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Complete moment convergence for maximal partial sums under NOD setup

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Abstract

The sufficient and necessary conditions of complete moment convergence for negatively orthant dependent (NOD) random variables are obtained, which improve and extend the well-known results.

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1 Introduction

The concept of complete convergence of a sequence of random variables was introduced by Hsu and Robbins [1] as follows. A sequence $\{U_n, n \geq 1\}$ of random variables converges completely to the constant θ if

$$\sum_{n=1}^{\infty} P\bigl(|U_n - \theta| > \varepsilon\bigr) < \infty \quad \text{for all } \varepsilon > 0.$$

Moreover, they proved that the sequence of arithmetic means of independent and identically distributed (i.i.d.) random variables converges completely to the expected value if the variance of the summands is finite. This result has been generalized and extended in several directions, one can refer to [2–13] and so forth.

When $\{X_n, n \geq 1\}$ is a sequence of i.i.d. random variables with mean zero, Chow [14] first investigated the complete moment convergence, which is more exact than complete convergence. He obtained the following result.

Theorem A *Let $\{X_n, n \geq 1\}$ be a sequence of i.i.d. random variables with $EX_1 = 0$. For $1 \leq p < 2$ and $r > 1$, if $E\{|X_1|^{rp} + |X_1| \log(1 + |X_1|)\} < \infty$, then*

$$\sum_{n=1}^{\infty} n^{r-2-1/p} E\left\{\left|\sum_{k=1}^{n} X_k\right| - \varepsilon n^{1/p}\right\}_{+} < \infty \quad \textit{for all } \varepsilon > 0,$$

where (as in the following) $x_+ = \max\{0, x\}$.

Theorem A has been generalized and extended in several directions. One can refer to Wang and Su [15] and Chen [16] for random elements taking values in a Banach space, Wang and Zhao [17] for NA random variables, Chen *et al.* [5], Li and Zhang [18] for moving-average processes based on NA random variables, Chen and Wang [19] for φ-

mixing random variables, Qiu and Chen [20] for weighted sums of arrays of rowwise NA random variables.

The aim of this paper is to extend and improve Theorem A to negatively orthant dependent (NOD) random variables. The sufficient and necessary conditions are obtained. In fact, the paper is the continued work of Qiu *et al.* [11] in which the complete convergence is obtained for NOD sequence. It is worth to point that Sung [21] has discussed the complete moment convergence for NOD, but the main result in our paper is more exact and the method is completely different.

The concepts of negatively associated (NA) and negatively orthant dependent (NOD) were introduced by Joag-Dev and Proschan [22] in the following way.

Definition 1.1 A finite family of random variables $\{X_i, 1 \le i \le n\}$ is said to be negatively associated (NA) if for every pair of disjoint nonempty subset A_1, A_2 of $\{1, 2, \ldots, n\}$,

$$\operatorname{Cov}\bigl(f_1(X_i, i \in A_1), f_2(X_j, j \in A_2)\bigr) \le 0,$$

where f_1 and f_2 are coordinatewise nondecreasing such that the covariance exists. An infinite sequence of $\{X_n, n \ge 1\}$ is NA if every finite subfamily is NA.

Definition 1.2 A finite family of random variables $\{X_i, 1 \le i \le n\}$ is said to be

(a) negatively upper orthant dependent (NUOD) if

$$P(X_i > x_i, i = 1, 2, \ldots, n) \le \prod_{i=1}^{n} P(X_i > x_i)$$

$\forall x_1, x_2, \ldots, x_n \in R$,

(b) negatively lower orthant dependent (NLOD) if

$$P(X_i \le x_i, i = 1, 2, \ldots, n) \le \prod_{i=1}^{n} P(X_i \le x_i)$$

$\forall x_1, x_2, \ldots, x_n \in R$,

(c) negatively orthant dependent (NOD) if they are both NUOD and NLOD.

A sequence of random variables $\{X_n, n \ge 1\}$ is said to be NOD if for each n, $X_1, X_2, \ldots, X_n$ are NOD.

Obviously, every sequence of independent random variables is NOD. Joag-Dev and Proschan [22] pointed out NA implies NOD, neither NUOD nor NLOD implies being NA. They gave an example which possesses NOD, but does not possess NA. So we can see that NOD is strictly wider than NA. For more convergence properties about NOD random variables, one can refer to [2, 11, 20, 23–26], and so forth.

In order to prove our main result, we need the following lemmas.

Lemma 1.1 (Bozorgnia *et al.* [23]) *Let $X_1, X_2, \ldots, X_n$ be NOD random variables.*

(i) *If $f_1, f_2, \ldots, f_n$ are Borel functions all of which are monotone increasing (or all monotone decreasing), then $f_1(X_1), f_2(X_2), \ldots, f_n(X_n)$ are NOD random variables.*

(ii) $E\prod_{i=1}^{n}(X_i)_+ \le \prod_{i=1}^{n} E(X_i)_+$, $\forall n \ge 2$.

