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Magnetic Rossby waves in the stratified ocean of the core, and topographic core-mantle coupling

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A new model of the stably stratified layer at the top of the core is proposed. The existence of a stably stratified layer (we name it the stratified ocean) at the top of the core makes possible the propagation of the waves akin to the Rossby waves (also named "planetary waves"), well known in oceanology and meteorology. These waves are modified and experience significant decay, due to the core's magnetic field. The "magnetic Rossby waves" are considered here, using a simple planar model, to reveal their qualitative features without going into significant mathematical complications. The core-mantle coupling, which originates from the interaction of the surface flow with the topography of the core-mantle boundary, is strongly influenced by the stably stratified layer. We consider the topographic core-mantle coupling arising due to generation of motion resembling the magnetic Rossby waves in the stably stratified layer. A simple expression is obtained for the topographic tangential stress on the core-mantle boundary.

1. Introduction

This paper is part of a series of studies investigating different aspects of the dynamics of a stably stratified layer at the top of the core, adjacent to the core-mantle boundary (CMB). This layer has a density that differs very little from the one corresponding to the adiabatic density gradient; however, this small difference implies a very large Archimedean (buoyancy) force. Therefore, hydrodynamic properties of such a thin layer differ drastically from those of the bulk of the Earth's core. We call this layer the stratified ocean of the core (SOC).

Many authors have assumed that the top of the core is stably stratified: Whaler (1980), Fearn and Loper (1981), Yukutake (1981), Gubbins *et al.* (1982), Frank (1982), Braginsky (1984, 1987b, 1993, 1998), Bergman (1993), Braginsky and Le Mouël (1993), Lister and Buffett (1994), Loper and Lay (1995), Shearer and Roberts (1997). The convincing proof of the existence of a stably stratified layer at the top of the core with an accurate estimate of its parameters is still lacking. Only recently Braginsky (1993) has shown that the observed 65-year variations of both the geomagnetic field and the speed of the Earth's rotation can be explained by the axisymmetric oscillation (akin to MAC-waves) in the stably stratified layer at the top of the core. Two main parameters of the layer were estimated in Braginsky (1993) by comparing the theory of this 65-year MAC-oscillation with the observed variations, namely the thickness of the layer, $H \approx 80$ km, and its Brunt-Väisälä frequency, $N \approx 2\Omega$, where $\Omega = 0.729 \cdot 10^{-4}$ s^{-1} is the angular velocity of Earth's daily rotation. These estimates provided the hypothesis of existence of the SOC with an observational support.

The stably stratified layer is characterized by its (negative) density excess, $C = (\rho - \rho_a)/\rho_a$. Here ρ is the fluid true density, ρ_a is the equilibrium density corresponding to the adiabatic gradient; we substitute ρ_a below by the constant ρ_0 = 10 g cm^{-3}. Braginsky (1993) assumed the following model of the stratified layer with the linear $C(r)$ dependence

$$C = -C_S - (C_H/H)(r - R_S). \quad (1)$$

The CMB placed at the radius R_1 is the bottom of the SOC, and its top merges with the bulk of the core at the radius $R_S = R_1 - H$, where the layers' density excess, C, drops to a negligibly small value, $\sim C_0$, corresponding to the bulk of the core. The density excess, $\sim C_0$, driving the geodynamo in the bulk of the core is very small and we ignore it here. It can be estimated from $C_0 \sim 2\Omega V/g$, where V is the fluid velocity, and thus $C_0 \sim 10^{-8}$ follows for $V \sim 7 \cdot 10^{-2}$ cm s^{-1}. The three characteristic parameters of the above model of the SOC are its thickness, H, the inner change of density excess, C_H, and the jump of density excess, C_S, at the top surface of the ocean, $r = R_S$. It is convenient to use also two corresponding Brunt-Väisälä frequencies:

$$N = (\boldsymbol{g} \cdot \nabla C)^{1/2} = (g_1 C_H/H)^{1/2}, \quad (2a)$$

$$N_S = (g_1 C_S/H)^{1/2}, \quad (2b)$$

where $\boldsymbol{g} = -\mathbf{1}_r g_1$ is the gravity acceleration, $\mathbf{1}_r$ is the unit vector in r-direction, and g_1 = 10 m s^{-2} in the SOC.

