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Two kinds of Hilbert-type integral inequalities in the whole plane

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Abstract
By the way of using real analysis and estimating the weight functions, two kinds of Hilbert-type integral inequalities in the whole plane with a non-homogeneous kernel and a homogeneous kernel are given. The constant factor related to the triangle functions is proved to be the best possible. We also consider the equivalent forms, the reverses, some particular cases, and two kinds of operator expressions.
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1 Introduction

Assuming that $f(x), g(y) \geq 0$, $0 < \int_0^\infty f^2(x)\,dx < \infty$ and $0 < \int_0^\infty g^2(y)\,dy < \infty$, we have the following Hilbert integral inequality (*cf.* [1]):

$$\int_0^\infty \int_0^\infty \frac{f(x)g(y)}{x+y}\,dx\,dy < \pi \left(\int_0^\infty f^2(x)\,dx \int_0^\infty g^2(y)\,dy \right)^{\frac{1}{2}}, \tag{1}$$

where the constant factor π is the best possible. Inequality (1) together with the discrete form is important in analysis and its applications (*cf.* [1, 2]). In recent years, applying weight functions and introducing parameters, many extensions of (1) were given by Yang (*cf.* [3]). Noticing that inequality (1) is a homogeneous kernel of degree –1, in 2009, a survey of the study of Hilbert-type inequalities with the homogeneous kernels of negative number degrees was given by [4]. Recently, some inequalities with the homogeneous kernels of degree 0 and non-homogeneous kernels were studied (*cf.* [5–10]). The other kinds of Hilbert-type inequalities were provided by [11–15]. All of the above integral inequalities are built in the quarter plane of the first quadrant.

In 2007, Yang [16] first gave a Hilbert-type integral inequality in the whole plane as follows:

$$\int_{-\infty}^\infty \int_{-\infty}^\infty \frac{f(x)g(y)}{(1+e^{x+y})^\lambda}\,dx\,dy < B\left(\frac{\lambda}{2}, \frac{\lambda}{2}\right) \left(\int_{-\infty}^\infty e^{-\lambda x} f^2(x)\,dx \int_{-\infty}^\infty e^{-\lambda y} g^2(y)\,dy \right)^{\frac{1}{2}}, \tag{2}$$

where the constant factor $B(\frac{\lambda}{2},\frac{\lambda}{2})$ ($\lambda > 0$) is the best possible, and $B(u,v)$ ($u,v > 0$) is the beta function (*cf.* [17]). He *et al.* [18–28] also provided some Hilbert-type integral inequalities in the whole plane.

In this paper, by the way of using real analysis and estimating the weight functions, two kinds of Hilbert-type integral inequalities in the whole plane with a non-homogeneous kernel and a homogeneous kernel are given. The constant factor related to the triangle functions is proved to be the best possible. We also consider the equivalent forms, the reverses, some particular cases, and two kinds of operator expressions.

2 Some lemmas

In the following, we use the following formula (*cf.* [1]):

$$\int_0^\infty \frac{(\ln t)t^{a-1}}{t-1}\,dt = \bigl[B(1-a,a)\bigr]^2 = \left[\frac{\pi}{\sin a\pi}\right]^2 \quad (0 < a < 1). \tag{3}$$

Lemma 1 *If* $0 < \alpha_1 \le \alpha_2 < \pi$, $\mu,\sigma > 0$, $\mu+\sigma = \lambda$, $\gamma \in \{a; a = \frac{1}{2k+1}, 2k-1\ (k \in \mathbf{N} = \{1,2,\ldots\})\}$, $\delta \in \{-1,1\}$, *we define two weight functions* $\omega(\sigma,y)$ *and* $\varpi(\sigma,x)$ ($y,x \in \mathbf{R} = (-\infty,\infty)$) *as follows*:

$$\omega(\sigma,y) := \int_{-\infty}^\infty \max_{i\in\{1,2\}} \frac{\ln[|x^\delta y|^\gamma + (x^\delta y)^\gamma \cos\alpha_i]}{[|x^\delta y|^\gamma + (x^\delta y)^\gamma \cos\alpha_i]^{\lambda/\gamma} - 1} \frac{|y|^\sigma}{|x|^{1-\delta\sigma}}\,dx, \tag{4}$$

$$\varpi(\sigma,x) := \int_{-\infty}^\infty \max_{i\in\{1,2\}} \frac{\ln[|x^\delta y|^\gamma + (x^\delta y)^\gamma \cos\alpha_i]}{[|x^\delta y|^\gamma + (x^\delta y)^\gamma \cos\alpha_i]^{\lambda/\gamma} - 1} \frac{|x|^{\delta\sigma}}{|y|^{1-\sigma}}\,dy. \tag{5}$$

Then for $y,x \in \mathbf{R}\backslash\{0\}$, *we have*

$$\begin{aligned} &\omega(\sigma,y) = \varpi(\sigma,x) = K(\sigma), \\ &K(\sigma) := \frac{\gamma}{2^{\frac{\sigma}{\gamma}}}\left[\left(\sec\frac{\alpha_2}{2}\right)^{\frac{2\sigma}{\gamma}} + \left(\csc\frac{\alpha_1}{2}\right)^{\frac{2\sigma}{\gamma}}\right]\left[\frac{\pi}{\lambda\sin(\frac{\pi\sigma}{\lambda})}\right]^2 \in \mathbf{R}_+. \end{aligned} \tag{6}$$