Lemma 1.2 (Asadian *et al.* [27]) *For any $v \geq 2$, there is a positive constant $C(v)$ depending only on v such that if $\{X_n, n \geq 1\}$ is a sequence of NOD random variables with $EX_n = 0$ for every $n \geq 1$, then for all $n \geq 1$,*

$$E\left|\sum_{i=1}^{n} X_i\right|^{v} \leq C(v)\left\{\sum_{i=1}^{n} E|X_i|^{v} + \left(\sum_{i=1}^{n} EX_i^2\right)^{v/2}\right\}.$$

We reason by Lemma 1.2 and a similar argument to Theorem 2.3.1 of Stout [28].

Lemma 1.3 *For any $v \geq 2$, there is a positive constant $C(v)$ depending only on v such that if $\{X_n, n \geq 1\}$ is a sequence of NOD random variables with $EX_n = 0$ for every $n \geq 1$, then for all $n \geq 1$,*

$$E\max_{1\leq j\leq n}\left|\sum_{i=1}^{j} X_i\right|^{v} \leq C(v)\bigl(\log(4n)\bigr)^{v}\left\{\sum_{i=1}^{n} E|X_i|^{v} + \left(\sum_{i=1}^{n} EX_i^2\right)^{v/2}\right\},$$

where $\log x = \max\{1, \ln x\}$, and $\ln x$ denotes the natural logarithm of x.

Lemma 1.4 (Kuczmaszewska [8]) *Let β be positive constant, $\{X_n, n \geq 1\}$ be a sequence of random variables and X be a random variable. Suppose that*

$$\sum_{i=1}^{n} P\bigl(|X_i| > x\bigr) \leq DnP\bigl(|X| > x\bigr), \quad \forall x > 0, \forall n \geq 1, \tag{1.1}$$

holds for some $D > 0$, then there exists a constant $C > 0$ depending only on D and β such that

(i) *If $E|X|^{\beta} < \infty$, then $\frac{1}{n}\sum_{j=1}^{n} E|X_j|^{\beta} \leq CE|X|^{\beta}$;*
(ii) $\frac{1}{n}\sum_{j=1}^{n} E|X_j|^{\beta} I(|X_j| \leq x) \leq C\{E|X|^{\beta} I(|X| \leq x) + x^{\beta} P(|X| > x)\}$;
(iii) $\frac{1}{n}\sum_{j=1}^{n} E|X_j|^{\beta} I(|X_j| > x) \leq CE|X|^{\beta} I(|X| > x)$.

Throughout this paper, C will represent positive constants; their value may change from one place to another.

2 Main results and proofs

Theorem 2.1 *Let $\gamma > 0, \alpha > 1/2, p > 0, \alpha p > 1$. Let $\{X_n, n \geq 1\}$ be a sequence of NOD random variables and X be a random variables possibly defined on a different space satisfying the condition* (1.1). *Moreover, assume that $EX_n = 0$ for all $n \geq 1$ in the case $\alpha \leq 1$. Suppose that*

$$\begin{cases} E|X|^p < \infty, & \gamma < p, \\ E|X|^p \log(1 + |X|) < \infty, & \gamma = p, \\ E|X|^{\gamma} < \infty, & \gamma > p. \end{cases} \tag{2.1}$$

Then the following statements hold:

$$\sum_{n=1}^{\infty} n^{\alpha(p-\gamma)-2} E\Bigl\{\max_{1\leq k\leq n} |S_k| - \varepsilon n^{\alpha}\Bigr\}_{+}^{\gamma} < \infty, \quad \forall \varepsilon > 0, \tag{2.2}$$

$$\sum_{n=1}^{\infty} n^{\alpha(p-\gamma)-2} E\Big\{\max_{1\le k\le n} \big|S_n^{(k)}\big| - \varepsilon n^{\alpha}\Big\}_+^{\gamma} < \infty, \quad \forall \varepsilon > 0, \tag{2.3}$$

$$\sum_{n=1}^{\infty} n^{\alpha(p-\gamma)-2} E\Big\{\max_{1\le k\le n} |X_k| - \varepsilon n^{\alpha}\Big\}_+^{\gamma} < \infty, \quad \forall \varepsilon > 0, \tag{2.4}$$

$$\sum_{n=1}^{\infty} n^{\alpha p-2} E\Big\{\sup_{k\ge n} k^{-\alpha}|S_k| - \varepsilon\Big\}_+^{\gamma} < \infty, \quad \forall \varepsilon > 0, \tag{2.5}$$

$$\sum_{n=1}^{\infty} n^{\alpha p-2} E\Big\{\sup_{k\ge n} k^{-\alpha}|X_k| - \varepsilon\Big\}_+^{\gamma} < \infty, \quad \forall \varepsilon > 0, \tag{2.6}$$

where $S_n = \sum_{i=1}^{n} X_i$, $S_n^{(k)} = S_n - X_k$, $k = 1, 2, \ldots, n$.

