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Chordal-(k, ℓ) and strongly chordal-(k, ℓ) graph sandwich problems

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Abstract

Background: In this work, we consider the graph sandwich decision problem for property Π, introduced by Golumbic, Kaplan and Shamir: given two graphs $G^1 = (V, E^1)$ and $G^2 = (V, E^2)$, the question is to know whether there exists a graph $G = (V, E)$ such that $E^1 \subseteq E \subseteq E^2$ and G satisfies property Π. Particurlarly, we are interested in fully classifying the complexity of this problem when we look to the following properties Π: 'G is a chordal-(k, ℓ)-graph' and 'G is a strongly chordal-(k, ℓ)-graph', for all k, ℓ.

Methods: In order to do that, we consider each pair of positive values of k and ℓ, exhibiting correspondent polynomial algorithms, or NP-complete reductions.

Results: We prove that the STRONGLY CHORDAL-(k, ℓ) GRAPH SANDWICH PROBLEM is NP-complete, for $k \geq 1$ and $\ell \geq 1$, and that the CHORDAL-(k, ℓ) GRAPH SANDWICH PROBLEM is NP-complete, for positive integers k and ℓ such that $k + \ell \geq 3$. Moreover, we prove that both problems are in P when k or ℓ is zero and $k + \ell \leq 2$.

Conclusions: To complete the complexity dichotomy concerning these problems for all nonnegative values of k and ℓ, there still remains the open question of settling the complexity for the case $k + \ell \geq 3$ and one of them is equal to zero.

Keywords: Graph sandwich problem; Chordal-(k, ℓ)-graph; Strongly chordal-(k, ℓ)-graph

Background

In 1995, Golumbic, Kaplan and Shamir [1] introduced the SANDWICH PROBLEM:

> GRAPH SANDWICH PROBLEM FOR PROPERTY Π (Π−SP)
> *Instance*: $G^1 = (V, E^1)$ and $G^2 = (V, E^2)$, such that $E^1 \subseteq E^2$.
> *Question*: Is there a graph $G = (V, E)$ such that $E^1 \subseteq E \subseteq E^2$ and G satisfies property Π?

Observe that the graph G, if it exists, must be 'sandwiched' between the graphs G^1 and G^2 and must satisfy property Π. In order to avoid a trivial problem, we assume that G^1 and G^2 do not satisfy property Π.

Given two graphs $G^1 = (V, E^1)$ and $G^2 = (V, E^2)$ with $E^1 \subseteq E^2$, a graph $G = (V, E)$ is called a *sandwich* graph for the pair (G^1, G^2) if $E^1 \subseteq E \subseteq E^2$. Let $E^3 = E(\overline{G^2})$ denote the set of *forbidden* edges. We call E^1 the *mandatory* edge

set, and $E^2 \setminus E^1$ the *optional* edge set. Hence, any sandwich graph $G = (V, E)$ for the pair (G^1, G^2) must contain all mandatory edges and no forbidden edge.

The *recognition* problem for a class of graphs $\mathcal{C}$ is equivalent to the particular graph sandwich problem where $E^1 = E^2$, that is, the optional edge set is empty. Graph sandwich problems have attracted much attention lately because of many applications and by the fact that they naturally generalize recognition problems [2-6]. Note that Π−SP is clearly at least as hard as the problem of recognizing graphs with property Π, since given a polynomial-time algorithm for Π−SP, it is possible to use this algorithm with $E^1 = E^2 = E$ to recognize if a graph $G = (V, E)$ satisfies property Π. Thus, in such cases, we have problems in P that cannot be easier when regarded as sandwich problems.

A graph $G = (V, E)$ is *chordal* if every cycle of length at least four contains a chord, i.e, an edge between two non-consecutive vertices. It is well known [7,8] that chordal graphs can be recognized in polynomial time.

A *sun* is a chordal graph on $2n$ vertices ($n \geq 3$) whose vertex set can be partitioned into $W = \{w_1, \ldots, w_n\}$ and

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$U = \{u_1, \ldots, u_n\}$ such that W is an independent set, U is a clique and u_i is adjacent to w_j if and only if $i = j$ or $i = j+1$ (mod n). *Strongly chordal graphs* were defined by Farber [9] as chordal graphs that do not contain a sun. He also proved that strongly chordal graphs can be recognized in polynomial time.

A bipartite graph is *chordal bipartite* if each of its cycles of length, at least 6, has a chord.

Let ℓ and k be two non-negative integers. A graph G is (k, ℓ) if $V(G)$ can be partitioned into at most k independent sets and ℓ cliques (a (k, ℓ)-partition). The problem of recognizing if a graph G is (k, ℓ) was shown to be NP-complete if $k \geq 3$ or $\ell \geq 3$ and solvable in polynomial time otherwise [10-12]. In [10,12,13], polynomial-time algorithms have been developed for deciding if a graph admits a $(2, 1)$ or a $(2, 2)$-partition.

We denote by $(\ell + 1)K_{k+1}$ the graph obtained from the disjoint union of $(\ell + 1)$ copies of K_{k+1}. Chordal-(k, ℓ) graphs have been well studied in the literature. Hell et al. [14-16] have analyzed algorithmic and complexity aspects of chordal-(k, ℓ) graphs, proving that the recognition problems of chordal-(k, ℓ) and strongly chordal-(k, ℓ) graphs are in P, using the following characterization:

Theorem 1. *A chordal graph G is (k, ℓ) if and only if G does not contain a $(\ell + 1)K_{k+1}$ as an induced subgraph (Figure 1) [15].*

In [1], Golumbic et al. have presented a diagram showing the complexity status (at that time) of the sandwich problem for some subfamilies of perfect graphs like, for instance, chordal graphs. They have showed that CHORDAL GRAPH SANDWICH PROBLEM is NP-complete. Looking at this diagram, we can notice that the majority of the studied problems are NP-complete. Since then, many other problems, like $(2, 1)$-SP [17] and STRONGLY CHORDAL-SP [5], have been proved to be NP-complete.

