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Actuation Spaces Synthesis of Lower-Mobility Parallel Mechanisms Based on Screw Theory

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Abstract

The lower-mobility parallel mechanism has been widely used in the engineering field due to its numerous excellent characteristics. However, little work has been devoted to the actuator selection and placement that best satisfy the system's functional requirements during concept design. In this study, a unified approach for synthesizing the actuation spaces of both rigid and flexure parallel mechanisms has been presented, and all possible combinations of inputs could be obtained, laying a theoretical foundation for the subsequent optimization of inputs. According to the linear independence of actuation space and constraint space of the lower-mobility parallel mechanism, a general expression of actuation spaces in the format of screw systems is deduced, a unified synthesis process for the lower-mobility parallel mechanism is derived, and the efficiency of the method is validated with two selective examples based on screw theory. This study presents a theoretical framework for the input selection problems of parallel mechanisms, aiming to help designers select and place actuators in a correct and even optimal way after the configuration design.

Keywords: Lower-mobility parallel mechanism, Screw theory, Actuation space, Actuator placement, Input selection

1 Introduction

The lower-mobility parallel mechanism (PM) [1] generally refers to parallel mechanisms with two to five degrees of freedom (DOFs). With the development and wide application of parallel mechanisms, the lower-mobility parallel mechanism has become a focus of manipulators in international academic and industrial circles because of its high stiffness, high speed, high precision, simple structure, and being easily controlled. It has been successfully applied in many engineering fields, such as flexible micro-positioning platforms [2], precision attitude adjusting devices [3], the Z3 spindle head based on the 3-PRS PM [4], the Tricept hybrid parallel manipulator [5], the space docking mechanism [6], and so on.

The configuration synthesis [7–11] of lower-mobility parallel mechanisms is regarded as an effective tool to find multiple solutions in the conceptual design phase of PMs, a series of systematic approaches for achieving a comprehensive type synthesis have been proposed, and a large variety of PMs generating a specified motion pattern have been developed. On this basis, Yu et al. [12] presented a unified approach for synthesizing both rigid and flexure parallel mechanisms in the framework of screw theory. However, the input parameters were generally not considered as design parameters in the above synthesis progress. In fact, the selection and placement of actuators are considered as important aspects in the synthesis of both rigid and flexure PMs. Improper actuator selection or placement in a flexure PM may cause additional errors, such as parasitic errors and even lead to actuation invalidity or redundancy [13]. In addition, it may cause interference [14], actuator singularity [15], and different driving modes that could affect the workspace [16], stiffness characteristic [17], and motion/force transitivity performance [18] in a rigid PM. When selecting and placing actuators, the moving pairs or frame pairs should be selected for convenience in practical

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application. But this method of selecting actuators is not always correct, which may lead to the changes of screw system's linear dependence and consequently result in actuation singularity. Based on the linear dependence of the screw system and the concept of the constraint screw, Zhao et al. [19] presented a judgement method for the input selection of PMs. To identify the rationality of inputs intuitively, Li et al. [20] researched the linear dependence of screw systems using Klein mapping theory. Gao et al. [21] analyzed the actuating modes of the 5-UPS/PRPU parallel machine tool, and evaluated the relative merits of the five combinations of inputs utilizing the isotropy index of the constraining force or moment. The judgements about the rationality of input selection and the researches about actuation singularity mentioned above were all carried out by that whether the constraint matrix established by locking the selected actuated pairs is a full-rank matrix. To obtain the rational inputs, the constraint matrix often needed to be established more than once, and it could only verify whether the selected inputs were rational. However, any parallel mechanism could have several rational combinations of inputs that access the desired DOFs. Chablat et al. [22] introduced a novel three-degrees-of-freedom planar parallel manipulator with variable actuation modes, and its kinematic and dynamic performances were optimized by selecting the actuation mode on the basis of a certain strategy. Hopkins and Culpepper [23, 24] were perhaps the first to introduce actuation spaces to study the actuator selection and layout of flexure systems. In Refs. [23, 24], the relation between actuation and output motion was established based on screw theory, the selection and placement of linear actuators were considered on the basis of freedom space and constraint space, and a total of 26 actuation spaces were enumerated. Yu et al. [25] derived a synthesis criterion stated as "any actuation space of a flexure mechanism is always linearly independent of its constraint space", based upon an assumption of small deformation, and an analytical approach for synthesizing the line actuation spaces of a parallel flexure mechanism was presented. Conconi et al. [26] discussed the notion of actuation space in a leg when studying the singularities of general parallel kinematic chains, and pointed the condition that the actuation is well-chosen, is "if and only if actuation space and constraint space of a leg are independent". The above researches have provided us with an entirely new idea, i.e., that the concept of actuation space not only exists in flexure parallel mechanisms, but also is suitable for rigid parallel mechanisms, therefore the actuation spaces synthesis of rigid and flexure parallel mechanisms could be considered uniformly.

Therefore, a unified approach has been presented in this study for synthesizing the actuation spaces of both rigid and flexure parallel mechanisms, and all the rational actuation spaces of PMs could be obtained, with the aim of helping designers to consider all design concepts and to quickly select and place the actuators. An important synthesis criterion stating that "any actuation space of lower-mobility PMs without redundant actuation is always linearly independent of its constraint space" is improved, a general expression of actuation spaces in the format of screw systems is deduced, and the synthesis process is obtained. Based on the synthesis approach and process, the typical lower-mobility 3-RPS rigid PM and the 3-DOF flexure PM were selected as examples to validate the effectiveness of this unified method.

2 Criterion for Synthesizing Actuation Spaces

Screw theory is applicable for both rigid and flexure mechanisms. So, based on all the achievements in the application of these two classes of mechanisms, it can be concluded that establishing a unified approach for synthesizing the actuation spaces of both rigid and flexure mechanisms based on screw theory is wholly feasible.

2.1 Actuation Space

The branches of a lower-mobility PM can transmit motion, actuation force, and provide constraint force to the end effector. The motion, actuation force, and constraint force can be represented by motion twist, actuation wrench and constraint wrench in screw theory.

Definition 1. Twist and freedom space

The motion of the kinematic pair and the moving platform of the mechanism can be represented by a twist in a rigid parallel mechanism. In a flexure parallel mechanism, the micro-deformation of the flexure system and the motion of the moving platform can also be represented by a deformation twist.

