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Stochastic perturbation of the Lighthill–Whitham–Richards model via the method of stochastic characteristics

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Abstract

In this paper we apply the method of stochastic characteristics to a Lighthill–Whitham–Richards model. The stochastic perturbation can be seen as errors in measurement of the traffic density. For concrete examples we solve the equation perturbed by a standard Brownian motion and the geometric Brownian motion without drift.

Keywords: Method of stochastic characteristics; Lighthill–Whitham–Richards model; Explicit solutions

1 Introduction

Many traffic flow models go back to scalar conservation laws of generally non-linear type, i.e.

$$u_t + f(u)_x = 0, \quad (1)$$

see e.g. [4]. There are many different results, making the area of scalar conservation laws to an area of very active research in mathematics, see e.g. [3, 5, 15, 16, 21, 23], to name just a few of those.

The function u in (1) describes the density of vehicles on a road and thus has values on the compact set $[0, 1]$. A conservation law is derived under the assumption, that the time propagation of a mass on a certain interval is only affected by the flux at the boundary of the interval. Hence one often chooses $f(u) = u \cdot v(u)$, where v is the Eulerian velocity of the traffic. There are many different models for traffic flow discussed in the literature where the most famous ones are Aw–Rascle–Zhang model [1, 2, 24] and Lighthill–Whitham–Richards model, which uses $f(u) = u \cdot (1 - u)$, i.e. a velocity depending linearly on the density. The scalar conservation law (1) now reads

$$u_t + (1 - 2u) \cdot u_x = 0. \quad (2)$$

The flux function in the Lighthill–Whitham–Richards model is in a relatively good agreement with traffic measurements, see [19]. The main problem is that measurements show

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that data points are quite accurate for low and high densities but are noisy around the maximum point. The flux function is hence rather given by

$$f(u) = u \cdot (1 - u) + H(u) \circ \frac{dM_t}{dt},$$

where H is a function vanishing at $u = 0$ and $u = 1$ and M_t is a suitable nice enough stochastic process. Plugging this in the above conservation law (1) yields

$$u_t + f(u)_x = u_t + (1 - 2u) \cdot u_x + H(u)_x \circ \frac{dM_t}{dt} = 0. \quad (3)$$

The random perturbation is chosen in such a way that $H(u)_x$ controls the dependence of the noise with respect to the density in space. The stochastic process itself give a temporal noise, in order to represent fluctuation in time. Of course with the use of a random field the model may be more general. However, the method of characteristics can not be applied in the same way as in this article.

The choice of the Stratonovich integral is due to the existence of the chain rule in the same form as in the deterministic case. For Itô integral a correction term is needed, which leads in some cases to non-existing solutions.

General results for such stochastic conservation laws can be found in e.g. [3, 6–8, 10–14, 20]. The results base on Itô- and Stratonovich type noise in finite and infinite dimensional spaced as well as rough noise and going much further than the model studied in this short article. We hence can of course not give a comprehensive list and refer to the above works and their references.

Industrial application and especially the study of traffic flows are nowadays interested in shocks and nonlinearities. Such shocks occur if the corresponding characteristic intersect, see [4]. In the method of stochastic characteristics, which we treat in this article, this is exactly the same. Our focus in this paper, however, is to show how stochastically perturbed conservation laws behave until the time of shocks. The investigation of shocks would be an interesting study for the future.

In this manuscript, we use the method of stochastic characteristics to solve such equations for the stochastically perturbed Lighthill–Whitham–Richards model explicitly for different cases, where the driving process is given by a Brownian motion or a geometrical Brownian motion. This work can be seen as the starting point to the investigation of different stochastically perturbed hyperbolic equations in direct applications.

The main advantage of the applied method is that the solutions are given in an explicit expression. This makes tedious and highly involved numerical simulation unnecessary. The motivation of the method is given by traffic flow equations, however, it can be applied to other stochastically perturbed conservation laws.

2 Preliminaries

We look at some examples for the Lighthill–Whitham–Richards model for different initial conditions. Due to the underlying model the initial condition describes the density of our traffic problem at time $t = 0$ and at position $x \in [0, 1]$.