Proof Firstly, we prove (2.2). Note that for all $\varepsilon > 0$

$$\begin{aligned}
&\sum_{n=1}^{\infty} n^{\alpha(p-\gamma)-2} E\Big\{\max_{1\le k\le n} |S_k| - \varepsilon n^{\alpha}\Big\}_+^{\gamma} \\
&\quad = \sum_{n=1}^{\infty} n^{\alpha(p-\gamma)-2} \int_0^{\infty} P\Big(\max_{1\le k\le n} |S_k| - \varepsilon n^{\alpha} > t^{1/\gamma}\Big)\,dt \\
&\quad = \sum_{n=1}^{\infty} n^{\alpha(p-\gamma)-2} \int_0^{n^{\gamma\alpha}} P\Big(\max_{1\le k\le n} |S_k| - \varepsilon n^{\alpha} > t^{1/\gamma}\Big)\,dt \\
&\qquad + \sum_{n=1}^{\infty} n^{\alpha(p-\gamma)-2} \int_{n^{\gamma\alpha}}^{\infty} P\Big(\max_{1\le k\le n} |S_k| - \varepsilon n^{\alpha} > t^{1/\gamma}\Big)\,dt \\
&\quad \le \sum_{n=1}^{\infty} n^{\alpha p-2} P\Big(\max_{1\le k\le n} |S_k| > \varepsilon n^{\alpha}\Big) \\
&\qquad + \sum_{n=1}^{\infty} n^{\alpha(p-\gamma)-2} \int_{n^{\gamma\alpha}}^{\infty} P\Big(\max_{1\le k\le n} |S_k| > t^{1/\gamma}\Big)\,dt.
\end{aligned}$$

Hence by Theorem 2.1 of Qiu *et al.* [11], in order to prove (2.2), it is enough to show that

$$\sum_{n=1}^{\infty} n^{\alpha(p-\gamma)-2} \int_{n^{\gamma\alpha}}^{\infty} P\Big(\max_{1\le k\le n} |S_k| > t^{1/\gamma}\Big)\,dt < \infty.$$

Choose q such that $1/(\alpha p) < q < 1$. $\forall j \ge 1$, $t > 0$, let

$$\begin{aligned}
X_j^{(t,1)} &= -t^{q/\gamma} I\big(X_j < -t^{q/\gamma}\big) + X_j I\big(|X_j| \le t^{q/\gamma}\big) + t^{q/\gamma} I\big(X_j > t^{q/\gamma}\big), \\
X_j^{(t,2)} &= \big(X_j - t^{q/\gamma}\big) I\big(t^{q/\gamma} < X_j \le t^{q/\gamma} + t^{1/\gamma}\big) + t^{1/\gamma} I\big(X_j > t^{q/\gamma} + t^{1/\gamma}\big), \\
X_j^{(t,3)} &= \big(X_j - t^{q/\gamma} - t^{1/\gamma}\big) I\big(X_j > t^{q/\gamma} + t^{1/\gamma}\big), \\
X_j^{(t,4)} &= \big(X_j + t^{q/\gamma}\big) I\big(-t^{q/\gamma} - t^{1\gamma} \le X_j < -t^{q/\gamma}\big) - t^{1/\gamma} I\big(X_j < -t^{q/\gamma} - t^{1\gamma}\big), \\
X_j^{(t,5)} &= \big(X_j + t^{q/\gamma} + t^{1/\gamma}\big) I\big(X_j < -t^{q/\gamma} - t^{1/\gamma}\big),
\end{aligned}$$

then $X_j = \sum_{l=1}^{5} X_j^{(t,l)}$. Note that

$$\begin{aligned}
&\sum_{n=1}^{\infty} n^{\alpha(p-\gamma)-2} \int_{n^{\gamma\alpha}}^{\infty} P\Big(\max_{1\le k\le n} |S_k| > t^{1/\gamma}\Big)\,dt \\
&\quad\le \sum_{n=1}^{\infty} n^{\alpha(p-\gamma)-2} \int_{n^{\gamma\alpha}}^{\infty} P\left(\max_{1\le k\le n}\left|\sum_{j=1}^{k} X_j^{(t,1)}\right| > t^{1/\gamma}/5\right) dt \\
&\qquad + \sum_{l=2}^{3}\sum_{n=1}^{\infty} n^{\alpha(p-\gamma)-2} \int_{n^{\gamma\alpha}}^{\infty} P\left(\sum_{j=1}^{n} X_j^{(t,l)} > t^{1/\gamma}/5\right) dt \\
&\qquad + \sum_{l=4}^{5}\sum_{n=1}^{\infty} n^{\alpha(p-\gamma)-2} \int_{n^{\gamma\alpha}}^{\infty} P\left(-\sum_{j=1}^{n} X_j^{(t,l)} > t^{1/\gamma}/5\right) dt \\
&\quad\overset{\text{def}}{=} \sum_{l=1}^{5} I_l.
\end{aligned}$$

Therefore to prove (2.2), it suffices to show that $I_l < \infty$ for $l = 1, 2, 3, 4, 5$.