Correspondingly, an n-dimensional freedom space $\boldsymbol{S}_D$, reflecting a specified motion pattern, can be spanned by n independent motion twists $\$_{Di}$ of the moving platform, which is written as:

$$\boldsymbol{S}_D = \left[\$_{D1}^{\mathrm{T}} \ \ldots \ \$_{Di}^{\mathrm{T}} \ \ldots \ \$_{Dn}^{\mathrm{T}} \right]^{\mathrm{T}}, i = 1, 2, \cdots, n. \quad (1)$$

Definition 2. Constraint wrench and constraint space

The constraint wrench represents the constraint force acting on the mechanism, which is the reciprocal screw of the twist. Therefore, an m-constraint space $\boldsymbol{S}_C$ is spanned by m independent constraint wrenches $\$_j^r$, i.e.,

$$\boldsymbol{S}_c = \left[\$_1^{r\mathrm{T}} \cdots \$_j^{r\mathrm{T}} \cdots \$_m^{r\mathrm{T}} \right], j = 1,2, \cdots, m. \quad (2)$$

Definition 3. Actuation wrench and actuation space

The actuation wrench refers to a wrench that is not reciprocal to the twist corresponding to the actuated pair, but reciprocal to other twists in a branch [27]. Each actuated pair corresponds to an actuation wrench, in consequence a set of actuated pairs were selected, and a set of actuation wrenches were generated correspondingly.

Therefore, an n-dimensional actuation space $\boldsymbol{S}_A$ is spanned by n independent actuation wrenches $\$_{Ak}$, which is expressed as:

$$\boldsymbol{S}_A = \left[\$_{A1}^{\mathrm{T}} \ \ldots \ \$_{Ak}^{\mathrm{T}} \ \ldots \ \$_{An}{}^{\mathrm{T}} \right]^{\mathrm{T}}, k = 1, 2, \cdots, n. \quad (3)$$

In general, the dimension of the actuation space is equal to the number of DOF of the mechanism.

2.2 Criterion for Synthesizing Actuation Space

The sum of the dimension of the freedom space and the constraint space of the moving platform is 6, and the dimension of the actuation space is equal to the dimension of the freedom space. Therefore, the sum of the dimension of the actuation space and the constraint space is 6 [10]. It can be concluded that the actuation space is always linearly independent of its constraint space for the lower-mobility PMs without redundant actuation, when the mechanisms have definite motion. The proof is as follows:

For the lower-mobility PM, assuming that the actuation space is linearly dependent of its constraint space, there are coefficients $\alpha_k(k = 1, 2, \ldots, n)$ and $\beta_j(j = 1, 2, \ldots, 6 - n)$ that are not all zero, which enable the following formula to hold:

$$\sum_{k=1}^{n} \alpha_k \$_{Ak} = \sum_{j=1}^{6-n} \beta_j \$_j^r, \quad (4)$$

where n is the number of DOF of the mechanism.

According to classical screw theory [28, 29], the reciprocal product of the constraint wrench $\$_j^r$ and the motion twist $\$_{Di}$ is zero, i.e.,

$$\$_j^r \circ \$_{Di} = 0. \quad (5)$$

According to Eqs. (4) and (5), this leads to

$$\left(\sum_{k=1}^{n} \alpha_k \$_{Ak} \right) \circ \$_{Di} = \sum_{j=1}^{6-n} \beta_j \$_j^r \circ \$_{Di} = 0, i = 1, 2, \cdots, n. \quad (6)$$

Obviously, Eq. (6) is false. According to the relationships between motion, force and power, the reciprocal product of the motion twist and the actuation wrench is not zero when the actuation wrench works on the moving platform. Therefore, it can be concluded that the actuation wrenches are always linearly independent of the constraint wrenches.

Therefore, an important criterion for synthesizing the actuation space of both rigid and flexure parallel mechanisms is proposed, which is stated as: the actuation space is always linearly independent of its constraint space for the lower-mobility PMs without redundant actuation, when the mechanisms have definite motion.

3 Synthesizing Process for Actuation Space

According to linear algebra, V_1 and V_2 are non-zero subspaces of the n-dimensional space R^n. If for all $\alpha \in V_1$, $\beta \in V_2$, there exists $\alpha \perp \beta$ and $\dim V_1 + \dim V_2 = \dim R^n$, then V_2 is called the orthogonal complement of V_1 [30].

The actuation space is always linearly independent of its constraint space, and the orthogonal complement can satisfy the linearly independent characteristics, so the actuation space can be regarded as the complement space of the constraint space.

It is wholly feasible to establish a unified approach for synthesizing the actuation spaces of both rigid and flexure PMs based on screw theory. But due to the structure characteristics of the flexure parallel mechanism, an analysis method based on a compliance matrix is adopted during the freedom analysis in this paper.

Therefore, the synthesizing process of the actuation spaces of a lower-mobility parallel mechanism is detailed as follows.

(1) Solve the constraint space. The constraint space can be represented by a set of linearly independent constraint wrenches $\$_j^r$when the mechanism has a defined motion:

$$\boldsymbol{S}_C = \left[\$_1^{r\mathrm{T}} \ \$_2^{r\mathrm{T}} \ \cdots \$_{6-n}^{r\mathrm{T}} \right]^{\mathrm{T}}. \quad (7)$$

The solving procedures of the constraint wrench system for rigid and flexure PMs are different due to their different structures. The solving process of the rigid parallel mechanism [28] is: (1) Solve the motion twists of the motion pairs on each branch in turn. (2) Obtain the constraint wrenches according to the reciprocal screw theory. 3) Compute the union set of the constraint wrenches of all branches.

The solving process of the flexure parallel mechanism [13] is: (1) Calculate the global compliance matrix $\boldsymbol{C}$ of the mechanism. (2) Calculate the eigenvalues and the eigenvector matrices of the compliance matrix $\Delta C \Delta$. (3) Nondimensionalize the feature values according to the compliance type they represent. (4) Compare the dimensionless eigenvalues (compare the maximum value $\lambda_{\max}$ of the dimensionless eigenvalues with the remaining eigenvalues λ_i, and assign zero to this eigenvalue if $|\lambda_i/\lambda_{\max}| << 1$). (5) The feature vectors (screws) corresponding to the zero eigenvalue constitute the constraint space of the mechanism.

(2) Compute the freedom space of the mechanism. The freedom space, reciprocal to the constraint space, is spanned by n independent basis motion twists $\$_{Di}$ of the moving platform, which is written as:

$$\boldsymbol{S}_D \circ \boldsymbol{S}_C = 0, \tag{8}$$

$$\boldsymbol{S}_D = \left[\$_{D1}^{\mathrm{T}} \ldots \$_{Di}^{\mathrm{T}} \ \ldots \ \$_{Dn}^{\mathrm{T}} \right]^{\mathrm{T}}, i = 1, 2, \cdots, n. \tag{9}$$

The motion continuity of the rigid mechanism has to be distinguished, but it is not necessary to the flexure mechanism, since it is used for micro-motion or small deformation generally.