Proof Setting $u = x^\delta y$ in (4), for $y \in \mathbf{R}\backslash\{0\}$, we find $x = y^{\frac{-1}{\delta}} u^{\frac{1}{\delta}}$, $dx = \frac{1}{\delta} y^{\frac{-1}{\delta}} u^{\frac{1}{\delta}-1}\,du$, and

$$\begin{aligned} \omega(\sigma,y) &= \left|\frac{1}{\delta}\right| \int_{-\infty}^\infty \max_{i\in\{1,2\}} \frac{\ln(|u|^\gamma + u^\gamma\cos\alpha_i)}{(|u|^\gamma + u^\gamma\cos\alpha_i)^{\lambda/\gamma} - 1} |u|^{\sigma-1}\,du \\ &= \int_0^\infty \max_{i\in\{1,2\}} \frac{\ln[u^\gamma(1+\cos\alpha_i)]}{u^\lambda(1+\cos\alpha_i)^{\lambda/\gamma} - 1} u^{\sigma-1}\,du \\ &\quad + \int_{-\infty}^0 \max_{i\in\{1,2\}} \frac{\ln[(-u)^\gamma(1-\cos\alpha_i)]}{(-u)^\lambda(1-\cos\alpha_i)^{\lambda/\gamma} - 1} (-u)^{\sigma-1}\,du \\ &= \gamma\left\{\int_0^\infty \max_{i\in\{1,2\}} \frac{\ln[u(1+\cos\alpha_i)^{1/\gamma}]}{[u(1+\cos\alpha_i)^{1/\gamma}]^\lambda - 1} u^{\sigma-1}\,du \right. \\ &\quad \left. + \int_0^\infty \max_{i\in\{1,2\}} \frac{\ln[u(1-\cos\alpha_i)^{1/\gamma}]}{[u(1-\cos\alpha_i)^{1/\gamma}]^\lambda - 1} u^{\sigma-1}\,du\right\}. \end{aligned}$$

In view of the function $\frac{\ln t}{t^\lambda - 1}$ being strictly decreasing in $\mathbf{R}_+$ (*cf.* [3]), we have

$$\omega(\sigma, y) = \int_0^\infty \frac{\ln[u^\gamma(1+\cos\alpha_2)]}{[u(1+\cos\alpha_2)^{1/\gamma}]^\lambda - 1} u^{\sigma-1}\, du + \int_0^\infty \frac{\ln[u^\gamma(1-\cos\alpha_1)]}{[u(1-\cos\alpha_1)^{1/\gamma}]^\lambda - 1} u^{\sigma-1}\, du. \tag{7}$$

Setting $t = [u(1+\cos\alpha_2)^{1/\gamma}]^\lambda$ ($t = [u(1-\cos\alpha_1)^{1/\gamma}]^\lambda$) in the above first (second) integral, by calculations and (3), it follows that

$$\omega(\sigma, y) = \frac{\gamma}{\lambda^2}\left[\left(\frac{\sec^2\frac{\alpha_2}{2}}{2}\right)^{\frac{\sigma}{\gamma}} + \left(\frac{\csc^2\frac{\alpha_1}{2}}{2}\right)^{\frac{\sigma}{\gamma}}\right]\int_0^\infty \frac{\ln t}{t-1} t^{\frac{\sigma}{\lambda}-1}\, dt = K(\sigma).$$

Setting $u = x^\delta y$ in (5), for $x \in \mathbf{R}\backslash\{0\}$, we find $y = x^{-\delta}u$, $dy = x^{-\delta}\, du$ and

$$\varpi(\sigma, x) = \int_{-\infty}^\infty \max_{i\in\{1,2\}} \frac{\ln(|u|^\gamma + u^\gamma\cos\alpha_i)|u|^{\sigma-1}}{(|u|^\gamma + u^\gamma\cos\alpha_i)^{\lambda/\gamma} - 1}\, du = K(\sigma).$$

Hence we have (6). □

Remark 1 If we replace $\max_{i\in\{1,2\}}$ by $\min_{i\in\{1,2\}}$ in (4) and (5), then we must exchange α_1 and α_2 in (6).

Lemma 2 *If* $p > 1$, $\frac{1}{p} + \frac{1}{q} = 1$, $0 < \alpha_1 \le \alpha_2 < \pi$, $\mu, \sigma > 0$, $\mu + \sigma = \lambda$, $\gamma \in \{a; a = \frac{1}{2k+1}, 2k-1\ (k \in \mathbf{N})\}$, $\delta \in \{-1, 1\}$, $K(\sigma)$ *is indicated by* (6), $f(x)$ *is a non-negative measurable function in* $\mathbf{R}$, *then we have*

$$J := \int_{-\infty}^\infty |y|^{p\sigma-1}\left\{\int_{-\infty}^\infty \max_{i\in\{1,2\}} \frac{\ln[|x^\delta y|^\gamma + (x^\delta y)^\gamma\cos\alpha_i]f(x)}{[|x^\delta y|^\gamma + (x^\delta y)^\gamma\cos\alpha_i]^{\lambda/\gamma} - 1}\, dx\right\}^p dy \le K^p(\sigma)\int_{-\infty}^\infty |x|^{p(1-\delta\sigma)-1} f^p(x)\, dx. \tag{8}$$

Proof For simplifying in the following, we set

$$h^{(\delta)}(x, y) := \max_{i\in\{1,2\}} \frac{\ln[|x^\delta y|^\gamma + (x^\delta y)^\gamma\cos\alpha_i]}{[|x^\delta y|^\gamma + (x^\delta y)^\gamma\cos\alpha_i]^{\lambda/\gamma} - 1} \quad (x, y \in \mathbf{R}). \tag{9}$$

By Hölder's inequality (*cf.* [29]), we have

$$\begin{aligned}
&\left(\int_{-\infty}^\infty h^{(\delta)}(x, y) f(x)\, dx\right)^p \\
&= \left\{\int_{-\infty}^\infty h^{(\delta)}(x, y)\left[\frac{|x|^{(1-\delta\sigma)/q}}{|y|^{(1-\sigma)/p}} f(x)\right]\left[\frac{|y|^{(1-\sigma)/p}}{|x|^{(1-\delta\sigma)/q}}\right] dx\right\}^p \\
&\le \int_{-\infty}^\infty h^{(\delta)}(x, y)\frac{|x|^{(1-\delta\sigma)(p-1)}}{|y|^{1-\sigma}} f^p(x)\, dx\left[\int_{-\infty}^\infty h^{(\delta)}(x, y)\frac{|y|^{(1-\sigma)(q-1)}}{|x|^{1-\delta\sigma}}\, dx\right]^{p-1} \\
&= \big(\omega(\sigma, y)\big)^{p-1}|y|^{-p\sigma+1}\int_{-\infty}^\infty h^{(\delta)}(x, y)\frac{|x|^{(1-\delta\sigma)(p-1)}}{|y|^{1-\sigma}} f^p(x)\, dx.
\end{aligned} \tag{10}$$