Lemma 2.1 Consider the following partial differential equation on $[0, 1]$

$$\begin{cases} du = -(1 - 2u) \cdot u_x dt, \\ u(x, 0) = g(x), \end{cases} \tag{4}$$

where $g(x)$ is a smooth function. Let (ξ_t, η_t) be the solutions to the so called characteristic equations given by

$$\begin{cases} d\xi_t = (1 - 2\eta_t) dt, \\ \xi_0(x) = x, \end{cases} \quad \text{and} \quad \begin{cases} d\eta_t = 0 dt, \\ \eta_0(x) = g(x). \end{cases}$$

Hence we obtain

$$\eta_t(x) = g(x), \quad \xi_t(x) = x + \int_0^t 1 - 2g(x) ds = x + t - 2g(x)t. \tag{5}$$

Then the solution to (4) is given by $u(x, t) = g(\xi_t^{-1}(x))$, where ξ_t^{-1} denotes the inverse function of ξ_t .

Proof The proof is an direct consequence of Chap. 3 in [9], in particular [9, § 3.2, Theorem 2] with an application of the inverse mapping theorem [18, Chapter XIV, Theorem 1.2]. □

Example 2.2 Consider the following partial differential equation (PDE) for the Lighthill–Whitham–Richards model on $[0, 1]$

$$du = -(1 - 2u) \cdot u_x dt, \quad u(x, 0) = g(x). \tag{6}$$

Due to Lemma 2.1 we obtain with the initial condition $g(x) = 1 - x$:

$$\begin{aligned} \xi_t(x) &= x - t + 2xt, & \text{hence } \xi_t^{-1}(x) &= \frac{x + t}{1 + 2t}. \\ \eta_t(x) &= 1 - x, \end{aligned} \tag{7}$$

Thus the solution of the above PDE (6) is given by

$$u(x, t) = \frac{1 - x + t}{1 + 2t}. \tag{8}$$

If we change the initial condition to be $g(x) = 1 - x^2$, we obtain

$$\begin{aligned} \xi_t(x) &= x - t + 2x^2t, & \text{and hence } \xi_t^{-1}(x) &= \frac{\sqrt{1 + 8t^2 + 8tx} - 1}{4t}. \\ \eta_t(x) &= 1 - x^2, \end{aligned} \tag{9}$$

Thus the corresponding solution is equal to

$$u(x, t) = \begin{cases} 1 - \frac{(\sqrt{1+8t^2+8tx}-1)^2}{16t^2}, & \text{for } t \neq 0, \\ 1 - x^2, & \text{for } t = 0. \end{cases} \tag{10}$$

One can easily verify that (10) solves indeed the PDE (6).

The main advantage of this method is the precise expression of a solution to a PDE – provided that the corresponding initial condition $g(x)$ and coefficient functions are explicitly given. Due to this fact and for a better comparison between the deterministic and stochastic case we present a collection of solutions in Appendix A.

Along the characteristics the solution remains constant. In the case of the traffic problem and under the considered initial conditions the characteristics never cross each other which means that no shocks appear and hence the solutions are global. As written in the introduction we will study the perturbed case (3) for $H(u) \neq 0$ and for M_t to be the standard Brownian motion (Bm) as well as the so called geometric Brownian motion defined in the following way.

Definition 2.3 A stochastic process $S_t, t \geq 0$, is said to be a *geometric Brownian motion* (gBm), if it satisfies

$$dS_t = \mu S_t dt + \sigma S_t dW_t = S_t dW_t, \tag{11}$$

where W_t is a Brownian motion. Hence the geometric Brownian motion without drift is given by

$$S_t := \exp\left(-\frac{t}{2} + W_t\right).$$

Definition 2.4 Let W_t be a standard one-dimensional Brownian motion on a complete separable probability space $(\Omega, \mathcal{F}, P, \mathcal{F}_t)$, with right-continuous filtration $(\mathcal{F}_t)_{t>0}$. Then we define for any smooth function $H(x, u, p, t), x, u \in [0, 1], p$ bounded, $t \in [0, T]$, for $0 < T < \infty$ the following integral expression

$$\int_0^t F(x, u, p, \circ ds) := \begin{cases} \int_0^t -(1 - 2u)p ds + \int_0^t H(x, u, p, s) \circ dW_s, & \text{for Bm,} \\ \int_0^t -(1 - 2u)p ds + \int_0^t H(x, u, p, s) \circ dS_s, & \text{for gBm.} \end{cases}$$

The integrals are given in the sense of Stratonovich.