(3) Derive the actuation space orthogonal to the constraint space. The actuation space and the constraint space satisfy the relationship of orthogonal complement. Therefore, by calculating the null space of the constraint space, actuation spaces orthogonal to the constraint space can be obtained:

$$\boldsymbol{S}_A \cdot \boldsymbol{S}_C = 0. \tag{10}$$

On the basis of linear algebra, the reciprocal product of two screws can be converted into their dot product. Therefore, the linearly independent vectors from the actuation space are obtained by swapping the rotational vector and the translational vector of these twists $\$_{Di}$ and multiplying the coefficient ε_i:

$$\boldsymbol{S}_A = \$'_D = \left[\$'^{\mathrm{T}}_{D1} \cdots \$'^{\mathrm{T}}_{Di} \cdots \$'^{\mathrm{T}}_{Dn} \right]^{\mathrm{T}}, i = 1, 2, \cdots, n, \tag{11}$$

where $\$'_{Di} = \varepsilon_i \$_{Di} \Delta$, $\Delta = \begin{bmatrix} 0 & \boldsymbol{E}_{3\times3} \\ \boldsymbol{E}_{3\times3} & 0 \end{bmatrix}$, $\boldsymbol{E}_{3\times3}$ is a unit matrix. ε_i is a coefficient for ensuring that the screw $\$'_{Di}$ is a unit screw.

(4) Derive the general formula of the actuation wrenches. According to the criterion of comprehensive actuation space proposed in Section 2.2, the actuation space is always linearly independent of its constraint space for the lower-mobility PMs without redundant actuation when the mechanisms have definite motion. From linear algebra, the constraint wrenches and the actuation wrenches could constitute a set of 6-dimensional vector space. From this the general formula of actuation wrenches is obtained in terms of the linear combination of n wrenches $\$'_{Di}$ and $6-n$ constraint wrenches $\$_j^r$:

$$\$_{Ak} = \sum_{i=1}^{n} \mu_{ik} \$'_{Di} + \sum_{j=1}^{6-n} \lambda_{jk} \$_j^r = \boldsymbol{\mu}_k \$'_D + \sum_{j=1}^{6-n} \lambda_{jk} \$_j^r, \tag{12}$$

where $\boldsymbol{\mu}_k = \left(\boldsymbol{\mu}_{1k} \ \boldsymbol{\mu}_{2k} \cdots \boldsymbol{\mu}_{nk} \right) \neq 0$, $\boldsymbol{\mu}_k (k = 1, 2, \cdots, n)$ is the non-zero row vector which make sure that the constraint wrenches are linearly independent of the actuation wrenches.

(5) Derive the actuation spaces. An actuation wrench is selected arbitrarily on each branch, and these wrenches can be combined. According to the matrix transformation and the calculation of the ranks of matrix in linear algebra, the combination of actuation wrenches can be deduced to be linearly independent, and can constitute a reasonable actuation combination if the corresponding vector group $\boldsymbol{\mu} : \boldsymbol{\mu}_1, \boldsymbol{\mu}_2, \cdots, \boldsymbol{\mu}_n$ is always linearly independent in the whole workspace after the combination. Thereby the actuation spaces of the mechanism can be obtained. It is necessary to calculate the actuation wrenches corresponding to the motion pairs, since the driving forces of the rigid parallel mechanism are actuated by motion pairs.

The whole synthesis process of the actuation spaces can be uniformly described with a flowchart as shown in Figure 1.

By synthesizing the actuation spaces of the lower-mobility parallel mechanism, several groups of reasonable combinations of input can be obtained, and then the input optimization of the mechanisms can be achieved by further selecting the appropriate performance index and analyzing the performance of the mechanism with different combinations of inputs.

4 Case Analysis

In this section, one typical 3-RPS rigid PM and one flexure PM are used as examples to demonstrate the proposed synthesis approach.

4.1 Actuation Spaces Synthesis of the 3-RPS Rigid Parallel Mechanism

A typical 3-RPS PM [15], which has two rotational and one translational DOFs, is adopted in this section. As shown in Figure 2, the 3-RPS parallel mechanism is composed of a moving platform, a fixed platform and three identical RPS branches. The moving platform and the fixed platform are two similar equilateral triangles. Each RPS branch is made up of one revolute pair (*R* pair), one prismatic pair (*P* pair), and one spherical pair (*S* pair). Each spherical joint is kinematically equivalent to three revolute joints with intersecting axes. A_i is the center of the *R* pair connected to the fixed platform, B_i is the center of the spherical joint connected to the moving platform.

(1) Compute the constraint space of the mechanism

As shown in Figure 2, a branch coordinate system $O_i - X_i Y_i Z_i$ is established, whose origin is coincided with point A_i, the Y_i-axis is parallel to the axis of the branch revolute joint, the Z_i-axis is perpendicular to the fixed platform, and the X_i-axis is determined by the right-hand rule.