The estimates $H \approx 80$ km and $N \approx 2\Omega$ give $C_H \sim 10^{-4}$. Even the best available seismic models, e.g. the well known PREM model by Dziewonski and Anderson (1981) are not able to differentiate such a small deviation of the core density distribution from the adiabatic stratification. That is why the stratified layer was called in Braginsky (1993) "the hidden ocean of the core", or HOC. It should be noted, however, that evidence is now accumulated from seismic observations that ~1% decline in a seismic velocity exists in the uppermost layer of the core, about 50–100 km thick, see Lay and Young (1990), Souriau and Poupinet (1991), Garnero *et al.* (1993), and Sylvander and Souriau (1996). This effect was not precisely measured yet, because of the difficulties associated with complications introduced by inhomogeneity of the D'' layer in the mantle nearby, but if confirmed, it would provide a direct proof of the existence of a layer of light material at the top of the core. The deficit of ~1% in seismic velocity would imply, however, a similar change in density, $C \sim 10^{-2}$, which is two orders of magnitude greater than C_H, and corresponds to Brunt-Väisälä frequency one order of magnitude greater than N obtained by Braginsky (1993). This apparent contradiction can be resolved in the frame of the model (1) by assuming a sharp density jump, $C_S \sim 10^{-2}$, at the layer's boundary, $r = R_S$, and a much smaller change, $C_H \sim 10^{-4}$, inside the layer. The Brunt-Väisälä frequency, $N \approx 2\Omega$, estimated by Braginsky (1993) from the MAC-oscillation in the layer is determined by the density gradient, C_H/H, inside the SOC, while the seismic measurements are sensitive to the total change of density, $C \approx -C_S \sim 10^{-2}$. In anticipation of a future confirmation of this model of the layer of light material at the top of the core we remove the adjective "hidden" from the name of the layer, and call it simply the stratified ocean of the core (SOC).

The SOC can be observed by its dynamic effects which are rather strong because the value, $N = (gC_H/H)^{1/2}$, of Brunt-Väisälä frequency in the layer is about three orders of magnitude greater than the corresponding value, $\sim(gC_0/R_1)^{1/2}$, in the bulk of the core. The density jump, C_S, is about six orders of magnitude greater than the density excess, $\sim C_0$, in the bulk of the core, and this gives a strong rigidity to the surface of the layer, $r = R_S$. This density jump separates the SOC from the bulk of the core, like the density jump on the surface of the "common" ocean separates it from the atmosphere. It should be emphasized that even $C \sim C_S$ is still much smaller than unity.

There is a close similarity between the SOC and the Earth's "common" ocean, both in their geometry of a thin shell and in the magnitude of the Brunt-Väisälä frequency. Dynamics of the SOC is reminiscent of the rich dynamics of the ocean and the atmosphere. The SOC parameters are determined by the arrival of light fluid into the SOC and its redistribution under the action of so far unknown mechanisms. There are two ways of addition of the light fluid to the SOC. One is the arrival of fluid particles with higher concentration of light admixture from the bulk of the core, and another is the leakage of light material from the mantle due to its chemical interaction with the core. Both of these processes are quite different from the convectional mechanisms of formation of the stable layer considered by Fearn and Loper (1981), Gubbins *et al.* (1982), Lister and Buffett (1994), and Shearer and Roberts (1997).

We do not even try to consider here the obscure and complicated mechanisms of SOC formation. Instead we postulate the model (1) and investigate various waves and oscillations which can develop in such an ocean of conducting fluid penetrated by a magnetic field. Various oscillations are possible in the SOC. We investigate here the waves, which are similar to Rossby waves. They were considered previously by Braginsky (1984, 1987b) and Bergman (1993) but for a much thinner layer; the small conductivity approximation was used in these papers. We reconsider these waves in Section 3 using the above parameters of the SOC.