Based on these definitions we are able to apply the so called method of stochastic characteristics to the PDEs as (4) but perturbed by Brownian motion respectively geometric Brownian motion. Since we consider partial differential equations with perturbations by (geometric) Brownian motions we get an ω - dependence in the solutions. The idea of the method is nearly the same as before: now we fix $\omega \in \Omega$ and transform a stochastic partial differential equation (SPDE) into a system of stochastic differential equations (SDEs), solve it and determine the solution to the original SPDE by using stopping times. Hence the precisely determined solutions are given for almost all ω and all space and time variables (x, t) up to a certain stopping time denoted by $\sigma(x)$. In contrast to the deterministic case we will introduce in the following the method of stochastic characteristics in a more detailed way. Based on Definition 2.4 a perturbed Lighthill–Whitham–Richards model (3) is equivalent to the Cauchy problem

$$\begin{cases} du = F(x, u, u_x, \circ dt), \\ u = g \quad \text{on } \Gamma := \{x \in [0, 1] \times [0, T] | x = (x_1, t), t = 0\}. \end{cases} \tag{12}$$

Therefore the solution to equation (12) is denoted by $u(x, t, \omega)$, but for short notation we only write $u(x, t)$. Suppose u is a solution to (12) and at least one-times continuously differentiable with respect to space and time for fixed $\omega \in \Omega$. Furthermore, we assume that there exists a curve $\xi_s(r)$ which maps the point $r \in \Gamma$ to a point of a neighborhood in Γ at time s . Additionally, we assume $\xi_0(x) = x$ for all $x \in [0, 1]$ as the initial condition. Due to these assumptions we consider and define the following functions, now for fixed $\omega, r \in [0, 1]$ and $s \in [0, T]$:

$$\begin{aligned} &(\xi_s(r, \omega), s), \\ \eta_s(r, \omega) &:= u(\xi_s(r, \omega), s), \\ \chi_s(r, \omega) &:= u_{\xi_s}(\xi_s(r, \omega), s). \end{aligned} \tag{13}$$

In the next step we combine (12) with equations (13) and obtain

$$\frac{d}{dt} \left[u(\xi_t(r), t) - u(\xi_0(r), 0) - \int_0^t F(\xi_s(r), \eta_s(r), \chi_s(r), \circ ds) \right] = 0.$$

By similar calculations as in [9, § 3.2.1, equation (11)] we get

$$\begin{cases} d\xi_t = -F_{\chi_t}(\xi_t, \eta_t, \chi_t, \circ dt), \\ d\eta_t = F(\xi_t, \eta_t, \chi_t, \circ dt) - \chi_t \cdot F_{\chi_t}(\xi_t, \eta_t, \chi_t, \circ dt), \\ d\chi_t = F_{\xi_t}(\xi_t, \eta_t, \chi_t, \circ dt) + F_{\eta_t}(\xi_t, \eta_t, \chi_t, \circ dt)\chi_t. \end{cases} \tag{SCE}$$

The above stochastic differential equations (SCE) are called *stochastic characteristic equations*, for a more detailed description and proofs, see also [17]. Given a point $x \in [0, 1]$ and assuming that there exist unique solutions to (SCE) starting from x at time $t = 0$, these solutions solve the corresponding integral equation with initial function g :

$$\begin{aligned} \xi_t(x) &= x - \int_0^t F_{\chi_s}(\xi_s(x), \eta_s(x), \chi_s(x), \circ ds), \\ \eta_t(x) &= g(x) - \int_0^t \chi_s \cdot F_{\chi_s}(\xi_s(x), \eta_s(x), \chi_s(x), \circ ds) + \int_0^t F(\xi_s(x), \eta_s(x), \chi_s(x), \circ ds), \\ \chi_t(x) &= g_x(x) + \int_0^t F_{\xi_s}(\xi_s(x), \eta_s(x), \chi_s(x), \circ ds) + \int_0^t F_{\eta_s}(\xi_s(x), \eta_s(x), \chi_s(x), \circ ds)\chi_s. \end{aligned}$$

Let us assume that the solutions $(\xi_t(x), \eta_t(x), \chi_t(x))$ exist up to a stopping time $T(x)$. As mentioned above we have to work on different stopping times based on the following definition of an explosion time.

Definition 2.5 Let $X_t, t \in [0, \tau)$, be a local process. The stopping time τ is called *terminal time* of the local process X_t . If

$$\lim_{t \nearrow \tau} |X_t| = \infty,$$

then τ is called *explosion time*.

In the case of our solutions (ξ_t, η_t, χ_t) this yields to the following definitions of stopping times.