In a previous paper [18], we have proved that the STRONGLY CHORDAL-$(2, \ell)$ GRAPH SANDWICH PROBLEM is NP-complete, for $\ell \geq 1$, and that the CHORDAL-(k, ℓ) GRAPH SANDWICH PROBLEM is NP-complete, for $k \geq 2$, $\ell \geq 1$.

In this paper, we establish the status complexity of the CHORDAL-(k, ℓ)-SP and STRONGLY CHORDAL-(k, ℓ)-SP

for integers $k \geq 1$ and $\ell \geq 1$: all of them are NP-complete, except CHORDAL-$(1, 1)$-SP proved in P by Golumbic et al. [1]. We show some progress towards the P versus NP dichotomy when k and ℓ are non-negative integers by proving the following in P: CHORDAL-$(0, k)$-SP, STRONGLY CHORDAL-$(0, k)$-SP, CHORDAL-$(k, 0)$-SP, STRONGLY CHORDAL-$(k, 0)$-SP when $k = 1, 2$.

Methods

In order to prove that STRONGLY CHORDAL-(k, ℓ)-SP is NP-complete for $k \geq 1$ and $\ell \geq 1$ we proved first that STRONGLY CHORDAL-$(1, 1)$-SP is in NP-c making a polynomial reduction from the NP-complete sandwich problem for chordal bipartite graphs [6]. After, we proved that both STRONGLY CHORDAL-$(1, \ell)$-SP and STRONGLY CHORDAL-$(k, 1)$-SP are NP-complete by induction, showing beforehand, that STRONGLY CHORDAL-$(1, \ell+1)$-SP and STRONGLY CHORDAL-$(k + 1, 1)$-SP are also NP-complete.

To prove that CHORDAL-(k, ℓ)-SP is NP-complete for $k + \ell \geq 3$, k, ℓ positive integers, we proved first that $(2, 1)$ TRIANGULATING COLORED GRAPHS ($(2, 1)$ TCG) is NP-complete showing that the instance used to prove that the problem of Triangulating Colored Graphs (TCG) is NP-complete [19] is also a $(2, 1)$-graph. Using this result we proved that CHORDAL-$(2, 1)$-SP is in NP-c. Still using this instance, we proved that CHORDAL-$(1, 2)$-SP is NP-complete. To do that, we changed the set of added edges in order to obtain one independent set and two cliques instead of two independent sets and one clique. After, we proved the NP-completeness of CHORDAL-(k, ℓ)-SP, for $k, \ell > 0$ and $k + \ell \geq 3$ by induction on k and ℓ.

Finally we proved that (STRONGLY) CHORDAL-$(0, 2)$-SP can be solved in polynomial time by a polynomial reduction to the problem of 2-Satisfability [20]. With this result added to some trivial ones we showed that CHORDAL-$(0, k)$-SP, STRONGLY CHORDAL-$(0, k)$-SP, CHORDAL-$(k, 0)$-SP, STRONGLY CHORDAL-$(k, 0)$-SP when $k = 1, 2$ are polynomially solvable.

Results and discussions
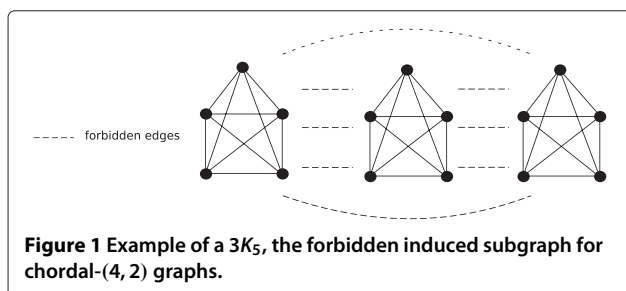### Strongly chordal-(k, ℓ) graph sandwich problem
The purpose of this section is to prove that the STRONGLY CHORDAL-(k, ℓ)-SP is NP-complete for all $k, \ell \geq 1$. This problem can be formulated as follows:

> STRONGLY CHORDAL-(k, ℓ) GRAPH SANDWICH PROBLEM
> *Instance*: $G^1 = (V, E^1)$ and $G^2 = (V, E^2)$, such that $E^1 \subseteq E^2$.
> *Question*: Is there a graph $G = (V, E)$ such that $E^1 \subseteq E \subseteq E^2$ and G is a strongly chordal-(k, ℓ) graph?

We notice that k and ℓ are not part of the input.

Figure 1 Example of a $3K_5$, the forbidden induced subgraph for chordal-$(4, 2)$ graphs.

In order to prove the main theorem, we will first present the STRONGLY CHORDAL-$(1,1)$-SP.

Strongly chordal-$(1,1)$ graph sandwich problem

We prove that the STRONGLY CHORDAL-$(1,1)$ SANDWICH PROBLEM is NP-complete by showing a reduction from the NP-complete problem CHORDAL BIPARTITE GRAPH SANDWICH PROBLEM [6].

Proposition 1. *Let $G = (X, Y, E)$ be a bipartite graph and G' be the graph obtained from the addition of edges to X in order to obtain a clique. Then G is chordal bipartite if and only if G' is strongly chordal* [21].

Notice that, since X is transformed into a clique, G is chordal bipartite if and only if G' is strongly chordal-$(1,1)$.

We can formulate the CHORDAL BIPARTITE GRAPH SANDWICH PROBLEM as follows:

CHORDAL BIPARTITE GRAPH SANDWICH PROBLEM
Instance: $G^1 = (V, E^1)$ and $G^2 = (V, E^2)$, such that $E^1 \subseteq E^2$.
Question: Is there a graph $G = (V, E)$ such that $E^1 \subseteq E \subseteq E^2$ and G is a chordal bipartite graph?