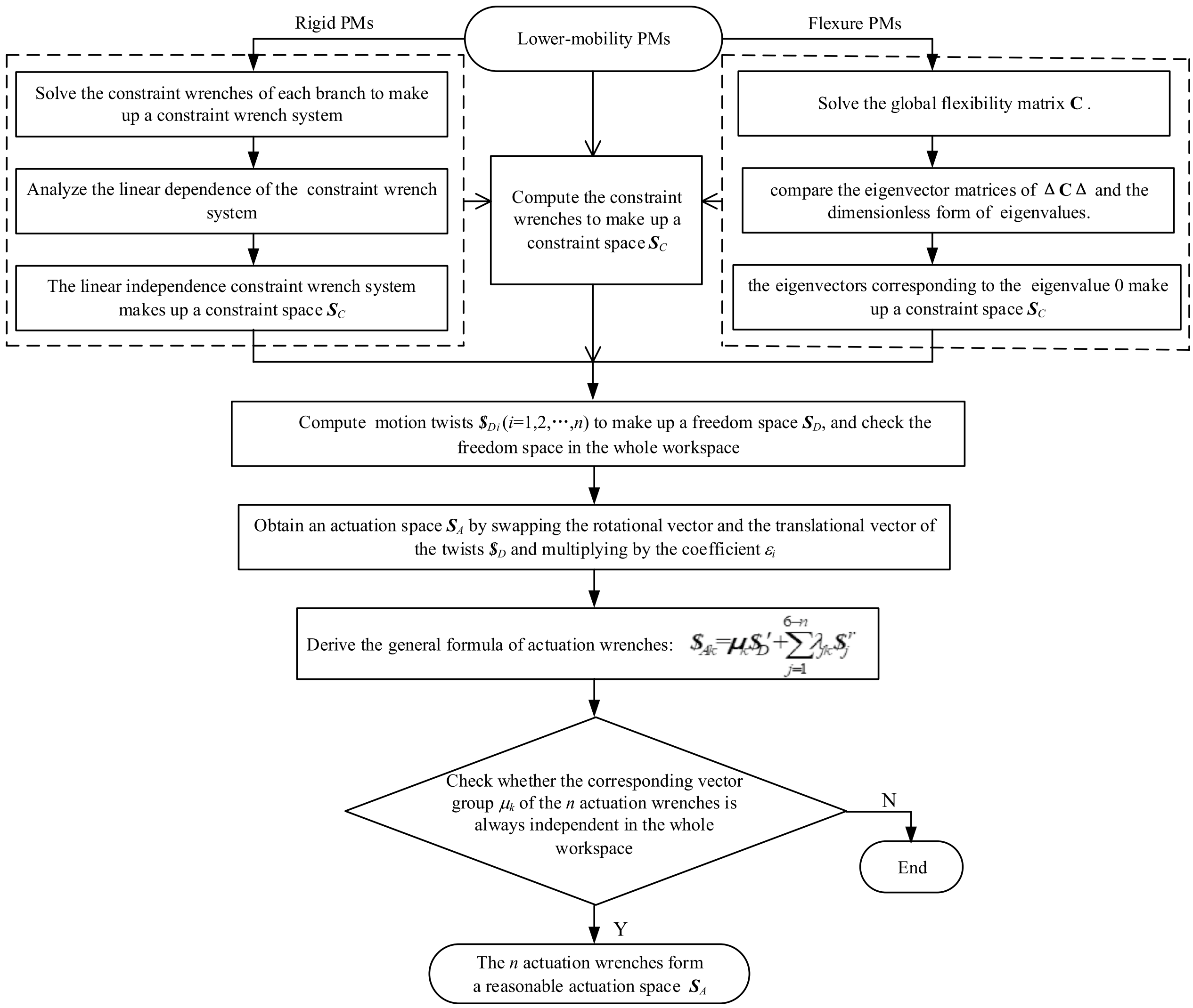


Fig. 1 Flow chart of synthesizing the actuation spaces of both rigid and flexure parallel mechanism

The kinematic screw system of each branch can be expressed as:

$$\begin{cases} \$_{i1} = \begin{pmatrix} 0 & 1 & 0; & 0 & 0 & 0 \end{pmatrix}, \\ \$_{i2} = \begin{pmatrix} 0 & 0 & 0; & q_{i2} & 0 & r_{i2} \end{pmatrix}, \\ \$_{i3} = \begin{pmatrix} 0 & 1 & 0; & r_{i2} & 0 & 0 \end{pmatrix}, \\ \$_{i4} = \begin{pmatrix} 1 & 0 & 0; & 0 & -r_{i2} & q_{i2} \end{pmatrix}, \\ \$_{i5} = \begin{pmatrix} 0 & 0 & 1; & -q_{i2} & 0 & 0 \end{pmatrix}, \end{cases} \tag{13}$$

where q_{i2} is the X-coordinate value of the center of the spherical joint B_i in the branch coordinate system; r_{i2} is the Y-coordinate value of the center of the spherical joint B_i in the branch coordinate system.

According to the screw theory, the constraint wrench is a constraint force that passes through the center of the spherical B_i and is parallel to the Y_i-axis. The constraint wrench of the branch can be obtained by calculating the reciprocal screw as follows:

$$\$_i^r = \begin{pmatrix} 0 & 1 & 0; & r_{i2} & 0 & 0 \end{pmatrix}. \tag{14}$$

Therefore, the constraint forces of the moving platform are located in the plane S passing through B_i and interlaced with each other. The distribution of the constraint forces under initial position is represented in Figure 3.

As shown in Figure 2, the coordinate system $O-XYZ$ is established, whose origin is located at the fixed platform center O, and the Y-axis is parallel to the axis of the branch revolute joint, the Z-axis is perpendicular to the fixed platform, and the X-axis is determined by the right-hand rule. According to the distribution of the constraint wrenches of each branch, the constraint space of the moving platform is

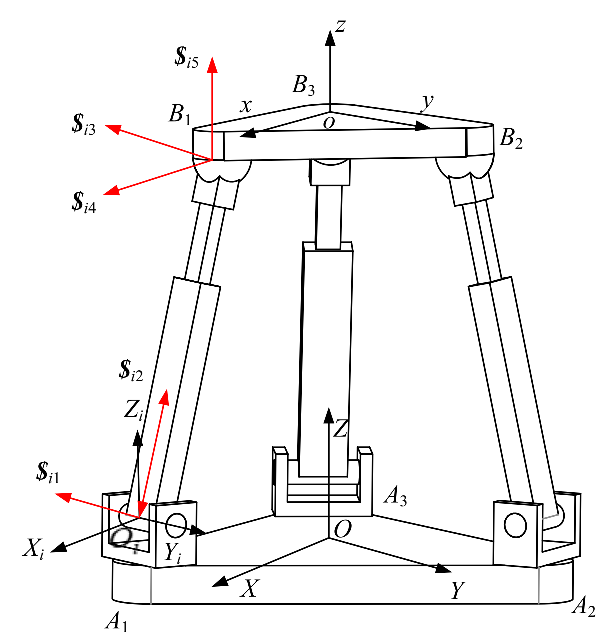


Fig. 2 Typical 3-RPS parallel mechanism

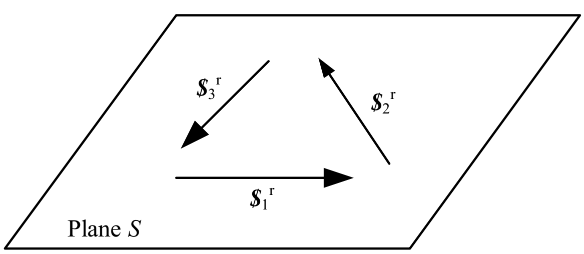


Fig. 3 Distribution of the constraint forces of the moving platform

$$\boldsymbol{S}_C=\begin{cases}\$_1^r=\begin{pmatrix}0 & 1 & 0; & -Z_{b1} & 0 & X_{b1}\end{pmatrix},\\ \$_2^r=\begin{pmatrix}\frac{\sqrt{3}}{2} & \frac{1}{2} & 0; & -\frac{1}{2}Z_{b2} & \frac{\sqrt{3}}{2}Z_{b2} & \frac{1}{2}X_{b2}-\frac{\sqrt{3}}{2}Y_{b2}\end{pmatrix},\\ \$_3^r=\begin{pmatrix}-\frac{\sqrt{3}}{2} & \frac{1}{2} & 0; & -\frac{1}{2}Z_{b3} & -\frac{\sqrt{3}}{2}Z_{b3} & \frac{1}{2}X_{b3}+\frac{\sqrt{3}}{2}Y_{b3}\end{pmatrix},\end{cases} \tag{15}$$

where $\begin{pmatrix}Xbi & Ybi & Zbi\end{pmatrix}$ are the coordinate values of b_i in the coordinate system $O-XYZ$, i=1,2,3.