Definition 2.6 Let $T(x)$ be the infimum of all explosion times of the solutions (ξ_t, η_t, χ_t) . Then we define for all $x, y \in [0, 1]$ the stopping times

$$\begin{aligned} \tau_{\text{inv}}(x) &:= \inf\{t > 0 \mid \det D\xi_t(x) = 0\}, \\ \tau(x) &:= \tau_{\text{inv}}(x) \wedge T(x), \\ \sigma(y) &:= \inf\{t > 0 \mid y \notin \xi_t(\{x \in [0, 1] \mid \tau(x) > t\})\}, \end{aligned}$$

where $D\xi_t$ denotes the Jacobian matrix.

Now let the inverse process ξ_t^{-1} of ξ_t exist up to some stopping time $\sigma(x)$. Then we define for almost all ω and for all (x, t) with $t < \sigma(x, \omega)$ the solution

$$u(x, t) := \eta_t(\xi_t^{-1}(x)). \tag{14}$$

Detailed derivations and introductions can be found in [22, Chap. 3]. Now we are able to solve different SPDEs concerning the Lighthill–Whitham–Richards model by using the method of stochastic characteristics.

3 Application & representation

In particular industrial applications the equations are much more involved than in the examples this played in here. Of course to use our method the characteristics have to be computed explicitly. After that standard ODE or SDE methods can be used, if needed.

Based on the flow rate function $H(u)$ and the continuity equation the most natural choice of the drift term is $H(u) = u - u^2$. In a first step we perturb the Lighthill–Whitham–Richards model by a standard Brownian motion. Hence we consider

$$\begin{cases} du = -(1 - 2u) \cdot u_x dt - (1 - 2u) \cdot u_x \circ dW_t, \\ u(x, 0) = g(x). \end{cases} \tag{15}$$

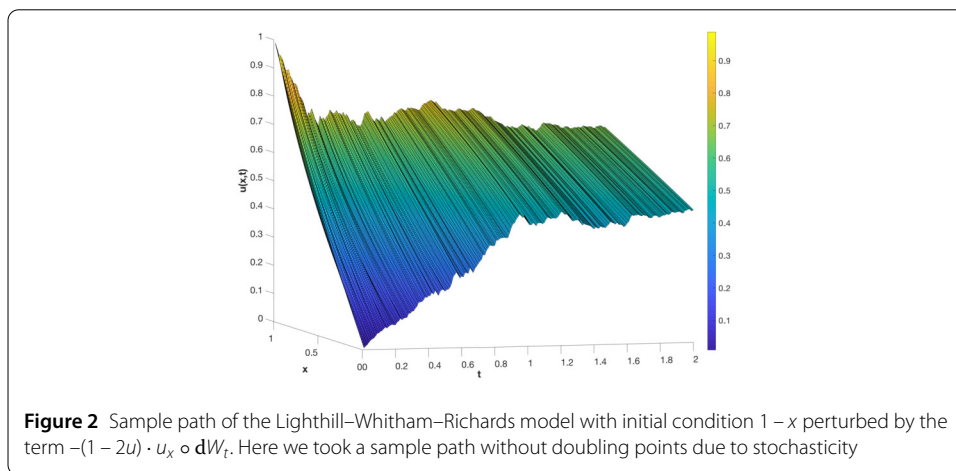
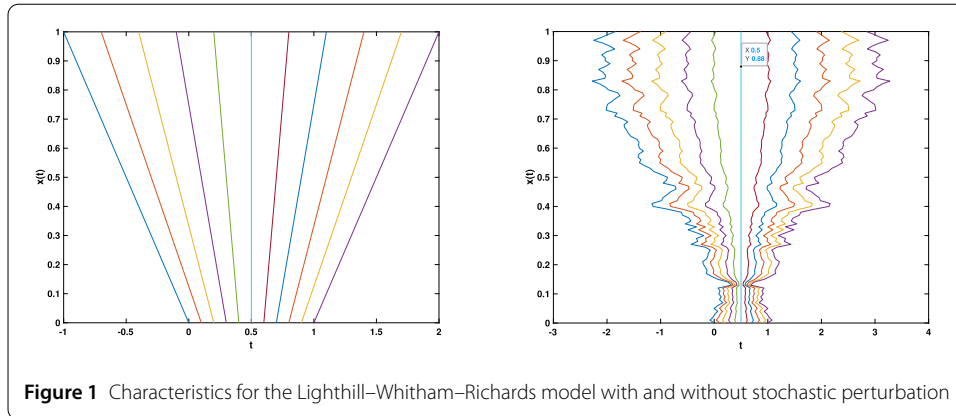
By using direct computation one can show that the corresponding stochastic characteristic equations are given for almost all ω and all (x, t) up to a stopping time $\sigma(x)$ by

$$\begin{cases} d\xi_t = (1 - 2\eta_t) dt + (1 - 2\eta_t) \circ dW_t, \\ \xi_0(x) = x, \end{cases} \quad \text{and} \quad \begin{cases} d\eta_t = 0 dt + 0 \circ dW_t, \\ \eta_0(x) = g(x). \end{cases} \tag{16}$$