In Section 4 we consider the core-mantle friction due to unevenness of the CMB. This mechanism of the "topographic coupling" was first suggested by Hide (1969), then it was considered (but without taking Archimedean forces into account) by many authors, e.g. Anufriev and Braginsky (1975, 1977a, 1977b), Moffatt (1978), Kuang and Bloxham (1993). Hide (1969) assumed that the CMB perturbations of small height, $h << R_1$, generate large perturbations of the fluid flow. Hide estimated the horizontal gradient of pressure perturbation due to the CMB topography as $\sim\rho_0 2\Omega V_\phi$, so that for an effective tangential stress on the CMB he obtained $\pi_{r\phi}{}^0 \sim \rho_0\, 2\Omega V_\phi h$. Here V_ϕ is the fluid velocity at the CMB, and for $V_\phi \sim 5{\cdot}10^{-4}$ m s^{-1} this estimate gives $\pi_{r\phi}{}^0 \sim 5{\cdot}10^{-2}\, h_{km}$ N m^{-2} (here the height h_{km} is in km). The Hide's estimate appeals to a similarity between the interaction of the atmospheric wind with the Earth's surface topography and the process in the core. The wind perturbation extends along the vector Ω like the "Taylor column", therefore the horizontal pressure perturbation is determined by the horizontal Coriolis force, and it is nearly independent of h, hence $\pi_{r\phi}{}^0$ is proportional to h. It was shown by Anufriev and Braginsky (1975, 1977a, 1977b) that the situation in the core is quite different from the above. The influence of the magnetic field levels out the flow perturbation, reduces it strongly, and makes the process nearly linear. In this case the pressure perturbation is proportional to h, or, more precisely, it is proportional to a small slope $k_\phi h << 1$. Here $k_\phi = \pi/L_\phi$ where L_ϕ is a horizontal dimension of the topography. Anufriev and Braginsky (1977a) assumed a horizontal (toroidal) magnetic field, B_ϕ, and considered the topography of a very small scale ("roughness") for which the magnetic Reynolds number is small, $V_\phi/k_\phi\eta << 1$. For this case they obtained the estimate $\pi_{r\phi} \sim \pi_{r\phi}{}^1 \sim \rho_0\Omega V_\phi k_\phi h^2$. Anufriev and Braginsky (1977b) considered also the topography of a large scale, $k_\phi \sim R_1{}^{-1}$, with the large magnetic Reynolds number. For this case they found an approximate solution assuming that the parameter $M = B_\phi{}^2(\mu_0\rho_0 2\Omega V_\phi L)^{-1}$ is small, and obtained the estimate of the stress, $\pi_{r\phi} \sim M^{-1}\pi_{r\phi}{}^1$. In both cases the core was modeled by a plane layer of thickness L. Kuang and Bloxham (1993) obtained a similar estimate, $\pi_{r\phi} \sim f_f\rho_0 2\Omega V_\phi h^2/L$, for large-scale perturbations, $k_\phi \sim L^{-1} \sim 10^{-6}$ m, and a large magnetic Reynolds number, $\sim 10^2$. They considered a magnetic field, B_ϕ, which varies inside the layer, and assumed the maximum value of parameter M to be $M_{\max} \sim 1$. A "friction coefficient", f_f, is determined by numerical integration; it depends on the form of B_ϕ. For a smooth field $f_f \sim 1$ but with concentration of B_ϕ near the CMB (thus the averaged M is reduced) it increases, and $f_f \sim 20$ was obtained for strongly concentrated B_ϕ. This is roughly comparable with the result of Anufriev and Braginsky (1977b).

It should be emphasized that for the largest bumps, $k_\phi \sim R_1{}^{-1}$, the plane layer model is not valid. Anufriev and Braginsky (1977b) considered also a spherical model for the case $M << 1$. They found that, at least in this case, the CMB sphericity drastically changes the influence of topography, reducing it by the factor $\sim h/R_1$. The cause of this strong effect is simple: a small radial shift, Δs, results in the same order change of the distance between the surfaces of north and south hemispheres, $\Delta z \sim \Delta s$. Thus a very small radial shift of the fluid particle path, $\Delta s \sim h \sim \Delta z$, can nearly cancel the influence of the bump on the geostrophic flow. This makes the problem of the largest bumps especially difficult to consider. In Section 4, where the problem of interaction of the flow in the SOC with the topography of the ocean's bottom (CMB) is investigated, we use the plane model and consider only the bumps which are not very large.