For I_1, we first prove that

$$\sup_{t\ge n^{\gamma\alpha}} t^{-1/\gamma} \max_{1\le k\le n}\left|E\sum_{j=1}^{k} X_j^{(t,1)}\right| \to 0, \quad n\to\infty. \tag{2.7}$$

When $\alpha \le 1$. Since $\alpha p > 1$ implies $p > 1$, by Lemma 1.4 and $EX_j = 0$, $j \ge 1$, we have

$$\begin{aligned}
&\sup_{t\ge n^{\gamma\alpha}} t^{-1/\gamma} \max_{1\le k\le n}\left|E\sum_{j=1}^{k} X_j^{(t,1)}\right| \\
&\quad\le \sup_{t\ge n^{\gamma\alpha}} t^{-1/\gamma} \sum_{j=1}^{n} E\big\{|X_j| I\big(|X_j| > t^{q/\gamma}\big) + t^{q/\gamma} I\big(|X_j| > t^{q/\gamma}\big)\big\} \\
&\quad\le 2 \sup_{t\ge n^{\gamma\alpha}} t^{-1/\gamma} \sum_{j=1}^{n} E|X_j| I\big(|X_j| > t^{q/\gamma}\big) \\
&\quad\le Cn \sup_{t\ge n^{\gamma\alpha}} t^{-1/\gamma} E|X| I\big(|X| > t^{q/\gamma}\big) \\
&\quad\le Cn^{1-\alpha} E|X| I\big(|X| > n^{\alpha q}\big) \\
&\quad\le Cn^{1-\alpha pq-\alpha(1-q)} E|X|^p \to 0, \quad n\to\infty.
\end{aligned}$$

When $\alpha > 1$ and $p \ge 1$.

$$\begin{aligned}
&\sup_{t\ge n^{\gamma\alpha}} t^{-1/\gamma} \max_{1\le k\le n}\left|E\sum_{j=1}^{k} X_j^{(t,1)}\right| \\
&\quad\le \sup_{t\ge n^{\gamma\alpha}} t^{-1/\gamma} \sum_{j=1}^{n} E|X_j| \le Cn \sup_{t\ge n^{\gamma\alpha}} t^{-1/\gamma} E|X| \le Cn^{1-\alpha} \to 0, \quad n\to\infty.
\end{aligned}$$

When $\alpha > 1$ and $p < 1$,

$$\begin{aligned}
&\sup_{t\ge n^{\gamma\alpha}} t^{-1/\gamma} \max_{1\le k\le n}\left|E\sum_{j=1}^{k} X_j^{(t,1)}\right| \\
&\quad\le \sup_{t\ge n^{\gamma\alpha}} t^{-1/\gamma} \sum_{j=1}^{n} E\bigl\{|X_j| I\bigl(|X_j|\le t^{q/\gamma}\bigr) + t^{q/\gamma} I\bigl(|X_j| > t^{q/\gamma}\bigr)\bigr\} \\
&\quad\le \sup_{t\ge n^{\gamma\alpha}} t^{-1/\gamma} \sum_{j=1}^{n} t^{q(1-p)/\gamma} E|X_j|^p \le Cn \sup_{t\ge n^{\gamma\alpha}} t^{\{q(1-p)-1\}/\gamma} E|X|^p \\
&\quad\le Cn^{1-\alpha pq-(1-q)\alpha} \to 0, \quad n\to\infty.
\end{aligned}$$

Therefore (2.7) holds. By (2.7), in order to prove $I_1 < \infty$, it is enough to show that

$$I_1^* := \sum_{n=1}^{\infty} n^{\alpha(p-\gamma)-2} \int_{n^{\gamma\alpha}}^{\infty} P\left(\max_{1\le k\le n}\left|\sum_{j=1}^{k}\bigl(X_j^{(t,1)} - EX_j^{(t,1)}\bigr)\right| > t^{1/\gamma}/10\right) dt < \infty.$$

Fix any $\nu \ge 2$ and $\nu > \max\{p/(1-q), \gamma/(1-q), 2\gamma/[2-(2-p)q], 2(\alpha p-1)/[2\alpha(1-q)+(\alpha pq-1)], (\alpha p-1)/(\alpha-1/2)\}$, by Markov's inequality, Lemma 1.1, Lemma 1.3, and C_r-inequality, we have

$$\begin{aligned}
I_1^* &\le C\sum_{n=1}^{\infty} n^{\alpha(p-\gamma)-2} \\
&\quad\times \int_{n^{\gamma\alpha}}^{\infty} t^{-\nu/\gamma}\bigl(\log(4n)\bigr)^{\nu}\left\{\sum_{j=1}^{n} E\bigl|X_j^{(t,1)}\bigr|^{\nu} + \left(\sum_{j=1}^{n} E\bigl(X_j^{(t,1)}\bigr)^2\right)^{\nu/2}\right\} dt \\
&\le C\sum_{n=1}^{\infty} n^{\alpha(p-\gamma)-2}\bigl(\log(4n)\bigr)^{\nu} \\
&\quad\times \int_{n^{\gamma\alpha}}^{\infty} t^{-\nu/\gamma} \sum_{j=1}^{n}\bigl\{E|X_j|^{\nu} I\bigl(|X_j|\le t^{\frac{q}{\gamma}}\bigr) + t^{q\nu/\gamma} P\bigl(|X_j| > t^{\frac{q}{\gamma}}\bigr)\bigr\} dt \\
&\quad + C\sum_{n=1}^{\infty} n^{\alpha(p-\gamma)-2}\bigl(\log(4n)\bigr)^{\nu} \\
&\quad\times \int_{n^{\gamma\alpha}}^{\infty} t^{\frac{-\nu}{\gamma}}\left\{\sum_{j=1}^{n}\bigl(EX_j^2 I\bigl(|X_j|\le t^{\frac{q}{\gamma}}\bigr) + t^{\frac{2q}{\gamma}} P\bigl(|X_j| > t^{\frac{q}{\gamma}}\bigr)\bigr)\right\}^{\frac{\nu}{2}} dt \\
&\stackrel{\text{def}}{=} I_{11} + I_{12}.
\end{aligned}$$

