax^{(k)}\|_2}{\|b - Ax^{(0)}\|_2} < 10^{-8}$$

or the prescribed iteration number $k_{\max} = n$ as a stopping criterion, where $x^{(k)}$ is the solution at the kth iterate.

For convenience, without loss of generality, we suppose $\|\cdot\| = \|\cdot\|_2$, $\alpha = \bigcup_{r=1}^{3} \alpha_{i_r}$ and $\alpha^c = \bigcup_{t=1}^{2} \alpha_{j_t}$, and we denote the block diagonal-Schur complement $A/_\circ A(\alpha)$ of $A(\alpha)$ in A by

$$A/_\circ A(\alpha) = \begin{pmatrix} \widetilde{A}_{11} & \widetilde{A}_{12} & \widetilde{A}_{13} \\ \widetilde{A}_{21} & \widetilde{A}_{22} & \widetilde{A}_{23} \\ \widetilde{A}_{31} & \widetilde{A}_{32} & \widetilde{A}_{33} \end{pmatrix}.$$

Let us consider the following linear system:

$$Ax = \begin{pmatrix} A(\alpha,\alpha) & A(\alpha,\alpha^c) \\ A(\alpha^c,\alpha) & A(\alpha^c,\alpha^c) \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} b_1 \\ b_2 \end{pmatrix}.$$

By solving the above linear system, we compare GMRES iteration method with the block triangular approximate Schur complement preconditioner (11) established in this paper with the ordinary GMRES iteration method. We have

$$P_1 = \begin{pmatrix} A(\alpha,\alpha) & \\ A(\alpha^c,\alpha) & A/_\circ A(\alpha) \end{pmatrix}. \tag{11}$$

In the following, we verify Theorem 3.1 and Theorem 3.2 by Example 4.1 and Example 4.2, respectively.

Example 4.1 Consider a linear equation system $Ax = b$ whose coefficient matrix A is denoted by

$$A = \begin{pmatrix} A_{11} & A_{12} & A_{13} & A_{14} & A_{15} \\ A_{21} & A_{22} & A_{23} & A_{24} & A_{25} \\ A_{31} & A_{32} & A_{33} & A_{34} & A_{35} \\ A_{41} & A_{42} & A_{43} & A_{44} & A_{45} \\ A_{51} & A_{52} & A_{53} & A_{54} & A_{55} \end{pmatrix},$$

where the submatrices of the coefficient matrix A take on structures of the forms

$$A_{11} = \begin{pmatrix} 150 & -50 & & & \\ -50 & 150 & -50 & & \\ & \ddots & \ddots & \ddots & \\ & & & -50 & 150 \end{pmatrix}_{20\times 20},$$

$$A_{22} = \begin{pmatrix} 1{,}500 & -50 & & & \\ -50 & 1{,}500 & -50 & & \\ & \ddots & \ddots & \ddots & \\ & & & -50 & 1{,}500 \end{pmatrix}_{20\times 20},$$

$$A_{33} = \begin{pmatrix} 1{,}500 & -50 & & & \\ -50 & 1{,}500 & -50 & & \\ & \ddots & \ddots & \ddots & \\ & & & -50 & 1{,}500 \end{pmatrix}_{30\times 30},$$

$$A_{44} = \begin{pmatrix} 1{,}500 & -50 & & & \\ -50 & 1{,}500 & -50 & & \\ & \ddots & \ddots & \ddots & \\ & & & -50 & 1{,}500 \end{pmatrix}_{15\times 15},$$

$$A_{12} = \begin{pmatrix} & & & -30 \\ & \ddots & \ddots & \ddots \\ -30 & & & \end{pmatrix}_{20\times 20}, \qquad A_{15} = \begin{pmatrix} & & & -50 \\ & \ddots & \ddots & \ddots \\ -50 & & & \end{pmatrix}_{20\times 15},$$

$$A_{34} = \begin{pmatrix} & & & -30 \\ & \ddots & \ddots & \ddots \\ -30 & & & \end{pmatrix}_{30\times 15}, \qquad A_{13} = \begin{pmatrix} & & & -30 \\ & \ddots & \ddots & \ddots \\ -30 & & & \end{pmatrix}_{20\times 30},$$