Due to the linearity in the space derivative u_x the solution $\eta_t(x) = g(x)$ is always valid. Therefore we receive the solution

$$\xi_t(x) = x + (1 - 2g(x))(t + W_t).$$

At this point we compare the characteristics in the deterministic case with the corresponding perturbed one, see Fig. 1. As the initial condition we use here $g(x) = 1 - x$.



In this case of $g(x) = 1 - x$ there exists obviously a process ξ_t^{-1} , such that the inverse property is fulfilled for almost all ω and all (x, t) up to stopping time $\sigma(x)$, i.e.

$$\xi_t^{-1}(x) = \frac{x + t + W_t}{1 + 2t + 2W_t}.$$

The solution to the considered SPDE (15) is given for almost all ω and all (x, t) up to stopping time $\sigma(x)$ by

$$u(x, t) = \frac{1 - x + t + W_t}{1 + 2t + 2W_t}, \tag{17}$$

which looks similar to the deterministic solution (8). Due to the explicit expression of the solution we are able to visualize a sample path easily, see Fig. 2. As introduced in Definition 2.6 the stopping time can be determined explicitly in this example by

$$\begin{aligned} \sigma(x) &= \inf \left\{ t > 0 \mid \frac{x + t + W_t}{1 + 2t + 2W_t} \notin [0, 1] \right\} \wedge \inf \{ t > 0 \mid 1 + 2t + 2W_t = 0 \} \\ &= \inf \left\{ t > 0 \mid \frac{x + t + W_t}{1 + 2t + 2W_t} \notin [0, 1] \right\} \wedge \infty. \end{aligned}$$

The perturbation by a geometric Brownian motion as given in Definition 2.3 is in this case straightforward. According to Definition 2.4 we practically can replace the Brownian motion W_t by $\exp(-t/2 + W_t) - 1$. Let us consider

$$\begin{cases} du = -(1 - 2u) \cdot u_x dt - (1 - 2u) \cdot u_x \circ d[\exp(-t/2 + W_t)], \\ u(x, 0) = 1 - x^2. \end{cases} \tag{18}$$

By an application of the method of stochastic characteristics we finally get the precise solution for almost all ω and (x, t) up to a stopping time $\sigma(x)$ by

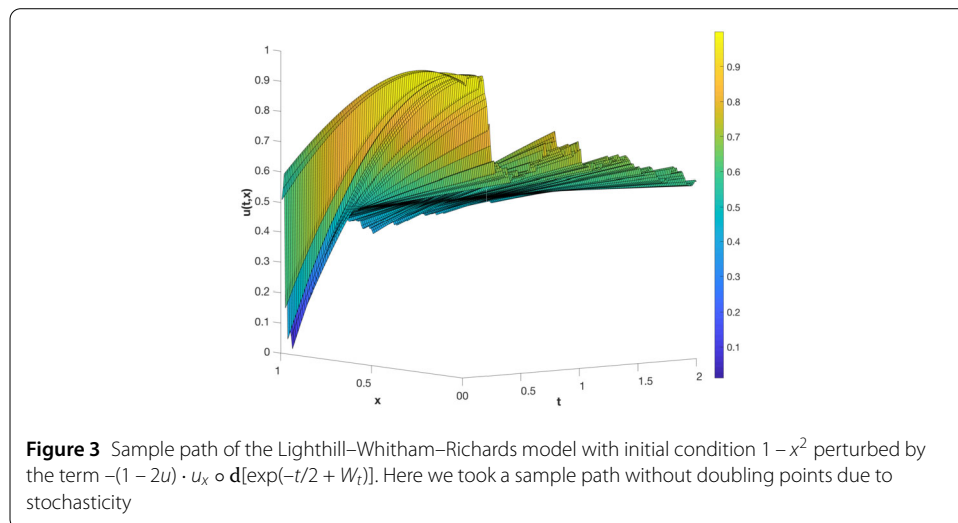
$$u(x, t) = \begin{cases} 1 - \frac{(\sqrt{8(t+e^{(-t/2+W_t)}-1)(t+x+e^{(-t/2+W_t)}-1)+1}-1)^2}{16(e^{(-t/2+W_t)}+t-1)^2}, & \text{if } t \neq 0, \\ 1 - x^2, & \text{if } t = 0, \end{cases} \tag{19}$$