Note that

$$\begin{aligned}
I_{11} &\le C\sum_{n=1}^{\infty} n^{\alpha(p-\gamma)-1}\bigl(\log(4n)\bigr)^{\nu}\int_{n^{\gamma\alpha}}^{\infty} t^{-(1-q)\nu/\gamma}\, dt \\
&\le C\sum_{n=1}^{\infty} n^{\alpha p-\alpha(1-q)\nu-1}\bigl(\log(4n)\bigr)^{\nu} < \infty.
\end{aligned}$$

If $\max\{p,\gamma\} < 2$, by Lemma 1.4, we have

$$\begin{aligned}
I_{12} &\le C\sum_{n=1}^{\infty} n^{\alpha(p-\gamma)-2}\bigl(\log(4n)\bigr)^{\nu}\int_{n^{\gamma\alpha}}^{\infty} t^{\frac{-\nu}{\gamma}}\left\{t^{(2-p)q/\gamma}\sum_{j=1}^{n}E|X_j|^p\right\}^{\frac{\nu}{2}}dt \\
&\le C\sum_{n=1}^{\infty} n^{\alpha(p-\gamma)-2+\nu/2}\bigl(\log(4n)\bigr)^{\nu}\int_{n^{\gamma\alpha}}^{\infty} t^{\frac{-[2-(2-p)q]\nu}{2\gamma}}\bigl(E|X|^p\bigr)^{\frac{\nu}{2}}dt \\
&\le C\sum_{n=1}^{\infty} n^{\alpha p-2-[\alpha(1-q)+(\alpha pq-1)/2]\nu}\bigl(\log(4n)\bigr)^{\nu} < \infty.
\end{aligned}$$

If $\max\{p,\gamma\} \ge 2$, note that $E|X|^2 < \infty$, by Lemma 1.4, we have

$$\begin{aligned}
I_{12} &\le C\sum_{n=1}^{\infty} n^{\alpha(p-\gamma)-2+\nu/2}\bigl(\log(4n)\bigr)^{\nu}\int_{n^{\gamma\alpha}}^{\infty} t^{-\nu/\gamma}\,dt \\
&\le C\sum_{n=1}^{\infty} n^{\alpha p-2-(\alpha-1/2)\nu}\bigl(\log(4n)\bigr)^{\nu} < \infty.
\end{aligned}$$

Therefore, $I_1^* < \infty$, so $I_1 < \infty$.

For I_2, we first prove

$$\sup_{t\ge n^{\gamma\alpha}}\left\{t^{-1/\gamma}\sum_{j=1}^{n}EX_j^{(t,2)}\right\} \to 0, \quad n\to\infty. \tag{2.8}$$

When $p > 1$, we have by Lemma 1.4 that

$$\begin{aligned}
&\sup_{t\ge n^{\gamma\alpha}}\left\{t^{-1/\gamma}\sum_{j=1}^{n}EX_j^{(t,2)}\right\} \\
&\quad\le \sup_{t\ge n^{\gamma\alpha}} t^{-1/\gamma}\sum_{j=1}^{n}\bigl\{EX_jI\bigl(X_j > t^{q/\gamma}\bigr) + t^{1/\gamma}P\bigl(X_j > t^{q/\gamma}+t^{1/\gamma}\bigr)\bigr\} \\
&\quad\le \sup_{t\ge n^{\gamma\alpha}} t^{-1/\gamma}\sum_{j=1}^{n}\bigl\{EX_jI\bigl(X_j > t^{q/\gamma}\bigr) + EX_jI\bigl(X_j > t^{q/\gamma}+t^{1/\gamma}\bigr)\bigr\} \\
&\quad\le Cn\sup_{t\ge n^{\gamma\alpha}} t^{-1/\gamma}E|X|I\bigl(|X| > t^{q/\gamma}\bigr) \le Cn^{1-\alpha}E|X|I\bigl(|X| > n^{q\alpha}\bigr) \\
&\quad\le Cn^{1-q\alpha p-\alpha(1-q)}E|X|^p \to 0, \quad n\to\infty.
\end{aligned}$$