2. Main Equations

Let us consider small oscillations about the stationary basic state. The main system of equations for the fluid velocity, $\boldsymbol{V} + \boldsymbol{v}$, magnetic field, $\boldsymbol{B} + \boldsymbol{b}$, and the density excess, $C + c$, is taken in the Boussinesq approximation. Here the capital and the small letters denote the basic and the oscillating quantities respectively. The linearized equations for $(\boldsymbol{v}, \boldsymbol{b}, c, p)$ in the Boussinesq approximation may be written as (see e.g. Brekhovskikh and Goncharov (1985))

$$d_t\boldsymbol{v} + 2\boldsymbol{\Omega} \times \boldsymbol{v} = -\nabla p + \boldsymbol{f}^\alpha + \boldsymbol{f}^b, \tag{3}$$

$$d_t\boldsymbol{b} - \eta\nabla^2\boldsymbol{b} = (\boldsymbol{B}\cdot\nabla)\boldsymbol{v}, \tag{4}$$

$$d_t c = -\boldsymbol{v}\cdot\nabla C, \tag{5}$$

$$\nabla\cdot\boldsymbol{v} = 0, \tag{6}$$

$$\nabla\cdot\boldsymbol{b} = 0. \tag{7}$$

The Archimedean force, $\boldsymbol{f}^\alpha$, and magnetic force, $\boldsymbol{f}^b$, per unit mass are

$$\boldsymbol{f}^\alpha = \boldsymbol{g}c, \tag{8}$$

$$\boldsymbol{f}^b = (\boldsymbol{B}\cdot\nabla)\boldsymbol{b}. \tag{9}$$

The magnetic field is divided by $(\mu_0\rho_0)^{1/2}$, where μ_0 is the magnetic permeability, so that $\boldsymbol{B}$ and $\boldsymbol{b}$ are measured in velocity units; 1 cm s^{-1} is equivalent to 11.2 G. It is assumed that the term corresponding to magnetic pressure, $\boldsymbol{B}\cdot\boldsymbol{b}$, is added to the (divided by ρ_0) thermodynamic pressure, p_T. Both terms are absorbed in the effective pressure $p = p_\mathrm{T} + \boldsymbol{B}\cdot\boldsymbol{b}$, so that magnetic pressure gradient term is absent from Eq. (9).

We consider here decadal periods which are very short on the time scale of the core convection, and we assume that the basic velocity has only V_ϕ component, so that material time derivative is $d_t = \partial_t + V_\phi\nabla_\phi$. We ignore also the terms having the gradients of $\boldsymbol{V}$ and $\boldsymbol{B}$ in Eqs. (3) and (4): $\boldsymbol{L}^v = (\boldsymbol{b}\cdot\nabla)\boldsymbol{B} - (\boldsymbol{v}\cdot\nabla)\boldsymbol{V}$ and $\boldsymbol{L}^b = (\boldsymbol{b}\cdot\nabla)\boldsymbol{V} - (\boldsymbol{v}\cdot\nabla)\boldsymbol{B}$. We assume these terms to be negligibly small as compared with the retained terms, $(\boldsymbol{B}\cdot\nabla)\boldsymbol{b}$ and $(\boldsymbol{B}\cdot\nabla)\boldsymbol{v}$ having the gradients of oscillating quantities. This assumption is valid for perturbations with a short wave length.

The boundary conditions should be prescribed on the bottom of the ocean ($r = R_1$) and on its surface ($r = R_S$). On the solid CMB the normal (approximately radial) velocity component, v_n, should disappear, and an arbitrary pressure is admissible. Jumps in tangential velocities also are admissible because we neglect viscosity and disregard the Ekman layers which are very thin. The top boundary of the ocean is a moving surface, however a large magnitude of C_S makes the surface $r = R_S$ very rigid. It is well known (Landau and Lifshitz, 1987) that gravitational waves can propagate at a density jump inside a fluid, like the common surface waves do at the external surface of the fluid. For the assumed large magnitude of C_S in the SOC the surface waves frequency is much greater then the frequency of the considered decadal variations. This means that the surface $r = R_S$ is very rigid, and we can ignore the motion of the "upper" surface of the SOC. A similar simplification is often used in oceanology, where it is called a "rigid lid approximation". We assume, therefore, the same boundary conditions both at the solid CMB and at the boundary $r = R_S$:

$$v_n = 0, \tag{10}$$

$$[[\boldsymbol{b}]] = 0. \tag{11}$$

A double square brackets [[...]] denote the jump in a quantity.