Theorem 2. *The* STRONGLY CHORDAL-$(1,1)$-SP *is NP-complete.*

Proof. The problem is clearly in NP. In order to prove its NP-completeness, we will consider the following special instance $(G^{1'}, G^{2'})$ of the STRONGLY CHORDAL-$(1,1)$-SP obtained from (G^1, G^2), an instance of the NP-complete problem CHORDAL BIPARTITE-SP [6], such that there is a chordal bipartite sandwich graph G for (G^1, G^2) if and only if there is a strongly chordal-$(1,1)$ sandwich graph G' for $(G^{1'}, G^{2'})$.

First, we observe that CHORDAL BIPARTITE-SP is NP-complete even when G^1 is connected. Let G^1 be (X, Y, E^1) and we define $G^{1'}$ and $G^{2'}$ as follows: $G^{1'} = (X, Y, E^{1'})$, where $E^{1'} = E^1 \cup \{(x_i, x_j) | x_i, x_j \in X\}$ and $G^{2'} = (V, E^2 \cup \{(x_i, x_j) | x_i, x_j \in X\})$. This finishes the construction of $(G^{1'}, G^{2'})$.

The proof of the NP-completeness of the STRONGLY CHORDAL-$(1,1)$ follows from Proposition 1. $\square$

Strongly chordal-(k, ℓ) graph sandwich problem, $k \geq 1, \ell \geq 1$

After solving the STRONGLY CHORDAL-$(1,1)$-SP, the question about the complexity of the STRONGLY CHORDAL-(k, ℓ)-SP, for $k \geq 1$ and $\ell \geq 1$, naturally arises. We have succeeded to prove that the STRONGLY CHORDAL-$(1, \ell)$-SP, for $\ell \geq 1$ and the STRONGLY CHORDAL-$(k, 1)$-SP, for $k \geq 1$, are NP-complete.

Lemma 1. *Given $k \geq 1$ and $\ell \geq 1$, if* STRONGLY CHORDAL-(k, ℓ)-SP *is NP-complete, then* STRONGLY CHORDAL-$(k, \ell + 1)$-SP *is NP-complete.*

Proof. We remark that the STRONGLY CHORDAL-(k, ℓ) GRAPH SANDWICH PROBLEM, $k \geq 1, \ell \geq 1$ is in NP, since we can check in polynomial time whether a graph G is a sandwich for a pair (G^1, G^2) and whether it is strongly chordal-(k, ℓ) [14-16].

We consider the following special instance $(G^{1'}, G^{2'})$ of STRONGLY CHORDAL-$(k, \ell + 1)$-SP obtained from (G^1, G^2), an instance of the NP-complete problem STRONGLY CHORDAL-(k, ℓ)-SP, such that there is a strongly chordal-(k, ℓ) sandwich graph G for (G^1, G^2) if and only if there is a strongly chordal-$(k, \ell + 1)$ sandwich graph G', $k \geq 1$, $\ell \geq 1$, for $(G^{1'}, G^{2'})$.

From (G^1, G^2), we define one additional clique K, such that $|K| = k + 1$. We set $V(G^{1'}) = V(G^{2'}) = V(G^1) \cup V(K), E(G^{1'}) = E^1 \cup E(K)$ and $E(G^{2'}) = E^2 \cup E(K)$. This concludes the construction of the instance $(G^{1'}, G^{2'})$ (see Figure 2 as an example).

Suppose there is a strongly chordal-(k, ℓ) sandwich graph G for (G^1, G^2). Consider G' formed by G plus the forced edges of $(G^{1'}, G^{2'})$. In order to prove that graph G' is a strongly chordal graph, we consider the strong elimination sequence started by any sequence of the vertices of K, followed by a strong elimination sequence of the strongly chordal graph G. In order to prove that G' is $(k, \ell + 1), k \geq 1$ and $\ell \geq 1$, we consider a (k, ℓ)-partition for G, and we set a $(k, \ell + 1)$-partition for G'
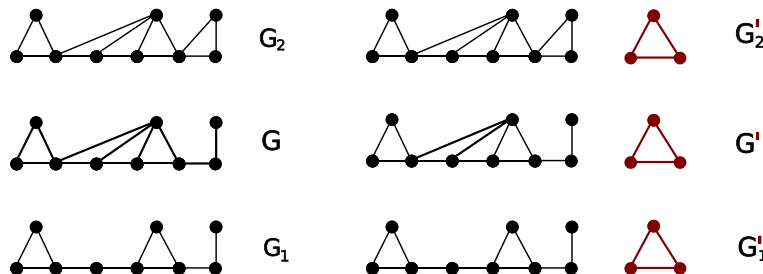

Figure 2 Example of the instance when $k = 2$ and $\ell = 1$. Note that when G has two isolated triangles, G' will have three isolated triangles.

formed by the k independent sets, the ℓ cliques of G, and K.

Suppose there is a strongly chordal-$(k, \ell + 1)$ sandwich graph G' for $(G^{1'}, G^{2'})$, $k \geq 1$ and $\ell \geq 1$. Given $G = G' - K$, we will prove that G is a strongly chordal-(k, ℓ) sandwich graph for (G^1, G^2). Suppose by contradiction that G is not a strongly chordal-(k, ℓ) graph. First, note that, since being a strongly chordal graph is an hereditary property, G must be strongly chordal. Thus, if G is not strongly chordal-(k, ℓ) then it is because G is not a (k, ℓ)-graph. It follows from Theorem 1 that G contains a $(\ell + 1)(K_{k+1})$ as an induced subgraph. Since G' is the disjoint union of G and K, there is an induced subgraph $(\ell + 2)K_{k+1}$ of G' formed by K and the induced subgraph $(\ell+1)(K_{k+1})$ of G. By Theorem 1, G' is not strongly chordal-$(k, \ell + 1)$, a contradiction. Hence, G is a strongly chordal-(k, ℓ) graph. $\qquad \square$

Theorem 3. STRONGLY CHORDAL-$(1, \ell)$-SP, *for* $\ell \geq 1$, *is NP-complete.*

Proof. The proof of Theorem 3 is done by induction using Theorem 2 and Lemma 1. $\qquad \square$

Lemma 2. *Given* $k \geq 1$, STRONGLY CHORDAL-$(k, 1)$-SP *is NP-complete.*

Proof. We remark that the STRONGLY CHORDAL-$(k, 1)$ GRAPH SANDWICH PROBLEM, $k \geq 1$ is in NP, since we can check in polynomial time whether a graph G is a sandwich for a pair (G^1, G^2) and whether it is strongly chordal-(k, ℓ) [9,15].