When $0 < p \le 1$, we have by Lemma 1.4

$$\begin{aligned}
&\sup_{t\ge n^{\gamma\alpha}}\left\{t^{-1/\gamma}\sum_{j=1}^{n}EX_j^{(t,2)}\right\} \\
&\quad\le \sup_{t\ge n^{\gamma\alpha}} t^{-1/\gamma}\sum_{j=1}^{n}\bigl\{E|X_j|I\bigl(|X_j| \le 2t^{1/\gamma}\bigr) + t^{1/\gamma}P\bigl(|X_j| > 2t^{q/\gamma}\bigr)\bigr\} \\
&\quad\le Cn\sup_{t\ge n^{\gamma\alpha}} t^{-1/\gamma}\bigl\{E|X|I\bigl(|X| \le 2t^{1/\gamma}\bigr) + 2t^{1/\gamma}P\bigl(|X| > 2t^{1/\gamma}\bigr) + t^{1/\gamma}P\bigl(|X| > 2t^{q/\gamma}\bigr)\bigr\}
\end{aligned}$$

$$
\begin{aligned}
&\leq Cn \sup_{t\geq n^{\gamma\alpha}} \left\{ t^{-p/\gamma} E|X|^p + t^{-pq/\gamma} E|X|^p \right\} \\
&\leq Cn^{1-\alpha pq} \to 0, \quad n \to \infty.
\end{aligned}
$$

Therefore (2.8) holds. By (2.8), in order to prove $I_2 < \infty$, it is enough to show that

$$
I_2^* := \sum_{n=1}^{\infty} n^{\alpha(p-\gamma)-2} \int_{n^{\gamma\alpha}}^{\infty} P\left(\sum_{j=1}^{n} \left(X_j^{(t,2)} - EX_j^{(t,2)} \right) > t^{1/\gamma}/10 \right) dt < \infty.
$$

Fix any $\nu \geq 2$ (to be specified later), by Markov's inequality, Lemma 1.1, Lemma 1.2, C_r-inequality, Jensen's inequality, and Lemma 1.4, we have

$$
\begin{aligned}
I_2^* &\leq C \sum_{n=1}^{\infty} n^{\alpha(p-\gamma)-2} \int_{n^{\gamma\alpha}}^{\infty} t^{-\nu/\gamma} \left\{ \sum_{j=1}^{n} E\left|X_j^{(t,2)}\right|^{\nu} + \left(\sum_{j=1}^{n} E\left(X_j^{(t,2)}\right)^2 \right)^{\nu/2} dt \right\} \\
&\leq C \sum_{n=1}^{\infty} n^{\alpha(p-\gamma)-2} \int_{n^{\gamma\alpha}}^{\infty} t^{-\nu/\gamma} \sum_{j=1}^{n} \left\{ E|X_j|^{\nu} I\left(|X_j| \leq 2t^{1/\gamma}\right) + t^{\nu/\gamma} P\left(X_j > t^{1/\gamma}\right) \right\} dt \\
&\quad + C \sum_{n=1}^{\infty} n^{\alpha(p-\gamma)-2} \int_{n^{\gamma\alpha}}^{\infty} t^{-\nu/\gamma} \left\{ \sum_{j=1}^{n} E\left(X_j^2 I\left(|X_j| \leq 2t^{1/\gamma}\right) + t^{2/\gamma} P\left(X_j > t^{1/\gamma}\right) \right) \right\}^{\nu/2} dt \\
&\leq C \sum_{n=1}^{\infty} n^{\alpha(p-\gamma)-1} \int_{n^{\gamma\alpha}}^{\infty} t^{-\nu/\gamma} E|X|^{\nu} I\left(|X| \leq 2t^{1/\gamma}\right) dt \\
&\quad + C \sum_{n=1}^{\infty} n^{\alpha(p-\gamma)-1} \int_{n^{\gamma\alpha}}^{\infty} P\left(|X| > t^{1/\gamma}\right) dt \\
&\quad + C \sum_{n=1}^{\infty} n^{\alpha(p-\gamma)-2+\nu/2} \int_{n^{\gamma\alpha}}^{\infty} t^{-\nu/\gamma} \left\{ E|X|^2 I\left(|X| \leq 2t^{1/\gamma}\right) \right\}^{\nu/2} dt \\
&\quad + C \sum_{n=1}^{\infty} n^{\alpha(p-\gamma)-2+\nu/2} \int_{n^{\gamma\alpha}}^{\infty} \left(P\left(|X| > t^{1/\gamma}\right) \right)^{\nu/2} dt \\
&\stackrel{\text{def}}{=} I_{21} + I_{22} + I_{23} + I_{24}.
\end{aligned}
$$

We get by the mean-value theorem and a standard computation

$$
\begin{aligned}
I_{22} &= C \sum_{n=1}^{\infty} n^{\alpha(p-\gamma)-1} \sum_{j=n}^{\infty} \int_{j^{\gamma\alpha}}^{(j+1)^{\gamma\alpha}} P\left(|X| > t^{1/\gamma}\right) dt \\
&\leq C \sum_{n=1}^{\infty} n^{\alpha(p-\gamma)-1} \sum_{j=n}^{\infty} j^{\gamma\alpha-1} P\left(|X| > j^{\alpha}\right) \\
&= C \sum_{j=1}^{\infty} j^{\gamma\alpha-1} P\left(|X| > j^{\alpha}\right) \sum_{n=1}^{j} n^{\alpha(p-\gamma)-1} \\
&\leq \begin{cases} C \sum_{j=1}^{\infty} j^{\alpha p-1} P(|X| > j^{\alpha}), & \gamma < p, \\ C \sum_{j=1}^{\infty} j^{\alpha p-1} \log j P(|X| > j^{\alpha}), & \gamma = p, \\ \sum_{j=1}^{\infty} j^{\gamma\alpha-1} P(|X| > j^{\alpha}), & \gamma > p \end{cases}
\end{aligned}
$$

$$\le \begin{cases} CE|X|^p, & \gamma < p, \\ CE|X|^p \log(1+|X|), & \gamma = p, \\ CE|X|^\gamma, & \gamma > p \end{cases}$$
$$< \infty.$$