Then by (6) and the Fubini theorem (*cf.* [30]), it follows that

$$J \le K^{p-1}(\sigma)\int_{-\infty}^{\infty}\left[\int_{-\infty}^{\infty} h^{(\delta)}(x,y)\frac{|x|^{(1-\delta\sigma)(p-1)}}{|y|^{1-\sigma}} f^p(x)\,dx\right]dy$$
$$= K^{p-1}(\sigma)\int_{-\infty}^{\infty}\varpi(\sigma,x)|x|^{p(1-\delta\sigma)-1}f^p(x)\,dx.$$

Hence, still in view of (6), inequality (8) follows. □

3 Main results and applications

Theorem 3 *If* $p > 1$, $\frac{1}{p}+\frac{1}{q}=1$, $0<\alpha_1\le\alpha_2<\pi$, $\mu,\sigma>0$, $\mu+\sigma=\lambda$, $\gamma\in\{a;a=\frac{1}{2k+1},2k-1\ (k\in\mathbf{N})\}$, $\delta\in\{-1,1\}$, $K(\sigma)$ *is indicated by* (6), $f(x),g(y)\ge 0$, *satisfying* $0<\int_{-\infty}^{\infty}|x|^{p(1-\delta\sigma)-1}f^p(x)\,dx<\infty$ *and* $0<\int_{-\infty}^{\infty}|y|^{q(1-\sigma)-1}g^q(y)\,dy<\infty$, *then we have*

$$I := \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\max_{i\in\{1,2\}}\frac{\ln[|x^\delta y|^\gamma + (x^\delta y)^\gamma\cos\alpha_i]}{[|x^\delta y|^\gamma + (x^\delta y)^\gamma\cos\alpha_i]^{\lambda/\gamma}-1} f(x)g(y)\,dx\,dy$$
$$< K(\sigma)\left[\int_{-\infty}^{\infty}|x|^{p(1-\delta\sigma)-1}f^p(x)\,dx\right]^{\frac{1}{p}}\left[\int_{-\infty}^{\infty}|y|^{q(1-\sigma)-1}g^q(y)\,dy\right]^{\frac{1}{q}}, \tag{11}$$
$$J := \int_{-\infty}^{\infty}|y|^{p\sigma-1}\left\{\int_{-\infty}^{\infty}\max_{i\in\{1,2\}}\frac{\ln[|x^\delta y|^\gamma + (x^\delta y)^\gamma\cos\alpha_i]}{[|x^\delta y|^\gamma + (x^\delta y)^\gamma\cos\alpha_i]^{\lambda/\gamma}-1} f(x)\,dx\right\}^p dy$$
$$< K^p(\sigma)\int_{-\infty}^{\infty}|x|^{p(1-\delta\sigma)-1}f^p(x)\,dx, \tag{12}$$

where the constant factors $K(\sigma)$ *and* $K^p(\sigma)$ *are the best possible. Inequalities* (11) *and* (12) *are equivalent.*

In particular, for $\alpha_1=\alpha_2=\alpha\in(0,\pi)$, $\gamma=1$ in (11) and (12), we find

$$K(\sigma)=k(\sigma):=\frac{1}{2^\sigma}\left[\left(\sec\frac{\alpha}{2}\right)^{2\sigma}+\left(\csc\frac{\alpha}{2}\right)^{2\sigma}\right]\left[\frac{\pi}{\lambda\sin(\frac{\pi\sigma}{\lambda})}\right]^2, \tag{13}$$

and the following equivalent inequalities:

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{\ln(|x^\delta y|+x^\delta y\cos\alpha)}{(|x^\delta y|+x^\delta y\cos\alpha)^\lambda-1}f(x)g(y)\,dx\,dy$$
$$< k(\sigma)\left[\int_{-\infty}^{\infty}|x|^{p(1-\delta\sigma)-1}f^p(x)\,dx\right]^{\frac{1}{p}}\left[\int_{-\infty}^{\infty}|y|^{q(1-\sigma)-1}g^q(y)\,dy\right]^{\frac{1}{q}}, \tag{14}$$
$$\int_{-\infty}^{\infty}|y|^{p\sigma-1}\left[\int_{-\infty}^{\infty}\frac{\ln(|x^\delta y|+x^\delta y\cos\alpha)}{(|x^\delta y|+x^\delta y\cos\alpha)^\lambda-1}f(x)\,dx\right]^p dy$$
$$< k^p(\sigma)\int_{-\infty}^{\infty}|x|^{p(1-\sigma)-1}f^p(x)\,dx. \tag{15}$$

Proof If (10) takes the form of an equality for a $y\neq 0$, then there exist constants A and B, such that they are not all zero, and

$$A\frac{|x|^{(1-\delta\sigma)(p-1)}}{|y|^{1-\sigma}}f^p(x) = B\frac{|y|^{(1-\sigma)(q-1)}}{|x|^{1-\delta\sigma}} \quad \text{a.e. in } \mathbf{R}.$$

We suppose that $A \neq 0$ (otherwise $B = A = 0$). Then it follows that

$$|x|^{p(1-\delta\sigma)-1} f^p(x) = |y|^{q(1-\sigma)} \frac{B}{A|x|} \quad \text{a.e. in } \mathbf{R},$$

which contradicts the fact that $0 < \int_{-\infty}^{\infty} |x|^{p(1-\delta\sigma)-1} f^p(x)\,dx < \infty$. Hence (10) takes the form of a strict inequality. So does (8), and we have (12).