We consider the following special instance $(G^{1'}, G^{2'})$ of STRONGLY CHORDAL-$(k, 1)$-SP obtained from (G^1, G^2), a connect instance of the NP-complete problem CHORDAL BIPARTITE-SP [6], such that there is a chordal bipartite sandwich graph $G = (V, E)$ for (G^1, G^2) if and only if there is a strongly chordal-$(k, 1)$ sandwich graph G', $k \geq 1$ for $(G^{1'}, G^{2'})$.

Notice that if there exists G a chordal bipartite sandwich graph, then $G^1 = (V, E^1)$ must be bipartite. Let $G^1 = (X, Y, E^1)$. Given $Y = \{y_1, y_2, \ldots, y_q\}$, we describe

- $V(G^{1'}) = V(G^{2'}) = V \cup \{w_1, w_2, \ldots, w_{k-1}\}$ and
- $E(G^{1'}) = E(G^{2'}) = E^1 \cup \{(x_i, x_j)|x_i, x_j \in X\}$
  $\cup \{(w_i, w_j), (w_i, y_1)|i \neq j; i, j \in \{1, 2, \ldots, k - 1\}\}$.

This concludes the construction of the instance $(G^{1'}, G^{2'})$ (see Figure 3 as an example).

We will prove now that there exists G a chordal bipartite sandwich graph for (G^1, G^2) if and only if there exists G' a strongly chordal-$(k, 1)$ sandwich graph for $(G^{1'}, G^{2'})$.

Suppose that G is a chordal bipartite sandwich graph for (G^1, G^2). Let G' be the graph where $V(G') = V(G^{1'})$ and

Figure 3 Example of the construction of the instance when $k = 3$.

$E(G') = E(G) \cup \{(w_i, w_j), (w_i, y_1)|i \neq j; i, j \in \{1, 2, \ldots, k - 1\}\} \cup \{(x_i, x_j)|x_i, x_j \in X\}$. We will prove that G' is strongly chordal and we will exhibit the partition of its vertex set into k independent sets and one clique. Observe that a sun of G' entirely belongs to a block of G'. Since y_1 is a cut-vertex, we have that a sun of G' belongs to the graph $G'[V]$. By the Proposition 1, $G'[V]$ is strongly chordal. Hence, we can ensure that G' is strongly chordal. Moreover, we can exhibit the $(k, 1)$-partition of G': Each vertex of $\{y_1, w_1, w_2, \ldots, w_{k-1}\}$ participates of an independent set and the clique is induced by X.

Now suppose that G' is a strongly chordal-$(k, 1)$ sandwich graph for $(G^{1'}, G^{2'})$. We prove next that $G = (V, E)$, where $E = \{E(G') \backslash \{(x_i, x_j)|x_i, x_j \in X\}$. We observe that the instance (G^1, G^2) of the NP-complete sandwich problem for chordal bipartite graphs [6] can be assumed to satisfy that G^1 is connected and G^2 is bipartite. Suppose by contradiction, that G contains a C_6. In this case, we would have a sun in $G'[V]$, and this is a contradiction. $\qquad \square$

Theorem 4. STRONGLY CHORDAL-(k, ℓ)-SP *is NP-complete for* $k \geq 1$ *and* $\ell \geq 1$.

Proof. This proof follows from Lemma 2 and Theorem 3. □

Chordal-(k, ℓ) graph sandwich problem

This problem can be formulated as follows:

CHORDAL-(k, ℓ) GRAPH SANDWICH PROBLEM
Instance: $G^1 = (V, E^1)$ and $G^2 = (V, E^2)$, such that $E^1 \subseteq E^2$.
Question: Is there a graph $G = (V, E)$ such that $E^1 \subseteq E \subseteq E^2$ and G is a chordal-(k, ℓ) graph?

From the literature, we know that the CHORDAL-$(1, 1)$-SP is polynomially solvable [1]. In order to prove the main theorem of this section, we will first present the proof that CHORDAL-$(2, 1)$-SP is NP-complete. We observe that this result is stated in [18], but the proof was omitted.

Chordal-$(2, 1)$ graph sandwich problem

Theorem 5. CHORDAL-$(2, 1)$-SP *is NP-complete.*

Proof. This proof follows from the NP-completness proof of TRIANGULATING COLORED GRAPHS (TCG) [19] and of CHORDAL-SP [1]. We can fomally define TCG as follows:

TRIANGULATING COLORED GRAPHS (TCG)
Input: A graph $G = (V, E)$ and a proper vertex coloring $c : V \to \mathbb{Z}$.
Question: Does there exist a supergraph $G' = (V, E')$ of G that is chordal and properly colored on vertices by c?

First, we will prove that the $(2, 1)$-TRIANGULATING COLORING GRAPHS ($(2, 1)$-TCG) is NP-complete. This problem cam be formulated as follows:

$(2, 1)$-TRIANGULATING COLORED GRAPHS ($(2, 1)$TCG)
Input: A graph $G = (V, E)$ and a proper vertex coloring $c : V \to \mathbb{Z}$.
Question: Does there exist a supergraph $G' = (V, E')$ of G that is chordal-$(2, 1)$ and properly colored on vertices by c?

We will use the same polynomial reduction from THREE-SATIFIABILITY PROBLEM (3SAT) [20] that was used by Bodlaender et al. to prove that TCG is NP-complete, and we will show that the instance used is also a (2,1)-graph. So, we will first present here the particular chordal graph they have constructed.