(2) Compute the freedom space of the mechanism. By substituting Eq. (15) into Eq. (8), the motion twists within its freedom space are derived as:

$$\boldsymbol{S}_D=\begin{cases}\$_{D1}=\begin{pmatrix}0 & 0 & 0; & 0 & 0 & 1\end{pmatrix},\\ \$_{D2}=\begin{pmatrix}l_2 & m_2 & 0; & p_2 & q_2 & 0\end{pmatrix},\\ \$_{D3}=\begin{pmatrix}l_3 & m_3 & n_3; & p_3 & 0 & 0\end{pmatrix}.\end{cases} \tag{16}$$

It is known from Ref. [15] that the freedom space is not purely local and it holds in the whole workspace.

(3) Derive the actuation space orthogonal to the constraint space. According to Eq. (11) and Eq. (16), an actuation space orthogonal to the constraint space can be obtained by swapping the rotational vector and translational vector of all the twists $\$_{Di}$:

$$\boldsymbol{S}_A=\begin{cases}\$'_{D1}=\begin{pmatrix}1 & 0 & 0; & 0 & 0 & 0\end{pmatrix},\\ \$'_{D2}=\begin{pmatrix}p_2 & q_2 & 0; & l_2 & m_2 & 0\end{pmatrix},\\ \$'_{D3}=\begin{pmatrix}p_3 & 0 & 0; & l_3 & m_3 & n_3\end{pmatrix}.\end{cases} \tag{17}$$

(4) Derive the general formula of the actuation wrenches. This can be obtained in terms of the linear combination of three twists $\$'_{Dj}$ and three constraint wrenches $\$_k^r$ according to Eq. (12), where n=3.

$$\$_{Ak}=\sum_{i=1}^{3}\mu_{ik}\$'_{Di}+\sum_{j=1}^{3}\lambda_{jk}\$_j^r=\boldsymbol{\mu}_k\$'_D+\sum_{j=1}^{3}\lambda_{jk}\$_j^r. \tag{18}$$

(5) Derive the actuation spaces. Calculate the corresponding actuation wrenches when different motion pairs of the mechanism are selected as actuated pairs. By substituting the actuation wrenches into Eq. (18), the vector could be obtained, and the results are represented in Table 1. There are 4 actuation combinations when any of or is selected as the actuated pair of each branch, the reasonable actuation combinations constitute the actuation spaces of the mechanism.

Assuming that the radius of the circumscribed circle of the fixed platform is 92.38 mm, and the circumscribed

Table 1 Corresponding actuation wrenches and vector μ_k

Actuated pairs	Actuation wrench	Vector μ_k
R_i, i=1,2,3	$\$_{Ai1}=\begin{pmatrix}l_{i1} & m_{i1} & n_{i1}; & p_{i1} & q_{i1} & r_{i1}\end{pmatrix}$	$\begin{pmatrix}n_{i1} & \mu_{2k} & \mu_{3k}\end{pmatrix}$
P_i, i=1,2,3	$\$_{Ai2}=\begin{pmatrix}l_{i2} & m_{i2} & n_{i2}; & p_{i2} & q_{i2} & r_{i2}\end{pmatrix}$	$\begin{pmatrix}n_{i2} & \mu_{2k} & \mu_{3k}\end{pmatrix}$

circle of the movable platform is 57.74 mm, the movement range of the prismatic pair is 120–195 mm.

Taking an actuation combination of $3R_i$ pairs as an example, substitute the actuation wrenches of the R_i pair into Eq. (18), the following equations can be obtained as:

$$\begin{cases} p_2\mu_{2k}+p_3\mu_{3k}+\frac{\sqrt{3}}{2}\lambda_{2k}-\frac{\sqrt{3}}{2}\lambda_{3k}-l_{i1}=0, \\ q_2\mu_{2k}+\lambda_{1k}+\frac{1}{2}\lambda_{2k}+\frac{1}{2}\lambda_{3k}-m_{i1}=0, \\ l_2\mu_{2k}+l_3\mu_{3k}-Z_{b1}\lambda_{1k}-\frac{1}{2}Z_{b2}\lambda_{2k}-\frac{1}{2}Z_{b3}\lambda_{3k}-p_{i1}=0, \\ m_2\mu_{2k}+m_3\mu_{3k}+\frac{\sqrt{3}}{2}Z_{b2}\lambda_{2k}-\frac{\sqrt{3}}{2}Z_{b3}\lambda_{3k}-q_{i1}=0, \\ X_{b1}\lambda_{1k}+\frac{1}{2}X_{b2}\lambda_{2k}-\frac{\sqrt{3}}{2}Y_{b2}\lambda_{2k}+\frac{1}{2}X_{b3}\lambda_{3k}+\frac{\sqrt{3}}{2}Y_{b3}\lambda_{3k}+n_3\mu_{3k}-r_{i1}=0, \\ \mu_{1k}=n_{i1}. \end{cases} \tag{19}$$

The characters of the actuation wrenches, the motion twists, and the constraint wrenches would not change in the whole workspace, i.e., the linearly independent of the vector groups $\boldsymbol{\mu}:\boldsymbol{\mu}_1,\ \boldsymbol{\mu}_2,\ \boldsymbol{\mu}_3$ would not change, so the reasonability of the actuation combinations could be verified in a workspace arbitrarily selected. The vector $\boldsymbol{\mu}_k$ is calculated, and the ranks of the vector groups $\boldsymbol{\mu}:\boldsymbol{\mu}_1,\ \boldsymbol{\mu}_2,\ \boldsymbol{\mu}_3$ is computed in MATLAB, as shown in Figure 4. When the posture of the mechanism changes, the rank of the vector group is always 3, that is, when $3R_i$ pairs are selected as the actuation combinations, the branch actuation screws always linearly independent, which is a reasonable actuation combination. The movement of the mechanism does not affect the orientation of the actuation screw, so the figure does not show the posture change of the mechanism in the movement orientation.