and

$$A_{45} = \begin{pmatrix} -20 & & & -20 \\ & \ddots & \ddots & \ddots \\ -20 & & & -20 \end{pmatrix}_{15\times 15},$$

with $A_{44} = A_{55}$, $A_{21}^T = A_{12}$, $A_{14} = A_{15} = A_{24} = A_{25} = A_{41}^T = A_{51}^T = A_{42}^T = A_{52}^T$, $A_{54}^T = A_{45}$, $A_{43}^T = A_{34} = A_{53}^T = A_{35}$, and $A_{31}^T = A_{13} = A_{32}^T = A_{23}$.

We suppose $t_i = \|[A_{ii}]^{-1}\|^{-1}$ and $s_i = \sum_{m=1, m\neq i}^{5} \|A_{im}\|$, with $i = 1, 2, \ldots, 5$. After programming and computation by the use of Matlab software, from Table 1, it is straightforward to show that the matrix A is an I-BSDD$_s$ but is not an I-BSD$_s$.

Table 1 The experimental verification of I-BSDD$_s$

i	1	2	3	4	5
$t_i - s_i$	**−108.88**	1.24×10^3	3.97×10^3	1.23×10^3	1.23×10^3
(i,j)	**(1, 2)**	**(1, 3)**	**(1, 4)**	**(1, 5)**	**(2, 3)**
$t_it_j - s_is_j$	4.6×10^4	5.24×10^4	4.45×10^4	4.45×10^4	1.94×10^6
(i,j)	**(2, 4)**	**(2, 5)**	**(3, 4)**	**(3, 5)**	**(4, 5)**
$t_it_j - s_is_j$	1.94×10^6	1.94×10^6	1.94×10^6	1.94×10^6	1.94×10^6

Table 2 The experimental verification of I-BSD$_s$

i	1	2	3
$\widetilde{t}_i - \widetilde{s}_i$	1,341	1,332	1,332
$\widetilde{t_it_j} - \widetilde{s_is_j}$	1.96×10^6	1.96×10^6	1.96×10^6

Table 3 The experimental verification of I-BSD$_s$

i	1	2	3
$\widehat{t}_i - \widehat{s}_i$	**−8.8985**	1,341	1,341
$\widehat{t_it_j} - \widehat{s_is_j}$	6.8×10^4	6.8×10^4	1.96×10^6

Table 4 Number of iterations and solution time in seconds of PGMRES(l) iteration method with preconditioner $\mathcal{P}_1$ and the ordinary GMRES(l) iteration method, where $\alpha = \{1, 2\}$

l		30	40	50	60	70	80	90
PGMRES(l)	CPU	0.0106	0.0015	0.0024	0.0018	0.0014	0.0024	0.0015
	IT	4	4	4	4	4	4	4
GMRES(l)	CPU	0.0046	0.0054	0.0042	0.0018	0.0042	0.0156	0.0048
	IT	23	23	23	23	23	23	23

To verify the heritable properties of the block diagonal-Schur complements from the original matrix I-BSDD$_s$, we need to consider two conditions $i_0 \in \alpha$ and $i_0 \in \alpha^c$, where i_0 is defined in (9). From Table 1, it is easy to see $i_0 = 1$. Firstly, we consider the condition $i_0 \in \alpha = \{1, 2\}$ and $\alpha^c = \{3, 4, 5\}$, and assume $\widetilde{t}_i = \|[\widetilde{A}_{ii}]^{-1}\|^{-1}$ and $\widetilde{s}_i = \sum_{m=1, m \neq i}^{3} \|\widetilde{A}_{im}\|$, with $i = 1, 2, 3$, as follows from Table 2, we can easily see that the block diagonal-Schur complement $A/_\circ A(\alpha)$ of $A(\alpha)$ in A is an I-BSD$_s$; thereby, we verify the conclusion (i) of Theorem 3.1. Secondly, we consider the condition $i_0 \in \alpha^c = \{1, 2, 3\}$ and $\alpha = \{4, 5\}$, and assume $\widehat{t}_i = \|[\widetilde{A}_{ii}]^{-1}\|^{-1}$ and $\widehat{s}_i = \sum_{m=1, m \neq i}^{3} \|\widetilde{A}_{im}\|$, with $i = 1, 2, 3$, as follows from Table 3, we can easily see that the block diagonal-Schur complement $A/_\circ A(\alpha)$ of $A(\alpha)$ in A is an I-BSDD$_s$ but is not an I-BSD$_s$, accordingly, we validate the conclusion (ii) of Theorem 3.1.