For our purposes, we will consider a graph G_I obtained by an instance I of 3SAT that is composed by a decision component and some clause components. We will suppose that any clause contains both a literal and its complement. For each variable X or its complement $\overline{X}$ contained in a clause i of I we have a decision component like the one of Figure 4.
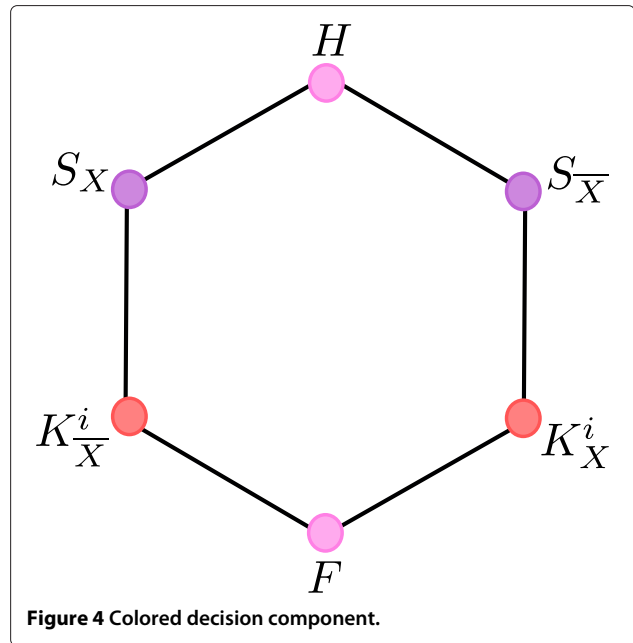
Figure 4 Colored decision component.

Each decison component has the vertex set: H (head), $S_X, S_{\overline{X}}$ (shoulders), $K_X^i, K_{\overline{X}}^i$ (knees) and F (foot). The instance G_I has only one head and one foot, a pair of shoulders for each variable X of I and a pair of knees for each occurence of X or $\overline{X}$ in a clause i of I. Moreover, as we can see in Figure 4, head and foot will receive the same color. We will do the same for each pair of shoulders $S_X, S_{\overline{X}}$ and for each pair of knees $K_X^i, K_{\overline{X}}^i$. Note that each color is assigned to exactly two vertices.

In order to triangulate the component related to the variable X, we can add either the set of edges $\left(H, K_X^i\right), \left(S_X, K_X^i\right), (S_X, F)$ or the set of edges $\left(H, K_{\overline{X}}^i\right), \left(S_{\overline{X}}, K_{\overline{X}}^i\right), (S_{\overline{X}}, F)$. We call the first set as *Mark of Zorro on positive orientation* and the second set as *Mark of Zorro on negative orientation* (Figure 5). Furthermore, we add either all edges $\left(H, K_X^i\right)$ or all edges $\left(H, K_{\overline{X}}\right)$.

If the Mark of Zorro is positively oriented in the decision component of X, the literal X will be set as TRUE. Otherwise, it will be set as FALSE.

To construct a clause component, we will not add vertices to G_I. We will just add some edges between some knees.

Let L be a literal of the clause i. We say that K_L^i is an active knee and $K_{\overline{L}}^i$ is an inactive knee.

Consider f as a truth assignment that satisfies the instance I of 3SAT. In this case, there exists at least one true literal in each clause. We will rename the variables in order to have X as a true literal and $\overline{X}$ as a false literal.

We have finally constructed the graph G_I (see Figure 6 as an example) and we will transform this graph in one that is chordal and $(2, 1)$. So, we will add some edges and prove
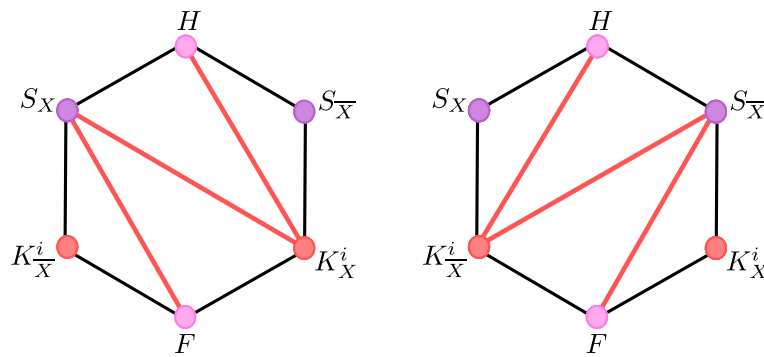
Figure 5 Marks of Zorro in positive and negative orientations, from left to right.

that these edges turn G_I into a (2, 1)-graph. The proof that it is also a chordal graph can be found in [19].

To transform G_I into a chordal-(2,1) graph, we will add

- The edges of the positive orientation of the Mark of Zorro;
- Edges that compose the complete graph formed by true shoulders and true knees; and
- Edges such that true shoulders are adjacent to all false knees.

Bodlaender et al. have proved that this modified instance is a chordal graph [19]. In order to show that this graph is also a (2, 1)-graph, we will prove that it does not contain a pair of isolated triangles [15].

Observing the set of edges we have added in order to transform G_I into a chordal-(2, 1) graph, we can see that each triangle formed by the addition of these edges has at least one true shoulder or at least one true knee. However, the clause components form three types of triangles in the graph:

1. Triangles composed by active knees.
   In this case, since the truth assignment must satisfy the instance I, all triangles must have at least one true knee.
2. Triangles composed by the foot and one true knee; and
3. Triangles composed by two false knees and the foot.

Note that the third type of triangles do not have true knees or true shoulders, but they have a common vertex: F. Also note that the foot is adjacent to all knees and to all true shoulders. Thus, this type of triangle is not isolated from any other triangle of the graph, and then we can state that this modified instance is a (2,1)-graph. Therefore, we can also exhibit the (2,1)-partition.