When $2R_i$ and $1P_i$ pairs, $1P_i$ and $2P_i$ pairs, and $3P_i$ pairs are selected as the actuation combinations, the rank of the vector group is the same with $3R_i$ pairs, that is, these three combinations are also reasonable actuation pair combinations. Therefore, there are four actuation combinations of 3-RPS parallel mechanism, and all of them constitute reasonable actuation space, as shown in Table 2.

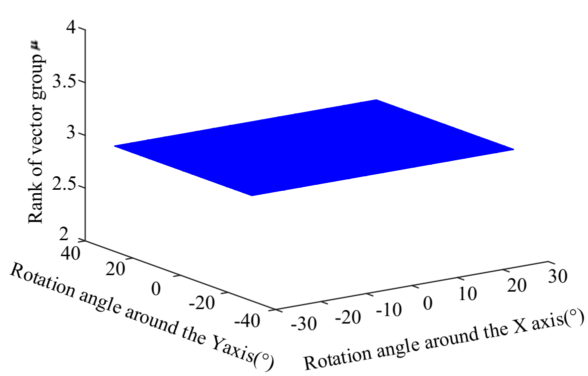


Fig. 4 Rank of each input combination vector group

All possible combinations of inputs were obtained by the synthesis of actuation spaces, and the subsequent optimization of inputs could be carried on based on the performance requirement of the mechanism.

4.2 Actuation Space Synthesis of 3R-DOFs Constraint Flexure Parallel Mechanism

A kind of simple flexible parallel mechanism [24] is adopted in this section, which consists of a moving platform connected with a fixed platform by three identical basic units of flexible links, as shown in Figure 5. The length of flexible links $l=82$ mm, the radius of

Table 2 Actuation spaces of 3-RPS parallel mechanism

Type	Reasonable actuation combination
2R1P	*RPS-RPS-RPS*
3P	*RPS-RPS-RPS*
2P1R	*RPS-RPS-RPS*
3R	*RPS-RPS-RPS*

Underlined letters show the actuated pairs

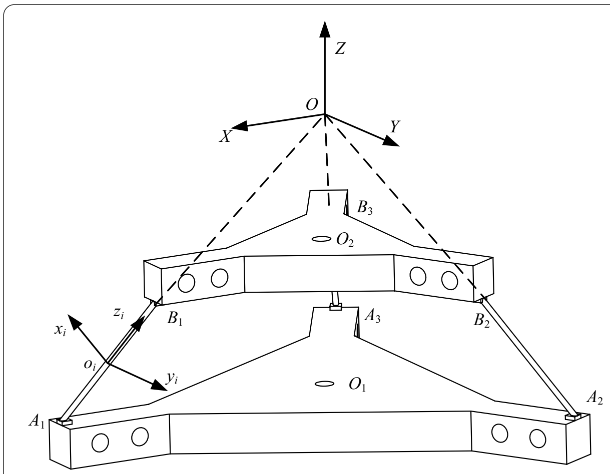


Fig. 5 Schematic diagram of 3R-DOFs constrained flexible parallel mechanism

cross-section $r = 1.5$ mm, elastic modulus $E = 70$ GPa, Poisson's ratio $\mu = 0.346$, the radius of the circumcircle of the equilateral triangle formed by the connection points of the moving platform and each flexible unit $R = 102$ mm, the tilt angle of each flexible unit $\theta = 45°$, A_i and B_i are the connection points of each flexible unit and the fixed platform/the moving platform respectively.

Compute the constraint space of the mechanism.

As shown in Figure 5, a branch coordinate system $o_i - x_i y_i z_i$ is established, whose origin o_i is located at the center of the flexible unit, the z_i-axis coincides with the axis A_iB_i, and the y_i-axis is parallel to the fixed platform and perpendicular to the axis O_1A_i, the x_i-axis is determined by the right-hand rule. The flexibility matrix of the flexible link in this coordinate system is written as:

$$C_i = \mathrm{diag}\left(\frac{l^3}{3E\pi r^2}, \frac{l^3}{3E\pi r^2}, \frac{l}{E\pi r^2}, \frac{4l}{E\pi r^2}, \frac{4l}{E\pi r^2}, \frac{2l}{G\pi r^2}\right). \tag{20}$$

As shown in Figure 5, a base coordinate system is established, whose origin O is the intersection of the axes A_1B_1, A_2B_2 and A_3B_3, the X-axis is parallel to the vector O_1A_i, the Z-axis is vertical upward, and the Y-axis is determined by the right-hand rule. The overall flexibility matrix of the mechanism is formulated as:

$$\mathbf{C} = \left(\sum_{i=1}^{3} (Ad_i C_i Ad_i)^{-1}\right)^{-1}, \tag{21}$$

where $Ad_i = \begin{bmatrix} R_i & 0 \\ T_i R_i & R_i \end{bmatrix}$ is the adjoint matrix of coordinate transformation, R_i is the rotation transformation matrix: $R_1 = R(Y, 90° - \theta)$, $R_2 = R_1 R(Z, 240°)$, $R_3 = R_1 R(Z, 120°)$. T_i is the oblique symmetric matrix determined by the motion vector: $T_1 = T\left(\frac{l}{2}\cos\theta - R, 0, R - \frac{l}{2}\sin\theta\right)$, $T_2 = T\left(\frac{R}{2} - \frac{l}{4}\cos\theta, \frac{\sqrt{3}l}{4}\cos\theta - \frac{\sqrt{3}R}{2}, R - \frac{l}{2}\sin\theta\right)$, $T_3 = T\left(\frac{R}{2} - \frac{l}{4}\cos\theta, \frac{\sqrt{3}R}{2} - \frac{\sqrt{3}l}{4}\cos\theta, R - \frac{l}{2}\sin\theta\right)$.