Table 4, for Example 4.1, lists the numerical results corresponding to the tolerance $\epsilon = 10^{-8}$, it means that the block diagonal Schur-based GMRES iteration method with the preconditioner P_1 is more efficient than the ordinary GMRES(l) iteration method.

Example 4.2 Consider a linear equation system $Ax = b$ whose coefficient matrix A is denoted by

$$A = \begin{pmatrix} A_{11} & A_{12} & A_{13} & A_{14} & A_{15} \\ A_{21} & A_{22} & A_{23} & A_{24} & A_{25} \\ A_{31} & A_{32} & A_{33} & A_{34} & A_{35} \\ A_{41} & A_{42} & A_{43} & A_{44} & A_{45} \\ A_{51} & A_{52} & A_{53} & A_{54} & A_{55} \end{pmatrix},$$

where the submatrices of the coefficient matrix A take on structures of the forms

$$A_{11} = \begin{pmatrix} 100 & -50 & & & \\ -50 & 100 & -50 & & \\ & \ddots & \ddots & \ddots & \\ & & & -50 & 100 \end{pmatrix}_{20\times 20},$$

$$A_{22} = \begin{pmatrix} 1{,}800 & -50 & & & \\ -50 & 1{,}800 & -50 & & \\ & \ddots & \ddots & \ddots & \\ & & & -50 & 1{,}800 \end{pmatrix}_{20\times 20},$$

$$A_{33} = \begin{pmatrix} 1{,}410 & -40 & & & \\ -40 & 1{,}420 & -40 & & \\ & \ddots & \ddots & \ddots & \\ & & & -40 & 1{,}700 \end{pmatrix}_{30\times 30},$$

$$A_{44} = \begin{pmatrix} 1{,}800 & -50 & & & \\ -50 & 1{,}800 & -50 & & \\ & \ddots & \ddots & \ddots & \\ & & & -50 & 1{,}800 \end{pmatrix}_{15\times 15},$$

$$A_{12} = \begin{pmatrix} & & & -30 \\ & \ddots & \ddots & \ddots \\ -30 & & & \end{pmatrix}_{20\times 20}, \quad A_{15} = \begin{pmatrix} & & & -50 \\ & \ddots & \ddots & \ddots \\ -50 & & & \end{pmatrix}_{20\times 15},$$

$$A_{34} = \begin{pmatrix} & & & -30 \\ & \ddots & \ddots & \ddots \\ -30 & & & \end{pmatrix}_{30\times 15}, \quad A_{13} = \begin{pmatrix} & & & -30 \\ & \ddots & \ddots & \ddots \\ -30 & & & \end{pmatrix}_{20\times 30},$$

and

$$A_{45} = \begin{pmatrix} -20 & & & -20 \\ & \ddots & \ddots & \ddots \\ -20 & & & -20 \end{pmatrix}_{15\times 15},$$

Table 5 The experimental verification of I-BSDD$_s$

i	1	2	3	4	5
ζ_i	**10.12**	0.089	0.085	0.095	0.095
(i,j)	**(1, 2)**	**(1, 3)**	**(1, 4)**	**(1, 5)**	**(2, 3)**
$\zeta_i\zeta_j$	0.9005	0.8622	0.9568	0.9568	0.0076
(i,j)	**(2, 4)**	**(2, 5)**	**(3, 4)**	**(3, 5)**	**(4, 5)**
$\zeta_i\zeta_j$	0.0084	0.0084	0.0081	0.0081	0.0089