Clique - True shoulders, true knees and the foot.
Independent sets

- S_1: The head, active false knees adjacent to inactive true knees, inactive false knees.
- S_2: False shoulders, active false knees adjacent to inactive false knees.
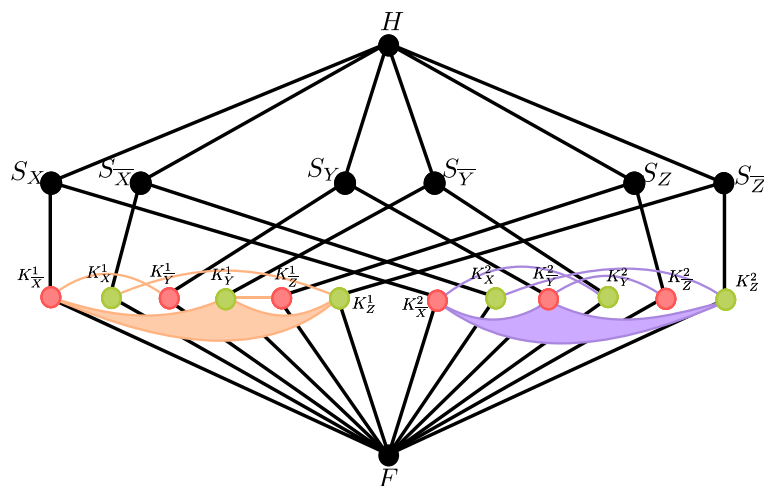
Figure 6 Example of graph G_I obtained from the instance $U = \{X, Y, Z\}, C = \{(\overline{X}, Y, Z), (\overline{X}, \overline{Y}, Z)\}$ of 3SAT.

Note that the head is not adjacent to false knees, and active false knees adjacent to inactive true knees are not adjacent to inactive false knees, even when they are from distinct clauses. Moreover, false shoulders are not adjacent to false knees. Hence, this (2,1)-partition is well defined. □

Chordal-(1, 2) graph sandwich problem

In order to prove that CHORDAL-(1,2) GRAPH SANDWICH PROBLEM is NP-complete, we will use the same instance G_I used to prove that CHORDAL-(2, 1)-SP is NP-complete.

Lemma 3. *The instance I of 3-SAT is satisfiable if and only if there is a $(1, 2)$-triangulation for G_I respecting the proper coloring of G_I.*

Proof. The sufficiency proof of Lemma 3 is already done in [19].

In order to prove its necesssity, suppose that there is a satisfiable truth assignment f for I. We will add the following set of edges in order to obtain a chordal-$(1, 2)$ graph respecting the proper coloring of G_I.

- The positive orientation of Mark of Zorro for each decision component;
- All edges between true knees and true shoulders, in order to obtain a clique;
- Edges such that every true shoulder sees every false knee;
- All edges between active knees, and
- Edges between inactive true knees adjacent to active false knees and active false knees adjacent to inactive true knees

Let G_1 be the instance G_I plus these additional edges and consider the following sets:

- $S_1 = \{$False shoulders, inactive false knees$\}$;
- $S_2 = \{$Head$\}$;
- $S_3 = \{$Inactive true knee adjacent to an active true knee$\}$;
- $S_4 = \{$Active false knee adjacent to an inactive false knee$\}$;
- $S_5 = \{$Active false knee adjacent to an inactive true knee (in the same clause component)$\}$, and
- $S_6 = \{$Active true knees, true shoulders, foot$\}$.

First, observe that these added edges are in the set of optional edges of G_I. After, lets analyze the neighborhoods of each vertex of these sets.

False shoulders are adjacent to the head and to some true knees. Since head and true knees form a clique, each false shoulder is a simplicial vertex that can be removed. Inactive false knees are adjacent to the foot, to true shoulders and to an active knee (true or false). This set is also a clique and inactive false knees are simplicial vertices as well, so they can be deleted. Let G_2 be the resulting graph after these remotions.

The neighborhood of the head in G_2 is formed by true shoulders and true knees, what induces a clique in G_2. Then the head is a simplicial vertex that can be removed of the graph. Let G_3 be the graph after the remotion of the Head.

The vertices of S_3 in G_3 are adjacent to the foot, to true shoulders and to true knees. Again, this set of vertices induces a clique in G_3. So, inactive true knees adjacent active true knees are simplicial vertices in G_3 that can be deleted forming G_4.

The vertices of S_4 in G_4 are adjacent to true shoulders, to the foot and to all active knees (true or false). Active knees form a clique as well as true shoulders and the foot sees every knee and every true shoulder. Moreover, true shoulders are adjacent to all knees. So, this neighborhood is also a clique what characterizes each active false knee adjacent to an inactive false knee as a simplicial vertex in G_4. Let G_5 be the graph after the remotion of all active false knee adjacent to an inactive false knee.

The vertices of S_5 are adjacent to all active knees, to all inactive true knees adjacent to an active false knee, to true shoulders and to the foot in G_5. This neighborhood induces a clique, and, because of that, active false knees adjacent to inactive true knees can be removed in this step. Let G_6 be the resulting graph, after the remotion of vertices of S_5.

Clearly G_6 is a clique.

Observe that, if we follow the order of the sets, any elimination ordering of vertices applied to each set creates a perfect elimination ordering for the graph G_1. Moreover, we can present a $(1, 2)$-partition for the vertex set of G_1:

Independent set - False shoulders and inactive false knees.
Clique 1 - Head, true shoulders and true knees.
Clique 2 - Foot and active false knees.

This concludes the proof of Lemma 3. □

Theorem 6. CHORDAL-(1,2)-SP *is NP-complete.*

Proof. Clearly, CHORDAL-(1,2)-SP is in NP. The NP-completeness proof follows from the polynomial redution done in Lemma 3. □

Chordal-(k, ℓ) graph sandwich problem, $k \geq 1$, $\ell \geq 2$

Lemma 4. *Given k and ℓ positive integers such that $k + \ell \geq 3$, if* CHORDAL-(k, ℓ)-SP *is NP-complete, then* CHORDAL-$(k + 1, \ell)$-SP *is NP-complete.*

Proof. Observe that the CHORDAL-(k, ℓ) GRAPH SANDWICH PROBLEM, $k, \ell > 0$ and $k + \ell \geq 3$, is in NP, since we can check in polynomial time whether a graph G is a sandwich for a pair (G^1, G^2) and whether it is chordal-(k, ℓ) [15].