The energy balance equation can be obtained if we multiply scalarly Eq. (3) by $\boldsymbol{v}$, Eq. (4) by $\boldsymbol{b}$, and Eq. (5) by the combination $cg_r/\partial_r C = c(g_r/N)^2$, add up the three results, and integrate over the space using boundary conditions (10) and (11). The energy balance takes the form

$$\partial_t \mathsf{E}_\Sigma = -Q_J \tag{12}$$

$$\mathsf{E}_\Sigma = \mathsf{E}_v + \mathsf{E}_b + \mathsf{E}_\alpha = \int\varepsilon_v dV + \int\varepsilon_b dV + \int\varepsilon_\alpha dV \tag{13}$$

$$\varepsilon_v = v^2/2, \tag{14a}$$

$$\varepsilon_b = b^2/2, \tag{14b}$$

$$\varepsilon_\alpha = (g_r/N)^2c^2/2. \tag{14c}$$

Here ε_v and ε_b are kinetic and magnetic energies (per unit mass), ε_α is an available potential energy, compare e.g. Cushman-Roisin (1994), Subsection 15-4. The terms ε_v and ε_α are integrated over the core's volume ($\varepsilon_\alpha \neq 0$ only in the SOC) while ε_b is integrated over the whole space. The total energy dissipation is mostly produced by the Joule heating Q_J, where

$$Q_J = \int\eta \boldsymbol{j}^2 dV. \tag{15}$$

It determines the oscillations' decay. A dissipation due to viscous friction, Q_ν, depends on the viscosity magnitude which is poorly known. The commonly accepted viscosity value $\nu \sim 10^{-6}$ m^2/s $\sim 10^{-6}\eta$ corresponds to the Ekman number of order of 10^{-15} and leads to $Q_\nu << Q_J$. We neglect Q_ν.

3. Magnetic Rossby Waves

Let us consider the waves in the SOC which are nonaxisymmetric, unlike one investigated in Braginsky (1993). We model the SOC by an infinite plane fluid layer of thickness H placed between two solid walls corresponding to $r = R_1$ and $r = R_S = R_1 - H$. The walls are perpendicular to the vertical unit vector $\mathbf{1}_r$ at the colatitude θ_0. The horizontal scale of perturbations in θ-direction, L_θ, is assumed to satisfy the conditions $H << L_\theta << R_1$, e.g. $L_\theta \sim 10H \sim 800$ km will be used in estimates below. This makes it possible to use the approximation of a thin layer and also to consider the perturbations as being essentially local. We introduce the local Cartesian coordinates (x_r, x_θ, x_ϕ) where $x_r = r - R_1$, $x_\theta = R_1(\theta - \theta_0)$, and $x_\phi = s_0\phi$; here $s_0 = R_1 \sin\theta_0$. The subscripts r, θ, ϕ indicate the corresponding Cartesian projections. Directions (θ, ϕ) are called horizontal or tangential and simbolized by the subscript τ. For example, a horizontal gradient operator is $\nabla_\tau = \mathbf{1}_\theta\nabla_\theta + \mathbf{1}_\phi\nabla_\phi$, where $\nabla_\theta = R_1^{-1}\partial_\theta$ and $\nabla_\phi = s_0^{-1}\partial_\phi$. Acceleration due to gravity $\boldsymbol{g} = -\mathbf{1}_r g_1$, Brunt-Väisälä frequency N, and magnetic field B_r are considered constant in the SOC. The "Coriolis parameter", $2\Omega_r = 2\Omega\cos\theta_0$, is taken at $\theta = \theta_0$; the value $\theta_0 = 45°$ is used below, assuming that the waves propagate at mid-latitudes. Thus we avoid the complicated effect of equatorial trapping of the Rossby waves considered by Bergman (1993). The change of the Coriolis parameter along the colatitude will be taken into account in the linear approximation. It is measured by the "β-parameter", $\beta = -\nabla_\theta 2\Omega_r = 2\Omega R_1^{-1} \sin\theta_0$, which is considered constant ($\beta \approx 3\cdot 10^{-11}$ m^{-1}s^{-1}). This model corresponds to the well known "β-plane" approximation, which is customary in oceanology and meteorology.

The perturbation equations of the model are greatly simplified by the fact that the frequencies $2\Omega_r \approx 10^{-4}$ s^{-1} and $N \approx 2\Omega$ are much greater than all other characteristic frequencies of the problem, therefore Coriolis and Archimedean forces are strongly dominating. The inertia force is much smaller than the Coriolis force because $d_t << \Omega$. The dimensionless number $B_r^2/2\Omega_r\eta$ compares magnetic and Coriolis forces; it is of order of 10^{-1} or smaller.