When $\max\{p,\gamma\} < 2$, let $\nu = 2$. We have $I_{24} = I_{22} < \infty$ and

$$\begin{aligned} I_{21} &= I_{23} = C\sum_{n=1}^{\infty} n^{\alpha(p-\gamma)-1} \sum_{j=n}^{\infty} \int_{j^{\gamma\alpha}}^{(j+1)^{\gamma\alpha}} t^{-2/\gamma} E|X|^2 I\bigl(|X| \le 2t^{1/\gamma}\bigr)\,dt \\ &\le C\sum_{n=1}^{\infty} n^{\alpha(p-\gamma)-1} \sum_{j=n}^{\infty} j^{(-2+\gamma)\alpha-1} E|X|^2 I\bigl(|X| \le 2(j+1)^\alpha\bigr) \\ &= C\sum_{j=1}^{\infty} j^{(-2+\gamma)\alpha-1} E|X|^2 I\bigl(|X| \le 2(j+1)^\alpha\bigr) \sum_{n=1}^{j} n^{\alpha(p-\gamma)-1} \\ &\le \begin{cases} C\sum_{j=1}^\infty j^{\alpha(p-2)-1} E|X|^2 I(|X| \le 2(j+1)^\alpha), & \gamma < p, \\ C\sum_{j=1}^\infty j^{\alpha(p-2)-1} \log j E|X|^2 I(|X| \le 2(j+1)^\alpha), & \gamma = p, \\ C\sum_{j=1}^\infty j^{\alpha(\gamma-2)-1} E|X|^2 I(|X| \le 2(j+1)^\alpha), & \gamma > p \end{cases} \\ &\le \begin{cases} CE|X|^p, & \gamma < p, \\ CE|X|^p \log(1+|X|), & \gamma = p, \\ CE|X|^\gamma, & \gamma > p \end{cases} \\ &< \infty. \end{aligned} \tag{2.9}$$

When $\max\{p,\gamma\} \ge 2$, let $\nu > \max\{\gamma, (\alpha p-1)/(\alpha - 1/2)\}$. Note that $E|X|^2 < \infty$

$$I_{23} \le C\sum_{n=1}^{\infty} n^{\alpha(p-\gamma)-2+\nu/2} \int_{n^{\gamma\alpha}}^{\infty} t^{-\nu/\gamma}\,dt = C\sum_{n=1}^{\infty} n^{\alpha p-2-(\alpha-1/2)\nu} < \infty,$$

and by the Markov inequality, we have

$$I_{24} \le C\sum_{n=1}^{\infty} n^{\alpha(p-\gamma)-2+\nu/2} \int_{n^{\gamma\alpha}}^{\infty} t^{-\nu/\gamma}\,dt < \infty.$$

The proof of $I_{21} < \infty$ is similar to that (2.9), so it is omitted. Therefore, $I_2^* < \infty$, so $I_2 < \infty$.

For I_3, we get

$$\begin{aligned} I_3 &\le \sum_{n=1}^{\infty} n^{\alpha(p-\gamma)-2} \int_{n^{\gamma\alpha}}^{\infty} P\left\{\bigcup_{j=1}^{n} \bigl(X_j^{(t,3)} > 0\bigr)\right\} dt \\ &= \sum_{n=1}^{\infty} n^{\alpha(p-\gamma)-2} \int_{n^{\gamma\alpha}}^{\infty} \sum_{j=1}^{n} P\bigl(X_j > t^{1/\gamma} + t^{q/\gamma}\bigr)\,dt \\ &\le C\sum_{n=1}^{\infty} n^{\alpha(p-\gamma)-1} \int_{n^{\gamma\alpha}}^{\infty} P\bigl(|X| > t^{1/\gamma}\bigr)\,dt = CI_{22} < \infty. \end{aligned}$$

By similar proofs to $I_2 < \infty$ and $I_3 < \infty$, we have $I_4 < \infty$ and $I_5 < \infty$, respectively. Therefore, (2.2) holds.