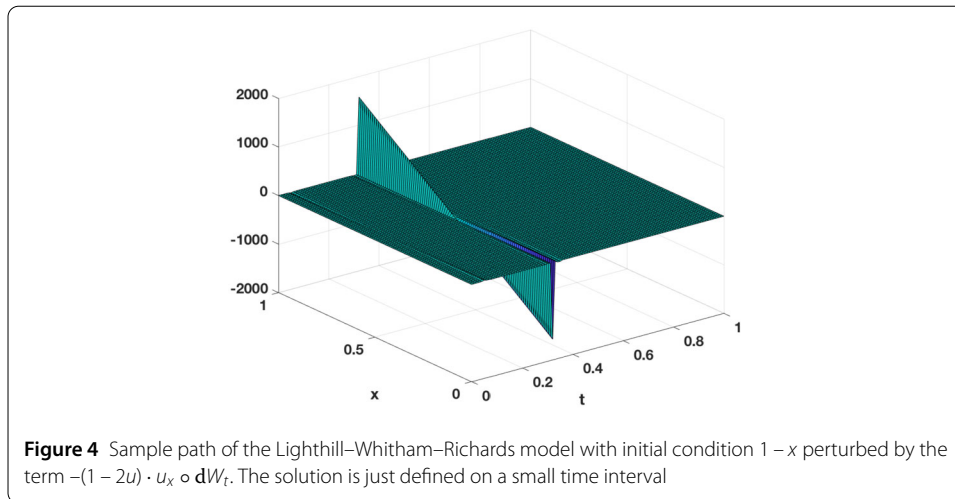
where we can use the classical l’Hospital argument. The corresponding stopping time is equal to

$$\begin{aligned} \sigma(x) &= \inf \left\{ t > 0 \mid \frac{\sqrt{8(e^{(-t/2+W_t)}+t-1)(t+x+e^{(-t/2+W_t)}-1)+1}-1}{4(e^{(-t/2+W_t)}+t-1)} \notin [0, 1] \right\} \\ &\wedge \inf \left\{ t > 0 \mid \frac{1}{\sqrt{8(t+e^{(-t/2+W_t)}-1)(t+x+e^{(-t/2+W_t)}-1)+1}} = 0 \right\} \\ &= \inf \left\{ t > 0 \mid \frac{\sqrt{8(e^{(-t/2+W_t)}+t-1)(t+x+e^{(-t/2+W_t)}-1)+1}-1}{4(e^{(-t/2+W_t)}+t-1)} \notin [0, 1] \right\} \wedge \infty. \end{aligned}$$

In Fig. 3 we display one sample path with initial condition $1 - x^2$ perturbed by the term $-(1 - 2u) \cdot u_x \circ d[\exp(-t/2 + W_t)]$.

Due to this approach we have to verify that the equations (17) as well as (19) really solve the underlying problems. For the sake of simplicity these necessary but lengthy calculation can be found in the Appendix C for equation (17).





For reader's convenience we add some other examples in Appendix B with precise expressions of solutions and different choices of $H(u)$, but which may not rigorously fit the Lighthill-Whitham-Richards model.

4 Conclusions and discussion

The method of stochastic characteristics can be used effectively to solve a stochastically perturbed Lighthill-Whitham-Richards model. The solutions are explicitly given up to a stopping time in closed form. Numerical simulations based on these models can hence be implemented straightforward. However one has to be careful, that the intersection of characteristics due to stochastic perturbation can lead to solutions which are only defined on a smaller time interval than the non-perturbed ones. On the other hand, it may be also possible, that the stochastic perturbations increase the time interval where solutions are defined. An example for a solution which is ill-defined due to intersecting characteristics can be seen in Fig. 4.

A collection for different examples of stochastic perturbations can be found in Appendix B. Note that with the considered perturbations measurement errors can be modelled effectively. This could be of high interest for more complicated traffic flow models.