The huge Archimedean force has only a vertical component which can be balanced only by the vertical pressure gradient. Therefore a quasistatic equilibrium of the greatest vertical forces is established. The horizontal gradient of pressure is $\sim L_\tau/H \sim 10$ times smaller than the vertical one. It is balanced (approximately) by the horizontal component of the Coriolis force which is about $2\Omega_r\eta/B_r^2 \sim 10$ times greater than the magnetic force. Thus the equation of motion in the leading approximation is reduced to the equation of equilibrium of the Coriolis force and the pressure gradient. This equation is called a "geostrophic balance" equation.

The vertical velocity is much less than the horizontal one, and estimates show that even $v_r/v_\tau << H/L_\tau << 1$. Hence in the leading approximation we have $\nabla_\tau\cdot\boldsymbol{v}_\tau = 0$. The quantity $\partial_r v_r$ enters the continuity equation only in the higher order approximations, therefore it should be found from the corresponding higher approximation of the equation of motion. Fortunately it is not necessary to construct a complicated perturbation scheme to derive the equation for v_r. Taking the *curl* of the equation of motion and looking on its vertical projection we eliminate both greatest terms, $\boldsymbol{f}^p = -\nabla p$ and $\boldsymbol{f}^\alpha = -\mathbf{1}_r g_1 c$, and obtain the equation which govern the small deviation from the geostrophy. A simple transformation using the complete continuity equation, $\nabla\cdot\boldsymbol{v} = 0$, gives $[\nabla \times (2\boldsymbol{\Omega} \times \boldsymbol{v})]_r = -2\Omega_r\partial_r v_r + (2\Omega_\theta/r)v_\theta$, and thus the small term $\partial_r v_r$ can be expressed through the quantities of leading order. We use here the approximation $-2\Omega_\theta/r \approx 2\Omega R_1^{-1}\sin\theta_0 \equiv \beta$. The rigor derivation of the "β-plane model" with explicit introduction of the relevant small parameters (Rossby number, and the ratio L/R_1) can be found in standard textbooks on geophysical fluid dynamics, e.g. Pedlosky (1987), Gill (1982).

With the above simplifications, Eqs. (3)–(8) can be reduced to a system of equations which is similar to the "quasigeostrophic" system well known in the oceanology:

$$g_1 c + \partial_r p = 0, \tag{16}$$

$$2\Omega_r \mathbf{1}_r \times \boldsymbol{v}_\tau + \nabla_\tau p = 0, \tag{17}$$

$$d_t(\nabla_\tau \times \boldsymbol{v}_\tau)_r - 2\Omega_r\partial_r v_r - \beta v_\theta = B_r\partial_r (\nabla_\tau \times \boldsymbol{b}_\tau)_r, \tag{18}$$

$$d_t\boldsymbol{b} - \eta\partial_r^2\boldsymbol{b} = B_r\partial_r\boldsymbol{v}, \tag{19}$$

$$g_1 d_t c - N^2 v_r = 0, \tag{20}$$

$$\nabla_\tau\cdot\boldsymbol{v}_\tau = 0, \tag{21a}$$

$$\nabla_\tau\cdot\boldsymbol{b}_\tau = 0. \tag{21b}$$

Equation (21a) is in accord with (17). Its validity follows from (18) and an estimate $\nabla_\theta \sim \pi/L_\theta = k_\theta$. This gives $\partial_r v_r/\nabla_\theta v_\theta \sim \beta/(2\Omega_r k_\theta) \sim (k_\theta R_1)^{-1} << 1$. E.g. for $L_\theta \sim 800$ km we have $k_\theta \sim 4\cdot 10^{-6}$ m^{-1} and $(k_\theta R_1)^{-1} \sim 10^{-1}$. Equation (21b) can be written now according to $b_r/b_\tau \sim v_r/v_\tau$, which follows from (19). The "inertia term", $d_t(\nabla_\tau \times \boldsymbol{v}_\tau)_r$, in (18) is small as compared with βv_θ, and it will be neglected below. With the assumption $H/L_\theta \sim 0.1$ the overall accuracy of the SOC quasigeostrophic theory is ~10%. It is of course rougher than corresponding approximations in oceanology applications because $H \sim 80$ km is much greater than a typical depth of the common ocean, and $R_1 = 3.48\cdot 10^3$ km is smaller than the Earth's core radius $R_0 = 6.37\cdot 10^3$ km. We hope, nevertheless, that the results obtained are at least qualitatively correct.