Equation (2.2) $\Rightarrow$ (2.3). Note that $|S_n^{(k)}| = |S_n - X_k| \le |S_n| + |X_k| = |S_n| + |S_k - S_{k-1}| \le |S_n| + |S_k| + |S_{k-1}| \le 3\max_{1\le j\le n}|S_j|$, $\forall 1 \le k \le n$, hence

$$\begin{aligned}
&\sum_{n=1}^{\infty} n^{\alpha(p-\gamma)-2} E\Big\{\max_{1\le k\le n}\big|S_n^{(k)}\big| - \varepsilon n^{\alpha}\Big\}_+^{\gamma} \\
&\quad= \sum_{n=1}^{\infty} n^{\alpha(p-\gamma)-2}\int_0^{\infty} P\Big(\max_{1\le k\le n}\big|S_n^{(k)}\big| - \varepsilon n^{\alpha} > t^{1/\gamma}\Big)\,dt \\
&\quad\le \sum_{n=1}^{\infty} n^{\alpha(p-\gamma)-2}\int_0^{\infty} P\Big(\max_{1\le k\le n}|S_n| > \varepsilon n^{\alpha}/3 + t^{1/\gamma}/3\Big)\,dt \\
&\quad= 3^{\gamma}\sum_{n=1}^{\infty} n^{\alpha(p-\gamma)-2}\int_0^{\infty} P\Big(\max_{1\le k\le n}|S_n| > \varepsilon n^{\alpha}/3 + t^{1/\gamma}\Big)\,dt \\
&\quad= 3^{\gamma}\sum_{n=1}^{\infty} n^{\alpha(p-\gamma)-2} E\Big\{\max_{1\le k\le n}|S_n| - \varepsilon n^{\alpha}/3\Big\}^{\gamma} < \infty.
\end{aligned} \tag{2.10}$$

Equation (2.3) holds.

Equation (2.3) $\Rightarrow$ (2.4). Since $\frac{1}{2}|S_n| \le \frac{n-1}{n}|S_n| = |\frac{1}{n}\sum_{k=1}^{n} S_n^{(k)}| \le \max_{1\le k\le n}|S_n^{(k)}|$, $\forall n \ge 2$, and $|X_k| = |S_n - S_n^{(k)}| \le |S_n| + |S_n^{(k)}| \le 3\max_{1\le k\le n}|S_n^{(k)}|$, we have (2.4) by a similar argument to (2.10).

Equation (2.2) $\Rightarrow$ (2.5). The proof of (2.2) $\Rightarrow$ (2.5) is similar to that (1.6) $\Rightarrow$ (1.7) of Chen and Wang [19], so it is omitted.

Equation (2.5) $\Rightarrow$ (2.6). Since $k^{-\alpha}|X_k| = k^{-\alpha}|S_k - S_{k-1}| \le k^{-\alpha}(|S_k| + |S_{k-1}|) \le \sup_{j\ge k} j^{-\alpha} \times (|S_j| + |S_{j-1}|) \le 2\sup_{j\ge k-1} j^{-\alpha}|S_j|$, $\forall k \ge 2$, we have (2.6) by the similar argument of (2.10). □

Theorem 2.2 *Let $\gamma > 0$, $\alpha > 1/2$, $p > 0$, $\alpha p > 1$. Let $\{X_n, n \ge 1\}$ be a sequence of NOD random variables and X be a random variables possibly defined on a different space. Moreover, assume that $EX_n = 0$ for all $n \ge 1$ when $\alpha \le 1$. If there exist constants $D_1 > 0$ and $D_2 > 0$ such that*

$$\frac{D_1}{n}\sum_{i=n}^{2n-1} P\big(|X_i| > x\big) \le P\big(|X| > x\big) \le \frac{D_2}{n}\sum_{i=n}^{2n-1} P\big(|X_i| > x\big), \quad \forall x > 0, n \ge 1.$$

Then (2.1)-(2.6) *are equivalent.*

Proof By Theorem 2.1, in order to prove Theorem 2.2, it is enough to show that (2.4) $\Rightarrow$ (2.1) and (2.6) $\Rightarrow$ (2.1). We only prove (2.4) $\Rightarrow$ (2.1), the proof of (2.6) $\Rightarrow$ (2.1) is similar and omitted. Note that

$$\begin{aligned}
\infty &> \sum_{n=1}^{\infty} n^{\alpha(p-\gamma)-2} E\Big\{\max_{1\le k\le n}|X_k| - \varepsilon n^{\alpha}\Big\}_+^{\gamma} \\
&= \sum_{n=1}^{\infty} n^{\alpha(p-\gamma)-2}\int_0^{\infty} P\Big(\max_{1\le k\le n}|X_k| - \varepsilon n^{\alpha} > t^{1/\gamma}\Big)\,dt
\end{aligned}$$

$$\geq \sum_{n=1}^{\infty} n^{\alpha(p-\gamma)-2} \int_0^{\varepsilon^\gamma n^{\alpha\gamma}} P\Big(\max_{1\le k\le n} |X_k| > \varepsilon n^\alpha + t^{1/\gamma}\Big)\, dt$$

$$\geq \gamma^\alpha \sum_{n=1}^{\infty} n^{\alpha p-2} P\Big(\max_{1\le k\le n} |X_k| > 2\varepsilon n^\alpha\Big)$$

by Theorem 2.2 of Qiu *et al.* [11], the proof of (2.4) ⇒ (2.1) is completed. □

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
All authors contributed equally to the writing of this paper. All authors read and approved the final manuscript.

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