We consider the following special instance $(G^{1'}, G^{2'})$ of the problem CHORDAL-$(k + 1, \ell)$-SP obtained from (G^1, G^2), an instance of the NP-complete problem CHORDAL-(k, ℓ)-SP, such that there is a (k, ℓ)-chordal sandwich graph G for (G^1, G^2) if and only if there is a $(k + 1, \ell)$-chordal sandwich graph G', $k, \ell > 0$ and $k + \ell \geq 3$, for $(G^{1'}, G^{2'})$.

We denote by n_1 and n_2, respectively, the number of K_{k+1}'s of G^1 and G^2. Let $K_{k+1}^1, K_{k+1}^2, \cdots, K_{k+1}^{n_2}$ be the subgraphs of G^2 isomorphic to K_{k+1}. For each K_{k+1}^i of G^2 we define one additional vertex u_i, such that $V(K_{k+1}^i) \cup \{u_i\}$ forms a clique K^i, $i = 1, \cdots, n_2$. We set $V(G^{1'}) = V(G^{2'}) = V(G^1) \bigcup_{i=1}^{n_2} \{u_i\}$, $E(G^{1'}) = E^1 \bigcup_{i=1}^{n_1} E(K^i)$, and $E(G^{2'}) = E^2 \bigcup_{i=1}^{n_2} E(K^i)$. This concludes the construction of the instance $(G^{1'}, G^{2'})$.

Suppose there is a chordal-(k, ℓ) sandwich graph G for (G^1, G^2). Consider G' formed by G plus the forced edges of $(G^{1'}, G^{2'})$. In order to prove that G' is a chordal graph, we consider the perfect elimination sequence started by the simplicial vertices u_i, $i = 1, \cdots, n_2$, and followed by a perfect elimination sequence of the chordal graph G. To prove that G' is a $(k+1, \ell)$-graph, $k, \ell > 0$ and $k + \ell \geq 3$, we consider a (k, ℓ)-partition of G, and we define a $(k + 1, \ell)$-partition for G' formed by the k independents sets of G, the independent set $\bigcup_{i=1}^{n_2} \{u_i\}$, and the ℓ cliques of G.

Suppose there is a chordal-$(k + 1, \ell)$ sandwich graph G' for $(G^{1'}, G^{2'})$, $k, \ell > 0$ and $k + \ell \geq 3$. We consider $G = G' - \bigcup_{i=1}^{n_2} \{u_i\}$ and we will prove that G is a chordal-(k, ℓ)-sandwich graph for (G^1, G^2). Suppose by contradiction that G is not a chordal-(k, ℓ) graph. Note that, since being a chordal graph is an hereditary property, if G is not chordal then G' is not chordal either. It follows from Theorem 1 that G contains a $(\ell + 1)K_{k+1}$ as an induced subgraph (see Figure 1). By construction of G', for each K_{k+1}^i of G^2 there is an additional vertex u_i forming a K_{k+2} in $G^{2'}$ (see Figure 7 for an example). Then, we have $(\ell + 1)$ copies of K_{k+2} in G' which, by the characterization in [15], implies that G' is not chordal-$(k + 1, \ell)$. Hence, G must be chordal-(k, ℓ). $\square$

Theorem 7. CHORDAL-(k, ℓ)-SP, *for* $k, \ell > 0$ *and* $k + \ell \geq 3$, *is NP-complete.*

Proof. The proof of Theorem 7 is done by a two-step mathematical induction on k and ℓ where the induction bases follow from Theorems 5 and 6 and the inductive steps follow from Lemma 1 and Lemma 4. $\square$

Chordal-(k, ℓ) graph sandwich problem when $k = 0$ or $\ell = 0$

Observe first that $(1, 0)$ and $(2, 0)$ are hereditary polynomial time checkable properties, and that a $(2, 0)$-graph G is chordal if and only if G is acyclic. Hence, in both cases, in order to have a positive answer for the sandwich problem, G^1 must satisfy the required property. Therefore, (STRONGLY) CHORDAL-$(1, 0)$-SP and (STRONGLY) CHORDAL-$(2, 0)$-SP are polynomially solvable. Moreover, since the existence of a chordal-$(0, 1)$ sandwich graph depends on G^2 being a clique, we have that (STRONGLY) CHORDAL-$(0, 1)$-SP is polynomially solvable as well.

Chordal-$(0, 2)$ graph sandwich problem

The following Lemma allows us to analyze the complexity of the (STRONGLY) CHORDAL-$(0, 2)$ GRAPH SANDWICH PROBLEM.

Lemma 5. *G is chordal-$(0, 2)$ if and only if G is strongly chordal-$(0, 2)$.*

Proof. If G is chordal-$(0, 2)$, then G cannot have a sun as an induced subgraph since, in this case, the size of an independent set is at most 2.

If G is strongly chordal-$(0, 2)$, then G is chordal-$(0, 2)$ by definition. $\square$

Theorem 8. CHORDAL-$(0, 2)$-SP *is polynomially solvable.*

Proof. In order to define an algorithm for CHORDAL-$(0,2)$ sandwich graph problem we use a polynomial reduction to the poly-time problem 2SAT. With this purpose,
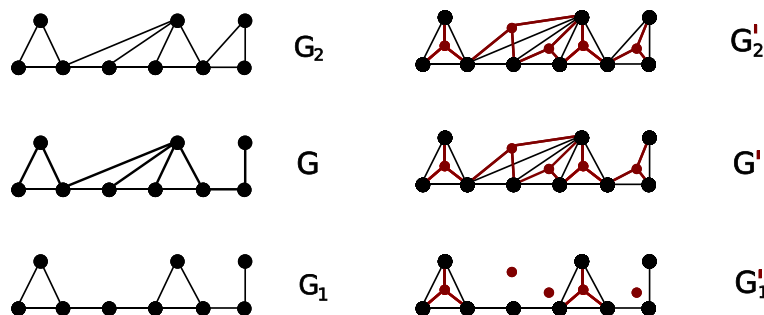