The system (16)–(21) allows a separation of variables. Let there be $\boldsymbol{v}_p = \mathbf{1}_r v_r + \mathbf{1}_\theta v_\theta$ and $\boldsymbol{b}_p = \mathbf{1}_r b_r + \mathbf{1}_\theta b_\theta$. The solution of the system can be sought in the form of a progressive wave in ϕ but a standing wave in θ:

$$(\boldsymbol{v}_p, \boldsymbol{b}_p, c, p) = (\boldsymbol{v}_p^c, \boldsymbol{b}_p^c, c^c, p^c)\cos(k_\theta x_\theta)\exp(i\Phi), \tag{22a}$$

$$(v_\phi, b_\phi) = (v_\phi^s, b_\phi^s)\sin(k_\theta x_\theta)\exp(i\Phi). \tag{22b}$$

We denote $\Phi = k_\phi x_\phi - \omega t$, $\varpi = \omega - k_\phi V_\phi$, and

$$k_\theta = \pi/L_\theta,\ k_\phi = m/s_0 = \pi/L_\phi,\ k_\tau^2 = k_\theta^2 + k_\phi^2, \tag{23}$$

where L_θ, L_ϕ, $L_\tau = \pi/k_\tau$ are the characteristic lengths. After the separation of variables we keep the same notation $\boldsymbol{v}$, $\boldsymbol{b}$, c, p for the unknown functions omitting the superscripts c and s, but we substitute $\nabla_\phi = ik_\phi$, $\nabla_\theta^2 = -k_\theta^2$, and replace d_t by $-i\varpi$. The transformation $\nabla_\phi(\nabla_\tau \times \boldsymbol{b}_\tau)_r = -\nabla_\tau^2 b_\theta$ exploiting

(21b) is made in (18). It can be derived from (16)–(21):

$$g_1 c = -i(2\Omega_r/k_\phi)\partial_r v_\theta, \tag{24a}$$

$$p = i(2\Omega_r/k_\phi)v_\theta, \tag{24b}$$

and

$$i\varpi(2\Omega_r/N)^2\partial_r^2 v_\theta - \beta i k_\phi v_\theta = k_\tau^2 B_r \partial_r b_\theta, \tag{25a}$$

$$i\varpi b_\theta + \eta\partial_r^2 b_\theta = -B_r\partial_r v_\theta. \tag{25b}$$

Equations (25a) and (25b) determine v_θ and b_θ, while (20) and (24a) give v_r:

$$v_r = \lambda_{r\theta}\partial_r v_\theta, \tag{26a}$$

$$\lambda_{r\theta} = -(\varpi/k_\phi)(2\Omega_r/N^2). \tag{26b}$$

The length $\lambda_{r\theta}$ is very small, e.g. for the frequency $\varpi \sim 10^{-8}$ s^{-1} (that is $2\pi/\varpi \sim 20$ yr) and $k_\theta \sim 4\cdot10^{-6}$ m^{-1} we have $\lambda_{r\theta} \sim 25$ m $\sim 10^{-4}\, k_\theta^{-1}$.

The bottom surface of the SOC is solid; its top surface, $r = R_S$, though fluid, is nevertheless very rigid (effectively solid) because C_S is very large. Therefore the boundary conditions for velocity are

$$v_r(R_1) = 0, \tag{27a}$$

$$v_r(R_S) = 0, \tag{27b}$$

which implies

$$\partial_r v_\theta(R_1) = 0, \tag{28a}$$

$$\partial_r v_\theta(R_S) = 0. \tag{28b}$$

The boundary conditions (11) for the magnetic field are more complicated. Neglecting the mantle conductivity and matching the field in the core with the potential field in the mantle we obtain for $r = R_1$:

$$\boldsymbol{b}_\tau = -i(\boldsymbol{k}_\tau/k_\tau)b_r,\ \ k_\tau = (\boldsymbol{k}_\tau^2)^{1/2}. \tag{29}$$