Appendix A: Collection of examples in the deterministic case

For reader's convenience the authors itemize the corresponding solutions to the deterministic Lighthill-Whitham-Richards model (4) for different initial functions $g(x)$. Based on the model a couple of initial conditions are possible apart from Example 2.2 with $g(x) = 1 - x$ and $g(x) = 1 - x^2$. The opposite to the above case is $g(x) = x$, i.e. the road at position $x = 0$ has empty density but at $x = 1$ there is e.g. a tailback or a red light, hence the initial density is maximal. We also want to consider a quadratic form by $g(x) = x - x^2$, which coincide with the behaviour of the drift part. Analogous calculations yield the following solutions (see Table 1).

Appendix B: Collection of examples in the stochastic case

Analogously to the observation in Appendix A we specify the solutions to the perturbed Lighthill-Whitham-Richards model for different choices of the initial function $g(x)$ as

Table 1 Solution for the Lighthill-Whitham-Richards model for different g

For	Solution to $\begin{cases} du = -(1 - 2u) \cdot u_x dt, \\ u(x, 0) = g(x), \quad x \in [0, 1]. \end{cases}$
$g(x) = x$	$u(x, t) = \frac{x-t}{1-2t}, t \neq \frac{1}{2}$
$g(x) = x - x^2$	$u(x, t) = \frac{\sqrt{-4t^2+t(8x-4)+1+2t-1}}{4t} - \frac{(\sqrt{-4t^2+t(8x-4)+1+2t-1})^2}{16t^2}, t \neq 0$

well as for different diffusion terms $H(u)$. Taking into account that these might not model the original traffic flow problem perfectly, the approach of the method of stochastic characteristics will give explicit solutions. Firstly we perturb the equation by standard Brownian motion.

- The solution to the equation

$$\begin{cases} du = -(1 - 2u) \cdot u_x dt + u_x \circ dW_t, \\ u(x, 0) = 1 - x^2, \end{cases} \tag{20}$$

is given for almost all ω and all (x, t) up to a certain stopping time by

$$u(x, t) = 1 - \frac{(\sqrt{8t(W_t + t + x) + 1} - 1)^2}{16t^2}.$$

- The solution to the equation

$$\begin{cases} du = -(1 - 2u) \cdot u_x dt + u \circ dW_t, \\ u(x, 0) = x, \end{cases} \tag{21}$$

is given for almost all ω and all (x, t) up to a certain stopping time by

$$u(x, t) = \frac{t - x}{2 \int_0^t \exp(W_s) ds - 1}.$$

- The solution to the equation

$$\begin{cases} du = -(1 - 2u) \cdot u_x dt + \sqrt{u - u^2} \cdot u_x \circ dW_t, \\ u(x, 0) = x, \end{cases} \tag{22}$$

is given for almost all ω and all (x, t) up to a certain stopping time by

$$u(x, t) = \frac{W_t \sqrt{W_t^2 + 4t^2 - 4t - 4x^2 + 4x} + W_t^2 + 4t^2 - 4xt - 2t + 2x}{2(1 - 4t + 4t^2 + W_t^2)}.$$

Replacing the standard Brownian motion by the geometric Brownian motion without drift we are able to determine also explicit solutions to different SPDEs.

- The solution to the equation

$$\begin{cases} du = -(1 - 2u) \cdot u_x dt + u_x \circ d[\exp(-t/2 + W_t)], \\ u(x, 0) = x. \end{cases} \tag{23}$$

is given for almost all ω and all (x, t) up to a certain stopping time by

$$u(x, t) = \frac{1 - x + t - \exp(-t/2 + W_t)}{2t - 1}.$$

- The solution to the equation

$$\begin{cases} du = -(1 - 2u) \cdot u_x dt + u \circ d[\exp(-t/2 + W_t)], \\ u(x, 0) = x, \end{cases} \tag{24}$$

is given for almost all ω and all (x, t) up to a certain stopping time by

$$u(x, t) = \frac{e(x - t)}{e^{-2 \int_0^t \exp(\exp(-s/2 + W_s)) ds}}.$$

- The solution to the equation

$$\begin{cases} du = -(1 - 2u) \cdot u_x dt + \sqrt{u - u^2} \cdot u_x \circ d[\exp(-t/2 + W_t)], \\ u(x, 0) = x, \end{cases} \tag{25}$$

is given for almost all ω and all (x, t) up to a certain stopping time by

$$\begin{aligned} u(x, t) = & (\exp(-t/2 + W_t) - 1) \\ & \cdot (\sqrt{4t^2 - 4t - 4x^2 + 4x + \exp(-t + 2W_t)} - 2 \exp(-t/2 + W_t) + 1 \\ & + 4t^2 - 4xt - 2t + 2x + \exp(-t + 2W_t) - 2 \exp(-t/2 + W_t) + 1) \\ & \cdot (2(4t^2 - 4t + \exp(-t + 2W_t) - 2 \exp(-t/2 + W_t) + 2))^{-1} \end{aligned}$$

Formally all given solutions need a verification, similarly to the proofs in Appendix C. But this should not be part of this manuscript.