Figure 7 Example of the addition of vertices u_i when $k = 2$ and $\ell = 1$. Note that when G has two isolated triangles, G' will have two isolated K_4's.
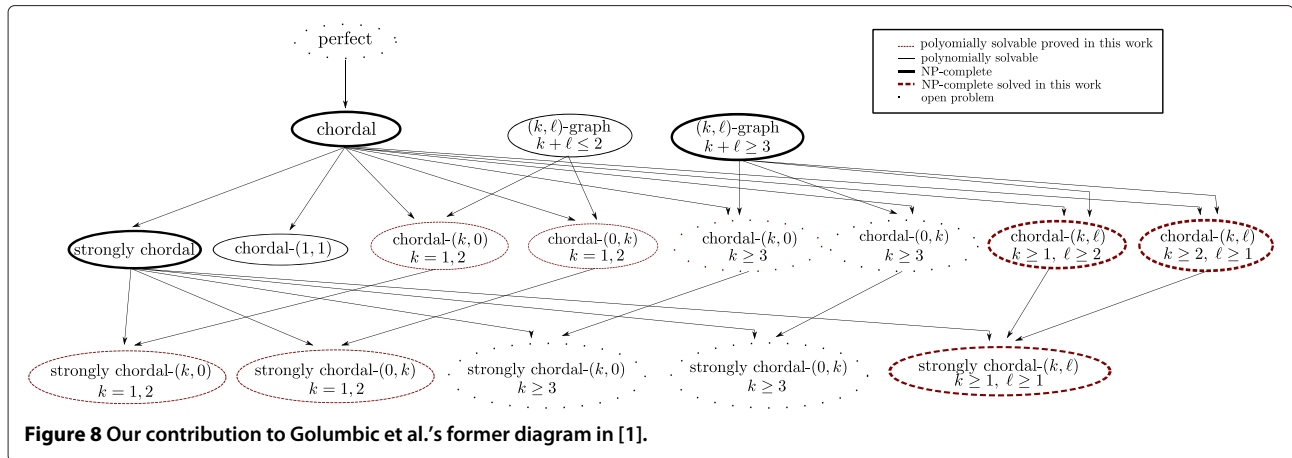
Figure 8 Our contribution to Golumbic et al.'s former diagram in [1].

from the vertex u and the partition (V_1, V_2) into two cliques for V, we define the Boolean variable:

$u^i = T \leftrightharpoons u$ belongs to the part $V_i, i \in \{1, 2\}$.

Given a CHORDAL-(0,2) sandwich instance (G^1, G^2).

We assume that there is no universal vertex in G^2 since in a $(0, 2)$-partition, this vertex can be added to any part. Next, we describe our polynomial time algorithm to CHORDAL-$(0, 2)$-SP.

Algorithm 1 CHORDAL-(0,2)-SP
___

Input: $G^1 = (V, E^1), G^2 = (V, E^2)$
Output: A chordal-(0,2) sandwich graph G or the answer NO if (G^1, G^2) is a NO instance for CHORDAL-$(0, 2)$-SP.
Begin

1. $U \leftarrow \emptyset; C \leftarrow \emptyset;$
2. For each $uw \in E^3$, a forbidden edge, add u^1, u^2, w^1, w^2 to U and add to C the clauses:

   (a) $(u^1, u^2), (w^1, w^2),$
   (b) $(\bar{u}^1, \bar{u}^2), (\bar{w}^1, \bar{w}^2),$
   (c) $(u^1, \bar{w}^2), (\bar{u}^1, w^2), (u^2, \bar{w}^1), (\bar{u}^2, w^1);$

3. For each quadruple of vertices t, u, v, w of V where $uw, tv \in E^3$, and $tu, vw \in E^1$ insert the clauses to C:
   $(u^1, \bar{v}^2), (\bar{u}^1, v^2), (u^2, \bar{v}^1), (\bar{u}^2, v^1);$
4. Run a polynomial algorithm for the instance $I = (U, C)$. If there is a truth assignment for I, then construct the sandwich graph G by setting into V_1 vertex v if and only if $v^1 = T$. Now add the optional edges between vertices placed into a same part in order to define E and return $G = (V, E), (V_1, V_2)$. Else return NO.

End.

Correctness: We notice that if $I = (U, C)$ is satisfiable, then a chordal-(0,2) sandwich graph G is obtained for

the CHORDAL-(0,2) instance (G_1, G_2) by setting vertex u on V_1 if and only if variable u^1 is true and on V_2 otherwise. The only optional edges added are the edges between vertices inside a same part either V_1 or V_2.

In order to see that, notice that clauses in 2a provide that the endpoints of each forbidden edge belong to a part of the partition (V_1, V_2). Clauses in 2b provide that each endpoint of each forbidden edge belong to exactly one part of the partition (V_1, V_2). Clauses in 2c provide that if one endpoint of a forbidden edge belong to one part, then the other endpoint belongs to the other part of the partition (V_1, V_2). Finally, clauses in 3 provide that from a satisfiable solution for $I = (U, C)$, there is no C_4 in G.

Suppose that there is a chordal sandwich graph G with partition into cliques (V_1, V_2) for the CHORDAL-(0,2) instance (G_1, G_2). We assume that all the optional edges of G belong to either $G[V_1]$ or $G[V_2]$. Hence, no forbidden edge is allowed inside V_1 or V_2, and no C_4 have two forced edges connecting vertices from V_1 to V_2. Hence, there is a satisfiable solution for $I = (U, C)$ such that the sandwich graph G is obtained for the CHORDAL-(0,2) instance (G_1, G_2). □

Conclusions

Figure 8 summarizes the results of this paper and presents open problems.

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