If the translation vector is (L, M, N), then the corresponding oblique symmetric matrix is

$$T(L, M, N) = \begin{bmatrix} 0 & -N & M \\ N & 0 & -L \\ -M & L & 0 \end{bmatrix}. \tag{22}$$

The eigenvalue matrix and eigenvector matrix of matrix $\Delta C \Delta$ can be obtained as:

$$\lambda = \mathrm{diag}(0.0982, 0, 0.0001, 0.0001, 0.1322, 0.0982), \tag{23}$$

$$V = \begin{bmatrix} 0 & 0.7071 & 0.7071 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0.0001 \\ 0 & 0.7071 & -0.7071 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0.0001 & 0.7071 & -0.7071 \\ -1 & 0 & -0.0001 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0.7071 & 0.7071 \end{bmatrix}. \tag{24}$$

The dimensionless feature matrix is written as:

$$\bar{\lambda} = \mathrm{diag}\left(1.41, 0, 1.51 \times 10^{-5}, 1.51 \times 10^{-5}, 1.90, 1.41\right) \times 10^{-2} \approx \mathrm{diag}(1.41, 0, 0, 0, 1.90, 1.41) \times 10^{-2}. \tag{25}$$

The eigenvectors corresponding to the zero eigenvalue are composed of the constraint space of the mechanism (the column vector represents the constraint wrench).

$$\boldsymbol{S}_C = \begin{bmatrix} 0.7071 & 0.7071 & 0 \\ 0 & 0 & 1 \\ 0.7071 & -0.7071 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}. \tag{26}$$

(2) Compute the freedom space of the mechanism. Calculate the reciprocal screws of Eq. (26), then the motion screws from the freedom space are obtained:

$$\boldsymbol{S}_D = \begin{cases} \$_{D1} = \left(1\ 0\ 0;\ 0\ 0\ 0\right), \\ \$_{D2} = \left(0\ 1\ 0;\ 0\ 0\ 0\right), \\ \$_{D3} = \left(0\ 0\ 1;\ 0\ 0\ 0\right). \end{cases} \tag{27}$$

(3) Derive the actuation space orthogonal to the constraint space. An actuation space orthogonal to the constraint space can be obtained by swapping the rotational vector and translational vector of all the screws in Eq. (27).

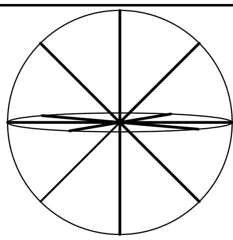
$$\boldsymbol{S}_A = \begin{cases} \$'_{D1} = \left(0\ 0\ 0;\ 1\ 0\ 0\right), \\ \$'_{D2} = \left(0\ 0\ 0;\ 0\ 1\ 0\right), \\ \$'_{D3} = \left(0\ 0\ 0;\ 0\ 0\ 1\right). \end{cases} \tag{28}$$

(4) A general formula of a reasonable actuation wrench is derived, which is represented in Eq. (12), where $n = 3$.

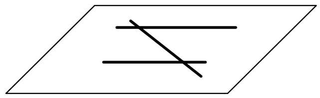
(5) Derive the actuation spaces. Because linear actuators which can generate large pushing or pulling forces are more likely to be used for an ultra-precision system, only the linear actuation spaces constituted of the actuation force are taken into consideration here.

It can be obtained from the general expression that any force vector that does not pass through the origin of the global coordinate system can be selected as a reasonable actuation wrench. Meanwhile, it can be concluded from Eqs. (12) and (24)–(26) that it is only

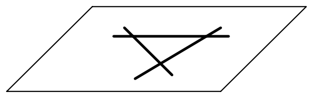
Table 3 Actuation space line maps



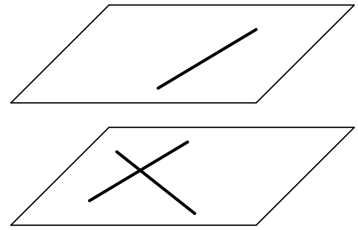
(a) Three force wrench lines intersecting at a point in space



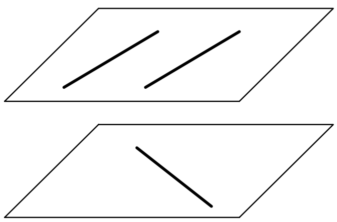
(b) Three force wrench lines in a plane, but non-intersecting



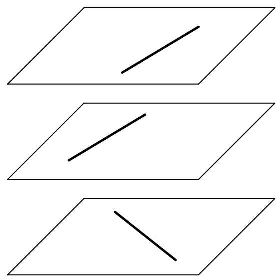
(c) Three force wrench lines in a plane, but non-intersecting



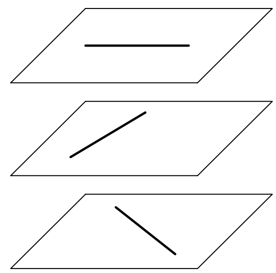
(d) Three force wrench lines belong to common vertical line



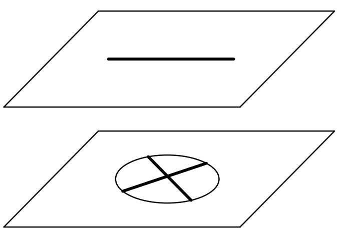
(e) Three force wrench lines belong to common vertical line



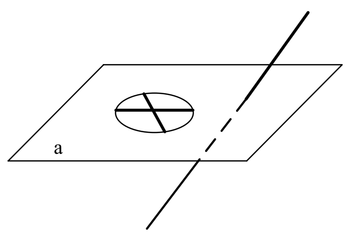
(f) Three force wrench lines belong to common vertical line



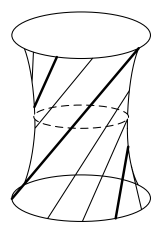
(g) Three force wrench lines belong to common vertical line



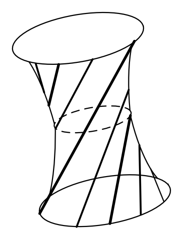
(h) Three force wrench lines belong to common vertical line



(i) One wrench oblique belongs to the plane that the other two wrench lines lie on



(j) Belong to a single-leaf hyperboloid



(k) Belong to a elliptic hyperboloid

necessary to ensure that the rotational vectors corresponding to the three actuation wrenches are linearly independent, then the vector groups corresponding to the three actuation wrenches are linearly independent. Therefore, any three actuation wrenches can constitute a reasonable actuation space if their rotational vectors are linearly independent. By linearly combining the force wrenches without passing through the origin, the actuation spaces can be obtained.

(1) Three spatial force wrenches intersect at one point

A set of special force wrenches which intersect at point (x, y, z) can be expressed as follows:

$$\begin{cases} \$_{A1} = \begin{pmatrix} 1 & 0 & 0; & 0 & z & -y \end{pmatrix}, \\ \$_{A2} = \begin{pmatrix} 0 & 1 & 0; & -z & 0 & x \end{pmatrix}, \\ \$_{A3} = \begin{pmatrix} 0 & 0 & 1; & y & -x & 0 \end{pmatrix}. \end{cases} \tag{29}$$

Obviously, the rotational vectors of the three wrenches are linearly dependent when $z = y = 0$, $z = x = 0$ or $x = y = 0$. Therefore, any three wrenches that do not intersect on the coordinate axis can constitute a reasonable actuation space, and the line graph of the actuation space is illustrated as (a) of Table 3.