Appendix C: Calculation and proofs

Claim (17) solves the stochastic partial differential equation (15).

Proof In a first step we determine the partial derivatives u_t and u_x by using $\frac{odW_t}{dt} = \dot{W}_t$. We obtain

$$\begin{aligned} \frac{du}{dt} &= \frac{d}{dt} \left[\frac{1 - x + t + W_t}{1 + 2t + 2W_t} \right] \\ &= \frac{(1 + 2t + 2W_t)(1 + \dot{W}_t) - (1 - x + t + W_t)(2 + 2\dot{W}_t)}{(1 + 2t + 2W_t)^2} \end{aligned}$$

and

$$\begin{aligned} \frac{du}{dx} &= \frac{d}{dx} \left[\frac{1 - x + t + W_t}{1 + 2t + 2W_t} \right] \\ &= -\frac{1}{(1 + 2t + 2W_t)}. \end{aligned}$$

Finally we have to verify that $u_t + (1 - 2u)u_x + (1 - 2u)u_x \dot{W}_t = 0$.

$$\begin{aligned}
 & u_t + (1 - 2u)u_x + (1 - 2u)u_x \dot{W}_t \\
 &= \frac{(1 + 2t + 2W_t)(1 + \dot{W}_t) - (1 - x + t + W_t)(2 + 2\dot{W}_t)}{(1 + 2t + 2W_t)^2} \\
 &\quad + \frac{(1 - 2x)}{(1 + 2t + 2W_t)^2} + \frac{(1 - 2x)\dot{W}_t}{(1 + 2t + 2W_t)^2} \\
 &= \frac{1 + \dot{W}_t + 2t + 2t\dot{W}_t + 2W_t + 2W_t\dot{W}_t - (2 + 2\dot{W}_t - 2x - 2x\dot{W}_t + 2t + 2t\dot{W}_t + 2W_t + 2W_t\dot{W}_t)}{(1 + 2t + 2W_t)^2} \\
 &\quad + \frac{1 - 2x + \dot{W}_t - 2x\dot{W}_t}{(1 + 2t + 2W_t)^2} \\
 &= \frac{1 + \dot{W}_t + 2t + 2t\dot{W}_t + 2W_t + 2W_t\dot{W}_t - 2 - 2\dot{W}_t + 2x + 2x\dot{W}_t - 2t - 2t\dot{W}_t - 2W_t - 2W_t\dot{W}_t}{(1 + 2t + 2W_t)^2} \\
 &\quad + \frac{1 - 2x + \dot{W}_t - 2x\dot{W}_t}{(1 + 2t + 2W_t)^2} \\
 &= \frac{\dot{W}_t - 2\dot{W}_t + 2x - 2x + \dot{W}_t}{(1 + 2t + 2W_t)^2} \\
 &= 0
 \end{aligned}$$

□

Acknowledgements

We would like to thank for the hospitality of CIMA-UMa at Madeira Math Encounters 2019 in Funchal. Without this meeting point, this publication would not have been possible. Also, we would like to extend our particular thanks to the CRC 1283 *Taming uncertainty and profiting from randomness and low regularity in analysis, stochastics and their applications* for support and generosity in attending conferences and workshops.

Funding

Open Access funding enabled and organized by Projekt DEAL.

Abbreviations

Bm, Brownian motion; gBm, geometric Brownian motion.

Availability of data and materials

The data is openly available and the sources can be found in the references.

Competing interests

The authors declare that they have no competing interests.

Authors' contributions

WB was involved in simulation and writing. NM was involved in exact computation and writing. All authors read and approved the final manuscript.

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Publisher's Note

Springer Nature remains neutral with regard to jurisdictional claims in published maps and institutional affiliations.

Received: 1 January 2021 Accepted: 29 April 2021 Published online: 07 May 2021

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