By the Hölder inequality (*cf.* [29]), we find

$$\begin{aligned} I &= \int_{-\infty}^{\infty} \left(|y|^{\sigma-\frac{1}{p}} \int_{-\infty}^{\infty} h^{(\delta)}(x,y) f(x)\,dx \right) \left(|y|^{\frac{1}{p}-\sigma} g(y) \right) dy \\ &\leq J^{\frac{1}{p}} \left[\int_{-\infty}^{\infty} |y|^{q(1-\sigma)-1} g^q(y)\,dy \right]^{\frac{1}{q}}. \end{aligned} \tag{16}$$

Then by (12), we have (11). On the other hand, suppose that (11) is valid. Setting

$$g(y) := |y|^{p\sigma-1} \left(\int_{-\infty}^{\infty} h^{(\delta)}(x,y) f(x)\,dx \right)^{p-1} \quad (y \in \mathbf{R}),$$

then it follows that $J = \int_{-\infty}^{\infty} |y|^{q(1-\sigma)-1} g^q(y)\,dy$. By (8), we have $J < \infty$. If $J = 0$, then (12) is trivially true; if $0 < J < \infty$, then by (11), we obtain

$$\begin{aligned} 0 &< \int_{-\infty}^{\infty} |y|^{q(1-\sigma)-1} g^q(y)\,dy = J = I \\ &< K(\sigma) \left[\int_{-\infty}^{\infty} |x|^{p(1-\delta\sigma)-1} f^p(x)\,dx \right]^{\frac{1}{p}} \left[\int_{-\infty}^{\infty} |y|^{q(1-\sigma)-1} g^q(y)\,dy \right]^{\frac{1}{q}} < \infty, \end{aligned} \tag{17}$$

$$J^{\frac{1}{p}} = \left[\int_{-\infty}^{\infty} |y|^{q(1-\sigma)-1} g^q(y)\,dy \right]^{\frac{1}{p}} < K(\sigma) \left[\int_{-\infty}^{\infty} |x|^{p(1-\delta\sigma)-1} f^p(x)\,dx \right]^{\frac{1}{p}}. \tag{18}$$

Hence we have (12), which is equivalent to (11).

We set $E_\delta := \{x \in \mathbf{R}; |x|^\delta \geq 1\}$, and $E_\delta^+ := E_\delta \cap \mathbf{R}_+ = \{x \in \mathbf{R}_+; x^\delta \geq 1\}$. For $\varepsilon > 0$, we define two functions $\tilde{f}(x)$, $\tilde{g}(y)$ as follows:

$$\tilde{f}(x) := \begin{cases} |x|^{\delta(\sigma-\frac{2\varepsilon}{p})-1}, & x \in E_\delta, \\ 0, & x \in \mathbf{R} \backslash E_\delta, \end{cases} \qquad \tilde{g}(y) := \begin{cases} 0, & y \in (-\infty,-1) \cup (1,\infty), \\ |y|^{\sigma+\frac{2\varepsilon}{q}-1}, & y \in [-1,1]. \end{cases}$$

Then we obtain

$$\begin{aligned} \tilde{L} &:= \left[\int_{-\infty}^{\infty} |x|^{p(1-\delta\sigma)-1} \tilde{f}^p(x)\,dx \right]^{\frac{1}{p}} \left[\int_{-\infty}^{\infty} |y|^{q(1-\sigma)-1} \tilde{g}^q(y)\,dy \right]^{\frac{1}{q}} \\ &= 2 \left(\int_{E_\delta^+} x^{-2\delta\varepsilon-1}\,dx \right)^{\frac{1}{p}} \left(\int_0^1 y^{2\varepsilon-1}\,dy \right)^{\frac{1}{q}} = \frac{1}{\varepsilon}. \end{aligned}$$

We find

$$I(x) := \int_{-1}^{1} \max_{i\in\{1,2\}} \frac{\ln[|x^\delta y|^\gamma + (x^\delta y)^\gamma \cos\alpha_i] |y|^{\sigma+\frac{2\varepsilon}{q}-1}}{[|x^\delta y|^\gamma + (x^\delta y)^\gamma \cos\alpha_i]^{\lambda/\gamma} - 1}\,dy = I(-x),$$

and then $I(x)$ is an even function. In fact, setting $Y = -y$, we obtain

$$\begin{aligned} I(-x) &= \int_{-1}^{1} \max_{i\in\{1,2\}} \frac{\ln[|(-x)^\delta y|^\gamma + ((-x)^\delta y)^\gamma \cos\alpha_i]|y|^{\sigma+\frac{2\varepsilon}{q}-1}}{[|(-x)^\delta y|^\gamma + ((-x)^\delta y)^\gamma \cos\alpha_i]^{\lambda/\gamma} - 1}\, dy \\ &= \int_{-1}^{1} \max_{i\in\{1,2\}} \frac{\ln[|x^\delta Y|^\gamma + (x^\delta Y)^\gamma \cos\alpha_i]|Y|^{\sigma+\frac{2\varepsilon}{q}-1}}{[|x^\delta Y|^\gamma + (x^\delta Y)^\gamma \cos\alpha_i]^{\lambda/\gamma} - 1}\, dY = I(x). \end{aligned}$$

It follows that

$$\begin{aligned} \tilde{I} &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} h^{(\delta)}(x,y)\tilde{f}(x)\tilde{g}(y)\,dx\,dy \\ &= \int_{E_\delta} |x|^{\delta(\sigma-\frac{2\varepsilon}{p})-1} I(x)\,dx = 2\int_{E_\delta^+} x^{\delta(\sigma-\frac{2\varepsilon}{p})-1} I(x)\,dx \\ &\overset{u=x^\delta y}{=} 2\int_{E_\delta^+} x^{-2\delta\varepsilon-1}\left[\int_{-x^\delta}^{x^\delta} \max_{i\in\{1,2\}} \frac{\ln(|u|^\gamma + u^\gamma\cos\alpha_i)|u|^{\sigma+\frac{2\varepsilon}{q}-1}}{(|u|^\gamma + u^\gamma\cos\alpha_i)^{\lambda/\gamma} - 1}\,du\right]dx. \end{aligned}$$

Setting $v = x^\delta$ in the above integral, by the Fubini theorem (*cf.* [30]), we find