Table 6 The experimental verification of II-BSDD$_s$

i	1	2	3
$\widetilde{s}_i$	0.0432	0.0401	0.0401
$\widetilde{s_i s_j}$	0.0017	0.0017	0.0016

Table 7 The experimental verification of II-BSDD$_s$

i	1	2	3
$\widehat{s}_i$	**4.0235**	0.0334	0.0426
$\widehat{s_i s_j}$	0.1345	0.1715	0.0014

Table 8 Number of iterations and solution time in seconds of PGMRES(l) iteration method with preconditioner $\mathcal{P}_1$ and GMRES(l) iteration method

l		30	40	50	60	70	80	90
PGMRES(l)	CPU	0.0021	0.0026	0.0016	0.0016	0.0033	0.0021	0.0016
	IT	4	4	4	4	4	4	4
GMRES(l)	CPU	0.0096	0.0159	0.0197	0.0141	0.0196	0.0186	0.0176
	IT	27	25	24	53	53	53	53

with $A_{44} = A_{55}$, $A_{21}^T = A_{12}$, $A_{14} = A_{15} = A_{24} = A_{25} = A_{41}^T = A_{51}^T = A_{42}^T = A_{52}^T$, $A_{54}^T = A_{45}$, $A_{43}^T = A_{34} = A_{53}^T = A_{35}$, and $A_{31}^T = A_{13} = A_{32}^T = A_{23}$.

After programming and computation by the use of Matlab software, denoting $\zeta_i = \sum_{m=1, m\neq i}^{5} \|[A_{ii}]^{-1}A_{im}\|$, with $i = 1, 2, \ldots, 5$, from Table 5, it is straightforward to show that the matrix A is an II-BSDD$_s$ but is not an II-BSD$_s$.

To verify the heritable properties of the block diagonal-Schur complements from the original matrix II-BSDD$_s$, we need to consider two conditions $i_0 \in \alpha$ and $i_0 \in \alpha^c$, where i_0 is defined in (10). From the first line of Table 5, it is easy to see that $i_0 = 1$. Firstly, we consider the condition $i_0 \in \alpha = \{1,2\}$ and $\alpha^c = \{3,4,5\}$, and assume $\widetilde{s}_i = \sum_{m=1, m\neq i}^{3} \|[\widetilde{A}_{ii}]^{-1}\widetilde{A}_{im}\|$, with $i = 1, 2, 3$, as follows from Table 6, we can easily see that the block diagonal-Schur complement $A/_\circ A(\alpha)$ of $A(\alpha)$ in A is an I-BSD$_s$, therefore, we verify the result (i) of Theorem 3.2. Secondly, we consider the condition $i_0 \in \alpha^c = \{1,2,3\}$ and $\alpha = \{4,5\}$, and assume $\widehat{s}_i = \sum_{m=1, m\neq i}^{3} \|[\widetilde{A}_{ii}]^{-1}\widetilde{A}_{im}\|$, with $i = 1, 2, 3$, as follows from Table 7, we can easily see that the block diagonal-Schur complement $A/_\circ A(\alpha)$ of $A(\alpha)$ in A is an II-BSDD$_s$ but is not an II-BSD$_s$, accordingly, we validate the result (ii) of Theorem 3.2.

Table 8, for Example 4.2, lists the numerical results corresponding to the tolerance $\epsilon = 10^{-8}$, it means that the block diagonal Schur-based GMRES(l) iteration method with the preconditioner P_1 is more efficient than the ordinary GMRES(l) iteration method, where $\alpha = \{1, 2\}$.

5 Conclusions

In this paper, the heritable properties of the block diagonal-Schur complements from the original matrix are presented. Numerical experiments further indicate the practical performance.

One of the advantages of the study of the heritable properties is that it is possible to estimate the sharp bounds of the eigenvalues of the original matrix and the corresponding block diagonal-Schur complements. Thereby we believe that the techniques presented here can be used to develop robust and efficient Schur complement preconditioning techniques for solving linear systems.

Competing interests

The author declares that he has no competing interests.

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