(2) Three wrenches are coplanar but not intersecting:

$$\begin{cases} \$_{A1} = (1\ 0\ 0;\ 0\ z\ -y_1), \\ \$_{A2} = (1\ 0\ 0;\ 0\ z\ -y_2), \\ \$_{A3} = (0\ 1\ 0;\ -z\ 0\ x_3). \end{cases} \tag{30}$$

Obviously, the rotational vectors corresponding to the three wrenches are linearly independent when $z \neq 0$. Therefore, any three coplanar but not intersecting wrenches that are not located on the *XOY*, *YOZ*, and *XOZ* planes can constitute a reasonable actuation space. The line graphs of the actuation space are illustrated in (b) and (c) of Table 3.

(3) Three wrenches are not coplanar but belong to a common vertical line.

Two sets of wrenches can be derived from the general expression of the mechanism's actuation wrench:

$$\begin{cases} \$_{A1} = (1\ 0\ 0;\ 0\ z_1\ -y_1), \\ \$_{A2} = (1\ 0\ 0;\ 0\ z_2\ -y_2), \\ \$_{A3} = (1\ 0\ 0;\ 0\ z_3\ -y_3), \end{cases} \tag{31}$$

$$\begin{cases} \$_{A1} = (1\ 0\ 0;\ 0\ z_1\ -y_1), \\ \$_{A2} = (1\ 0\ 0;\ 0\ z_2\ -y_2), \\ \$_{A3} = (0\ 1\ 0;\ -z_3\ 0\ x_3). \end{cases} \tag{32}$$

It can be deduced from Eq. (31) that any three actuation wrenches parallel to each other cannot constitute a reasonable actuation space.

Equation (32) shows that the rotational vectors corresponding to the three wrenches are linearly independent when $z_1y_2 \neq z_2y_1$ and $z_3 \neq 0$. Therefore, for the three wrenches including two parallel wrenches belonging to a common vertical line, only when the planes of the two parallel wrenches do not pass through the origin can they constitute a reasonable actuation space. The line graph of the actuation space is illustrated in (d), (e), and (f) of Table 3.

It is easy to know that the three wrenches that belong to a common vertical line, whose line graphs are illustrated in (g) and (h) of Table 3, can also constitute a reasonable actuation space from the above analysis and the base vector.

(4) Three wrenches in which one wrench intersects the plane of the other two wrenches:

$$\begin{cases} \$_{A1} = (1\ 0\ 0;\ 0\ z_1\ -y_1), \\ \$_{A1} = (1\ 0\ 0;\ 0\ z_1\ -y_2), \\ \$_{A1} = (0\ 0\ 1;\ y_3\ -x_3\ 0), \end{cases} \tag{33}$$

$$\begin{cases} \$_{A1} = (1\ 0\ 0;\ 0\ z_1\ -y_1), \\ \$_{A1} = (0\ 1\ 0;\ -z_1\ 0\ x_1), \\ \$_{A1} = (0\ 0\ 1;\ y_3\ -x_3\ 0). \end{cases} \tag{34}$$

The rotational vectors corresponding to the two wrenches are linearly independent when $z_1 \neq 0$ and $y_3 \neq 0$ in Eq. (33); the rotational vectors of the three wrenches are linearly independent when $z_1 \neq 0$, x_1 and y_1 are not zero in Eq. (34). Therefore, the three wrenches can also constitute a reasonable actuation space as illustrated in (i) of Table 3.

(5) Three force wrenches in which any two wrenches are in different planes:

$$\begin{cases} \$_{A1} = (1\ 0\ 0;\ 0\ z_1\ -y_1), \\ \$_{A1} = (0\ 1\ 0;\ -z_2\ 0\ x_2), \\ \$_{A1} = (0\ 0\ 1;\ y_3\ -x_3\ 0). \end{cases} \tag{35}$$

Equation (35) shows that the three wrenches can constitute a reasonable actuation space when z_2 and y_3,z_1 and x_3 or y_1 and x_2 are equal to 0 not simultaneously. Therefore, the three wrenches in which any two wrenches are in different planes can also constitute a reasonable actuation space. Any three wrenches distributed on the single-leaf hyperboloids or elliptical hyperboloids can constitute a reasonable actuation space as illustrated in (j) and (k) of Table 3.

All the analysis shows that 11 reasonable actuation combinations can be obtained to constitute the actuation spaces of the mechanism.

Comparing the synthesizing process and the results of the actuation spaces of the flexure and rigid parallel mechanisms, we find that reasonable actuation wrenches could be constituted by swapping the rotational vector and translational vector, and by the linear combination of the constraint spaces and the freedom spaces based on the important synthesis criteria. But the actuators can only be arranged on the motion pairs for the rigid PMs, while the actuators could be arranged more flexibly for the flexure PMs. This result not only demonstrates the consistency and the feasibility of the synthesizing process for actuation spaces of both rigid and flexure PMs with lower-mobility, but also indicates that the rigid mechanism is a special case of the flexible mechanism.

5 Conclusions and Future Work

(1) A unified approach has been presented for synthesizing the actuation spaces of both rigid and flexure parallel mechanisms, and all the rational actuation spaces of PMs could be obtained.

(2) On the basis of the synthesis criterion, a general expression of the actuation space in the format of screw systems has been derived, and the synthesis process of all the actuation spaces generating a specified motion pattern has been deduced.

(3) The typical 3-RPS rigid parallel mechanism and the 3R-DOFs flexible mechanism has been selected as examples to synthesize the actuation spaces, and all reasonable actuator placements has been obtained, which verify the feasibility of the united approach for synthesizing actuation spaces of both rigid and flexure parallel mechanisms.

The selection and placement of the actuators would be carried out according to experiences or principles generally, but it could not consider all the rational input combinations, and it may result in that the selected input combination is not the best-chosen. This research provides all the rational schemes for input selection, input optimization could be proceeded next according to a realistic engineering application.

How to express the actuation spaces more intuitively and figure out the problem of input optimization to find an optimal way to place actuators for both rigid and flexure parallel mechanisms will be very challenging tasks in the future.

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Authors' contributions
SL was in charge of the whole research; YS wrote the manuscript; JY discussed and read the manuscript; YK assisted with the analysis and validation. All authors read and approved the final manuscript.

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Competing interests
The authors declare no competing financial interests.

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