$$\begin{aligned} \tilde{I} &= 2\int_1^\infty v^{-2\varepsilon-1}\left[\int_{-v}^{v} \max_{i\in\{1,2\}} \frac{\ln(|u|^\gamma + u^\gamma\cos\alpha_i)|u|^{\sigma+\frac{2\varepsilon}{q}-1}}{(|u|^\gamma + u^\gamma\cos\alpha_i)^{\lambda/\gamma} - 1}\,du\right]dv \\ &= 2\int_1^\infty v^{-2\varepsilon-1}\left\{\int_0^v \left[\max_{i\in\{1,2\}} \frac{\ln[u^\gamma(1+\cos\alpha_i)]}{[u(1+\cos\alpha_i)^{1/\gamma}]^\lambda - 1}\right.\right. \\ &\quad \left.\left. + \max_{i\in\{1,2\}} \frac{\ln[u^\gamma(1-\cos\alpha_i)]}{[u(1-\cos\alpha_i)^{1/\gamma}]^\lambda - 1}\right] u^{\sigma+\frac{2\varepsilon}{q}-1}\,du\right\}dv \\ &= 2\int_1^\infty v^{-2\varepsilon-1}\left\{\int_0^v \left[\frac{\ln[u^\gamma(1+\cos\alpha_2)]}{[u(1+\cos\alpha_2)^{1/\gamma}]^\lambda - 1} + \frac{\ln[u^\gamma(1-\cos\alpha_1)]}{[u(1-\cos\alpha_1)^{1/\gamma}]^\lambda - 1}\right] u^{\sigma+\frac{2\varepsilon}{q}-1}\,du\right\}dv \\ &= 2\int_1^\infty v^{-2\varepsilon-1}\left\{\int_0^1 \left[\frac{\ln[u^\gamma(1+\cos\alpha_2)]}{[u(1+\cos\alpha_2)^{1/\gamma}]^\lambda - 1} + \frac{\ln[u^\gamma(1-\cos\alpha_1)]}{[u(1-\cos\alpha_1)^{1/\gamma}]^\lambda - 1}\right] u^{\sigma+\frac{2\varepsilon}{q}-1}\,du\right\}dv \\ &\quad + 2\int_1^\infty v^{-2\varepsilon-1}\left\{\int_1^v \left[\frac{\ln[u^\gamma(1+\cos\alpha_2)]}{[u(1+\cos\alpha_2)^{1/\gamma}]^\lambda - 1}\right.\right. \\ &\quad \left.\left. + \frac{\ln[u^\gamma(1-\cos\alpha_1)]}{[u(1-\cos\alpha_1)^{1/\gamma}]^\lambda - 1}\right] u^{\sigma+\frac{2\varepsilon}{q}-1}\,du\right\}dv \\ &= \frac{1}{\varepsilon}\int_0^1 \left[\frac{\ln[u^\gamma(1+\cos\alpha_2)]u^{\sigma+\frac{2\varepsilon}{q}-1}}{[u(1+\cos\alpha_2)^{1/\gamma}]^\lambda - 1} + \frac{\ln[u^\gamma(1-\cos\alpha_1)]u^{\sigma+\frac{2\varepsilon}{q}-1}}{[u(1-\cos\alpha_1)^{1/\gamma}]^\lambda - 1}\right]du \\ &\quad + 2\int_1^\infty \left(\int_u^\infty v^{-2\varepsilon-1}\,dv\right) \\ &\quad \times \left\{\frac{\ln[u^\gamma(1+\cos\alpha_2)]u^{\sigma+\frac{2\varepsilon}{q}-1}}{[u(1+\cos\alpha_2)^{1/\gamma}]^\lambda - 1} + \frac{\ln[u^\gamma(1-\cos\alpha_1)]u^{\sigma+\frac{2\varepsilon}{q}-1}}{[u(1-\cos\alpha_1)^{1/\gamma}]^\lambda - 1}\right\}du \\ &= \frac{1}{\varepsilon}\left\{\int_0^1 \left[\frac{\ln[u^\gamma(1+\cos\alpha_2)]u^{\sigma+\frac{2\varepsilon}{q}-1}}{[u(1+\cos\alpha_2)^{1/\gamma}]^\lambda - 1} + \frac{\ln[u^\gamma(1-\cos\alpha_1)]u^{\sigma+\frac{2\varepsilon}{q}-1}}{[u(1-\cos\alpha_1)^{1/\gamma}]^\lambda - 1}\right]du\right. \\ &\quad \left. + \int_1^\infty \left[\frac{\ln[u^\gamma(1+\cos\alpha_2)]u^{\sigma-\frac{2\varepsilon}{p}-1}}{[u(1+\cos\alpha_2)^{1/\gamma}]^\lambda - 1} + \frac{\ln[u^\gamma(1-\cos\alpha_1)]u^{\sigma-\frac{2\varepsilon}{p}-1}}{[u(1-\cos\alpha_1)^{1/\gamma}]^\lambda - 1}\right]du\right\}. \end{aligned}$$

If the constant factor $K(\sigma)$ in (11) is not the best possible, then there exists a positive number k, with $K(\sigma) < k$, such that (11) is valid when replacing $K(\sigma)$ by k. Then we have $\varepsilon\tilde{I} < \varepsilon k\tilde{L}$, and

$$\begin{aligned}&\int_0^1 \left\{\frac{\ln[u^\gamma(1+\cos\alpha_2)]u^{\sigma+\frac{2\varepsilon}{q}-1}}{[u(1+\cos\alpha_2)^{1/\gamma}]^\lambda-1}+\frac{\ln[u^\gamma(1-\cos\alpha_1)]u^{\sigma+\frac{2\varepsilon}{q}-1}}{[u(1-\cos\alpha_1)^{1/\gamma}]^\lambda-1}\right\}du\\&\quad+\int_1^\infty \left\{\frac{\ln[u^\gamma(1+\cos\alpha_2)]u^{\sigma-\frac{2\varepsilon}{p}-1}}{[u(1+\cos\alpha_2)^{1/\gamma}]^\lambda-1}+\frac{\ln[u^\gamma(1-\cos\alpha_1)]u^{\sigma-\frac{2\varepsilon}{p}-1}}{[u(1-\cos\alpha_1)^{1/\gamma}]^\lambda-1}\right\}du\\&\quad=\varepsilon\tilde{I}<\varepsilon k\tilde{L}=k.\end{aligned}\tag{19}$$

By (7) and the Levi theorem (*cf.* [30]), we have

$$\begin{aligned}K(\sigma)&=\int_0^\infty \frac{\ln[u^\gamma(1+\cos\alpha_2)]u^{\sigma-1}\,du}{[u(1+\cos\alpha_2)^{1/\gamma}]^\lambda-1}+\int_0^\infty \frac{\ln[u^\gamma(1-\cos\alpha_1)]u^{\sigma-1}\,du}{[u(1-\cos\alpha_1)^{1/\gamma}]^\lambda-1}\\&=\int_0^1 \lim_{\varepsilon\to0^+}\left\{\frac{\ln[u^\gamma(1+\cos\alpha_2)]u^{\sigma+\frac{2\varepsilon}{q}-1}}{[u(1+\cos\alpha_2)^{1/\gamma}]^\lambda-1}+\frac{\ln[u^\gamma(1-\cos\alpha_1)]u^{\sigma+\frac{2\varepsilon}{q}-1}}{[u(1-\cos\alpha_1)^{1/\gamma}]^\lambda-1}\right\}du\\&\quad+\int_1^\infty \lim_{\varepsilon\to0^+}\left\{\frac{\ln[u^\gamma(1+\cos\alpha_2)]u^{\sigma-\frac{2\varepsilon}{p}-1}}{[u(1+\cos\alpha_2)^{1/\gamma}]^\lambda-1}+\frac{\ln[u^\gamma(1-\cos\alpha_1)]u^{\sigma-\frac{2\varepsilon}{p}-1}}{[u(1-\cos\alpha_1)^{1/\gamma}]^\lambda-1}\right\}du\\&=\lim_{\varepsilon\to0^+}\left\{\int_0^1\left[\frac{\ln[u^\gamma(1+\cos\alpha_2)]u^{\sigma+\frac{2\varepsilon}{q}-1}}{[u(1+\cos\alpha_2)^{1/\gamma}]^\lambda-1}+\frac{\ln[u^\gamma(1-\cos\alpha_1)]u^{\sigma+\frac{2\varepsilon}{q}-1}}{[u(1-\cos\alpha_1)^{1/\gamma}]^\lambda-1}\right]du\right.\\&\quad\left.+\int_1^\infty\left[\frac{\ln[u^\gamma(1+\cos\alpha_2)]u^{\sigma-\frac{2\varepsilon}{p}-1}}{[u(1+\cos\alpha_2)^{1/\gamma}]^\lambda-1}+\frac{\ln[u^\gamma(1-\cos\alpha_1)]u^{\sigma-\frac{2\varepsilon}{p}-1}}{[u(1-\cos\alpha_1)^{1/\gamma}]^\lambda-1}\right]du\right\}\le k,\end{aligned}$$

which contradicts the fact that $k < K(\sigma)$. Hence the constant factor $K(\sigma)$ in (11) is the best possible.

If the constant factor in (12) is not the best possible, then by (16), we would reach a contradiction: that the constant factor in (11) is not the best possible. □

Theorem 4 *On the assumptions of Theorem* 3, *replacing* $p > 1$ *by* $0 < p < 1$, *we have the equivalent reverses of* (11) *and* (12) *with the same best constant factors*.

Proof By the reverse Hölder inequality (*cf.* [29]), we have the reverses of (9) and (16). It is easy to obtain the reverse of (12). In view of the reverses of (12) and (16), we obtain the reverse of (11). On the other hand, suppose that the reverse of (11) is valid. Setting the same $g(y)$ as Theorem 3, by the reverse of (9), we have $J > 0$. If $J = \infty$, then the reverse of (12) is trivially value; if $J < \infty$, then by the reverse of (11), we obtain the reverses of (17) and (18). Hence we have the reverse of (12), which is equivalent to the reverse of (11).

If the constant factor $K(\sigma)$ in the reverse of (11) is not the best possible, then there exists a positive constant k, with $k > K(\sigma)$, such that the reverse of (11) is still valid when replacing $K(\sigma)$ by k. By the reverse of (19), we have

$$\begin{aligned}&\int_0^1 \left\{\frac{\ln[u^\gamma(1+\cos\alpha_2)]u^{\sigma+\frac{2\varepsilon}{q}-1}}{[u(1+\cos\alpha_2)^{1/\gamma}]^\lambda-1}+\frac{\ln[u^\gamma(1-\cos\alpha_1)]u^{\sigma+\frac{2\varepsilon}{q}-1}}{[u(1-\cos\alpha_1)^{1/\gamma}]^\lambda-1}\right\}du\\&\quad+\int_1^\infty \left\{\frac{\ln[u^\gamma(1+\cos\alpha_2)]u^{\sigma-\frac{2\varepsilon}{p}-1}}{[u(1+\cos\alpha_2)^{1/\gamma}]^\lambda-1}+\frac{\ln[u^\gamma(1-\cos\alpha_1)]u^{\sigma-\frac{2\varepsilon}{p}-1}}{[u(1-\cos\alpha_1)^{1/\gamma}]^\lambda-1}\right\}du>k.\end{aligned}\tag{20}$$

For $\varepsilon \to 0^+$, by the Levi theorem (*cf.* [30]), we find

$$\begin{aligned}&\int_1^\infty \left\{ \frac{\ln[u^\gamma(1+\cos\alpha_2)]u^{\sigma-\frac{2\varepsilon}{p}-1}}{[u(1+\cos\alpha_2)^{1/\gamma}]^\lambda - 1} + \frac{\ln[u^\gamma(1-\cos\alpha_1)]u^{\sigma-\frac{2\varepsilon}{p}-1}}{[u(1-\cos\alpha_1)^{1/\gamma}]^\lambda - 1} \right\} du \\ &\quad\to \int_1^\infty \left\{ \frac{\ln[u^\gamma(1+\cos\alpha_2)]u^{\sigma-1}}{[u(1+\cos\alpha_2)^{1/\gamma}]^\lambda - 1} + \frac{\ln[u^\gamma(1-\cos\alpha_1)]u^{\sigma-1}}{[u(1-\cos\alpha_1)^{1/\gamma}]^\lambda - 1} \right\} du. \end{aligned} \tag{21}$$

There exists a constant $\delta_0 > 0$, such that $\sigma - \frac{1}{2}\delta_0 > 0$, and then $K(\sigma - \frac{\delta_0}{2}) \in \mathbf{R}_+$. For $0 < \varepsilon < \frac{\delta_0|q|}{4}$ ($q < 0$), since $u^{\sigma+\frac{2\varepsilon}{q}-1} \le u^{\sigma-\frac{\delta_0}{2}-1}$, $u \in (0,1]$, and

$$\begin{aligned}0 &< \int_0^1 \left\{ \frac{\ln[u^\gamma(1+\cos\alpha_2)]u^{\sigma-\frac{\delta_0}{2}-1}}{[u(1+\cos\alpha_2)^{1/\gamma}]^\lambda - 1} + \frac{\ln[u^\gamma(1-\cos\alpha_1)]u^{\sigma-\frac{\delta_0}{2}-1}}{[u(1-\cos\alpha_1)^{1/\gamma}]^\lambda - 1} \right\} du \\ &\le K\left(\sigma - \frac{\delta_0}{2}\right),\end{aligned}$$

then by the Lebesgue control convergence theorem (*cf.* [30]), for $\varepsilon \to 0^+$, we have

$$\begin{aligned}&\int_0^1 \left\{ \frac{\ln[u^\gamma(1+\cos\alpha_2)]u^{\sigma+\frac{2\varepsilon}{q}-1}}{[u(1+\cos\alpha_2)^{1/\gamma}]^\lambda - 1} + \frac{\ln[u^\gamma(1-\cos\alpha_1)]u^{\sigma+\frac{2\varepsilon}{q}-1}}{[u(1-\cos\alpha_1)^{1/\gamma}]^\lambda - 1} \right\} du \\ &\quad\to \int_0^1 \left\{ \frac{\ln[u^\gamma(1+\cos\alpha_2)]u^{\sigma-1}}{[u(1+\cos\alpha_2)^{1/\gamma}]^\lambda - 1} + \frac{\ln[u^\gamma(1-\cos\alpha_1)]u^{\sigma-1}}{[u(1-\cos\alpha_1)^{1/\gamma}]^\lambda - 1} \right\} du. \end{aligned} \tag{22}$$

By (20), (21), and (22), for $\varepsilon \to 0^+$, we find $K(\sigma) \ge k$, which contradicts the fact that $k > K(\sigma)$. Hence, the constant factor $K(\sigma)$ in the reverse of (11) is the best possible.

If the constant factor in reverse of (12) is not the best possible, then by the reverse of (16), we would reach a contradiction: that the constant factor in the reverse of (11) is not the best possible. □

Corollary 5 *For* $\delta = -1$ *in* (11) *and* (12), *replacing* $|x|^\lambda f(x)$ *by* $f(x)$, *we obtain*

$$0 < \int_{-\infty}^\infty |x|^{p(1-\mu)-1} f^p(x)\,dx < \infty,$$

and the following equivalent inequalities with the homogeneous kernel and the best possible constant factors:

$$\begin{aligned}&\int_{-\infty}^\infty \int_{-\infty}^\infty \max_{i\in\{1,2\}} \frac{\ln[(|y|^\gamma + \operatorname{sgn}(x)y^\gamma \cos\alpha_i)/|x|^\gamma]}{[|y|^\gamma + \operatorname{sgn}(x)y^\gamma\cos\alpha_i]^{\lambda/\gamma} - |x|^\lambda} f(x)g(y)\,dx\,dy \\ &\quad< K(\sigma)\left[\int_{-\infty}^\infty |x|^{p(1-\mu)-1} f^p(x)\,dx\right]^{\frac{1}{p}} \left[\int_{-\infty}^\infty |y|^{q(1-\sigma)-1} g^q(y)\,dy\right]^{\frac{1}{q}},\end{aligned} \tag{23}$$

$$\begin{aligned}&\int_{-\infty}^\infty |y|^{p\sigma-1}\left\{\int_{-\infty}^\infty \max_{i\in\{1,2\}} \frac{\ln[(|y|^\gamma + \operatorname{sgn}(x)y^\gamma \cos\alpha_i)/|x|^\gamma]}{[|y|^\gamma + \operatorname{sgn}(x)y^\gamma\cos\alpha_i]^{\lambda/\gamma} - |x|^\lambda} f(x)\,dx\right\}^p dy \\ &\quad< K^p(\sigma)\int_{-\infty}^\infty |x|^{p(1-\mu)-1} f^p(x)\,dx.\end{aligned} \tag{24}$$

In particular, for $\alpha_1 = \alpha_2 = \alpha \in (0,\pi)$, $\gamma = 1$ in (23) and (24), we obtain the following equivalent inequalities:

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{\ln[(|y|+\operatorname{sgn}(x)y\cos\alpha)/|x|]}{[|y|+\operatorname{sgn}(x)y\cos\alpha]^{\lambda}-|x|^{\lambda}}f(x)g(y)\,dx\,dy$$
$$<k(\sigma)\left[\int_{-\infty}^{\infty}|x|^{p(1-\mu)-1}f^{p}(x)\,dx\right]^{\frac{1}{p}}\left[\int_{-\infty}^{\infty}|y|^{q(1-\sigma)-1}g^{q}(y)\,dy\right]^{\frac{1}{q}}, \tag{25}$$

$$\int_{-\infty}^{\infty}|y|^{p\sigma-1}\left\{\int_{-\infty}^{\infty}\frac{\ln[(|y|+\operatorname{sgn}(x)y\cos\alpha)/|x|]}{[|y|+\operatorname{sgn}(x)y\cos\alpha]^{\lambda}-|x|^{\lambda}}f(x)\,dx\right\}^{p}dy$$
$$<k^{p}(\sigma)\int_{-\infty}^{\infty}|x|^{p(1-\mu)-1}f^{p}(x)\,dx, \tag{26}$$

where $k(\sigma)$ is indicated by (13).

4 Two kinds of operator expressions

Suppose that $p>1$, $\frac{1}{p}+\frac{1}{q}=1$, $0<\alpha_1\le\alpha_2<\pi$, $\mu,\sigma>0$, $\mu+\sigma=\lambda$, $\gamma\in\{a;a=\frac{1}{2k+1},2k-1\ (k\in\mathbf{N})\}$, $\delta\in\{-1,1\}$. We set the following functions:

$$\varphi(x):=|x|^{p(1-\delta\sigma)-1},\qquad \psi(y):=|y|^{q(1-\sigma)-1},\qquad \phi(x):=|x|^{p(1-\mu)-1}\quad(x,y\in\mathbf{R}),$$

therefore, $\psi^{1-p}(y)=|y|^{p\sigma-1}$. Define the following real normed linear space:

$$L_{p,\varphi}(\mathbf{R}):=\left\{f:\|f\|_{p,\varphi}:=\left(\int_{-\infty}^{\infty}\varphi(x)\left|f(x)\right|^{p}dx\right)^{\frac{1}{p}}<\infty\right\},$$
$$L_{p,\psi^{1-p}}(\mathbf{R})=\left\{h:\|h\|_{p,\psi^{1-p}}=\left(\int_{-\infty}^{\infty}\psi^{1-p}(y)\left|h(y)\right|^{p}dy\right)^{\frac{1}{p}}<\infty\right\},$$
$$L_{p,\phi}(\mathbf{R})=\left\{g:\|g\|_{p,\phi}=\left(\int_{-\infty}^{\infty}\phi(x)\left|g(x)\right|^{p}dx\right)^{\frac{1}{p}}<\infty\right\}.$$

(a) In view of Theorem 3, for $f\in L_{p,\varphi}(\mathbf{R})$, setting

$$H_1(y):=\int_{-\infty}^{\infty}\max_{i\in\{1,2\}}\frac{\ln[|x^{\delta}y|^{\gamma}+(x^{\delta}y)^{\gamma}\cos\alpha_i]}{[|x^{\delta}y|^{\gamma}+(x^{\delta}y)^{\gamma}\cos\alpha_i]^{\lambda/\gamma}-1}f(x)\,dx\quad(y\in\mathbf{R}),$$

by (12), we have

$$\|H_1\|_{p,\psi^{1-p}}=\left(\int_{-\infty}^{\infty}\psi^{1-p}(y)H_1^{p}(y)\,dy\right)^{\frac{1}{p}}<K(\sigma)\|f\|_{p,\varphi}<\infty. \tag{27}$$

Definition 1 Define a Hilbert-type integral operator with the non-homogeneous kernel in the whole plane $T_1:L_{p,\varphi}(\mathbf{R})\to L_{p,\psi^{1-p}}(\mathbf{R})$ as follows: For any $f\in L_{p,\varphi}(\mathbf{R})$, there exists a unique representation $T_1f=H_1\in L_{p,\psi^{1-p}}(\mathbf{R})$, satisfying, for any $y\in\mathbf{R}$, $T_1f(y)=H_1(y)$.

In view of (27), it follows that $\|T_1f\|_{p,\psi^{1-p}}=\|H_1\|_{p,\psi^{1-p}}\le K(\sigma)\|f\|_{p,\varphi}$, and then the operator T_1 is bounded satisfying

$$\|T_1\|=\sup_{f(\neq\theta)\in L_{p,\varphi}(\mathbf{R})}\frac{\|T_1f\|_{p,\psi^{1-p}}}{\|f\|_{p,\varphi}}\le K(\sigma).$$

Since the constant factor $K(\sigma)$ in (27) is the best possible, we have $\|T_1\|=K(\sigma)$.

If we define the formal inner product of $T_1 f$ and g as

$$(T_1 f, g) := \int_{-\infty}^{\infty} \left(\int_{-\infty}^{\infty} h^{(\delta)}(x,y) f(x)\,dx \right) g(y)\,dy$$
$$= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} h^{(\delta)}(x,y) f(x) g(y)\,dx\,dy,$$

then we can rewrite (11) and (12) as follows:

$$(T_1 f, g) < \|T_1\| \cdot \|f\|_{p,\varphi} \|g\|_{q,\psi}, \qquad \|T_1 f\|_{p,\psi^{1-p}} < \|T_1\| \cdot \|f\|_{p,\varphi}.$$

(b) In view of Corollary 5, for $f \in L_{p,\phi}(\mathbf{R})$, setting

$$H_2(y) := \int_{-\infty}^{\infty} \max_{i\in\{1,2\}} \frac{\ln[(|y|^\gamma + \operatorname{sgn}(x) y^\gamma \cos\alpha_i)/|x|^\gamma]}{[|y|^\gamma + \operatorname{sgn}(x) y^\gamma \cos\alpha_i]^{\lambda/\gamma} - |x|^\lambda} f(x)\,dx \quad (y \in \mathbf{R}),$$

by (24), we have

$$\|H_2\|_{p,\psi^{1-p}} = \left(\int_{-\infty}^{\infty} \psi^{1-p}(y) H_2^p(y)\,dy \right)^{\frac{1}{p}} < K(\sigma) \|f\|_{p,\phi} < \infty. \tag{28}$$

Definition 2 Define a Hilbert-type integral operator with the homogeneous kernel in the whole plane $T_2 : L_{p,\phi}(\mathbf{R}) \to L_{p,\psi^{1-p}}(\mathbf{R})$ as follows: For any $f \in L_{p,\phi}(\mathbf{R})$, there exists a unique representation $T_2 f = H_2 \in L_{p,\psi^{1-p}}(\mathbf{R})$, satisfying, for any $y \in \mathbf{R}$, $T_2 f(y) = H_2(y)$.

In view of (28), it follows that $\|T_2 f\|_{p,\psi^{1-p}} = \|H_2\|_{p,\psi^{1-p}} \le K(\sigma)\|f\|_{p,\phi}$, and then the operator T_2 is bounded satisfying

$$\|T_2\| = \sup_{f(\neq\theta)\in L_{p,\phi}(\mathbf{R})} \frac{\|T_2 f\|_{p,\psi^{1-p}}}{\|f\|_{p,\phi}} \le K(\sigma).$$

Since the constant factor $K(\sigma)$ in (28) is the best possible, we have $\|T_2\| = K(\sigma)$.

If we define the formal inner product of $T_2 f$ and g as

$$(T_2 f, g) := \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \max_{i\in\{1,2\}} \frac{\ln[(|y|^\gamma + \operatorname{sgn}(x) y^\gamma \cos\alpha_i)/|x|^\gamma]}{[|y|^\gamma + \operatorname{sgn}(x) y^\gamma \cos\alpha_i]^{\lambda/\gamma} - |x|^\lambda} f(x) g(y)\,dx\,dy,$$

then we can rewrite (23) and (24) as follows:

$$(T_2 f, g) < \|T_2\| \cdot \|f\|_{p,\phi} \|g\|_{q,\psi}, \qquad \|T_2 f\|_{p,\psi^{1-p}} < \|T_2\| \cdot \|f\|_{p,\phi}.$$

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
BY carried out the mathematical studies, participated in the sequence alignment and drafted the manuscript. QC participated in the design of the study and performed the numerical analysis. All authors read and approved the final manuscript.

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