It follows that b_τ on the boundary ($b_\tau \sim b_r$) is much smaller than inside the SOC where $b_\tau >> b_r$. We may, therefore, take, as a good approximation, the simpler condition, $\boldsymbol{b}_\tau(R_1) = 0$. The plain reason for this simplification is that a potential magnetic perturbation in the insulating mantle, $\boldsymbol{b}^M$, changes in all directions on the same characteristic length, $L \sim 10^3$ km, so that throughout the mantle we have $b^M{}_\tau \sim b_r{}^M$, and the same is true at the boundary $r = R_1$. The exact form of relation (29) is not significant; the simplification $b_\tau(R_1) = 0$ follows from $b^M{}_\tau \sim b_r{}^M = b_r(R_1) \sim b_r << b_\tau$, therefore it is valid for a weakly conducting mantle as well.

Magnetic, $\boldsymbol{b}^L$, and velocity, $\boldsymbol{v}^L$, perturbations in the bulk of the core generated by perturbations in the SOC have a complicated form of superposition of MAC-waves. There is no specific reason for them to have $b_r{}^L$-component much smaller than $b^L{}_\tau$, hence one may anticipate $b^L{}_\tau \sim b_r{}^L$. If this is confirmed, the simplification $b^L{}_\tau = 0$ can be assumed for $r = R_S$ as well as for $r = R_1$. It is difficult to find the field in the bulk, $\boldsymbol{b}^L$, in general case to match it with the solution in the SOC. We will only consider a simple example of a constant $\boldsymbol{B}$ and a plane wave proportional to $\exp(i\boldsymbol{k}\cdot\boldsymbol{r} - i\omega t)$. In this case Eqs. (3)–(7) with the Archimedean force omitted can be reduced to

$$\omega_\eta(2\boldsymbol{\Omega}\cdot\boldsymbol{k})\boldsymbol{v} = (\omega_A{}^2 - \omega\omega_\eta)i\boldsymbol{k}\times\boldsymbol{v}, \tag{30a}$$

$$\omega_\eta\boldsymbol{b} = -\omega_A\boldsymbol{v}, \tag{30b}$$

where $\omega_A = \boldsymbol{k}\cdot\boldsymbol{B}$ is Alfvén frequency, and $\omega_\eta = \omega + i\eta k^2$. A solvability condition for (30a) gives two (±) frequency relations for magnetic waves

$$\omega_A{}^2 - \omega(\omega + i\eta k^2) = \pm(\omega + i\eta k^2)(2\boldsymbol{\Omega}\cdot\boldsymbol{k}/k). \tag{31}$$

Substitution of (31) into (30a) and (30b) transforms them to

$$\boldsymbol{v} = \pm i k^{-1}\boldsymbol{k}\times\boldsymbol{v}, \tag{32a}$$

$$\boldsymbol{b} = \pm i k^{-1}\boldsymbol{k}\times\boldsymbol{b}, \tag{32b}$$

then a simple manipulation using (6), (7) gives for the components of magnetic waves in the bulk of the core: $v_r{}^2 = -v_\tau{}^2$ and $b_r{}^2 = -b_\tau{}^2$.

To work with the simplest model we assume $b^L{}_\tau \sim b_r{}^L$, and accept both for $r = R_1$ and for $r = R_S$ the same simplified boundary conditions:

$$\boldsymbol{b}_\tau(R_1) = 0, \tag{33a}$$

$$\boldsymbol{b}_\tau(R_S) = 0. \tag{33b}$$

Equations (25a) and (25b) should be solved with the boundary conditions (28a), (28b), (33a), and (33b). The equation of fourth order for v_θ and b_θ can be obtained from (25a) and (25b), and the same equation is valid for other components due to (26) and (20):

$$\{(\varpi - i\eta\partial_r{}^2)(\omega_\beta k_H{}^2 - \varpi\partial_r{}^2) + \omega_B{}^2\partial_r{}^2\}(\boldsymbol{v}, \boldsymbol{b}, c, p) = 0. \tag{34}$$

Here $k_H = \pi/H$, and the quantities ω_β and ω_B with dimension of frequency are introduced:

$$\omega_\beta = \beta k_\phi k_H{}^{-2}(N/2\Omega_r)^2, \tag{35a}$$

$$\omega_B{}^2 = k_\tau{}^2 B_r{}^2(N/2\Omega_r)^2. \tag{35b}$$

Let us assume $L_\theta = 10H = 8\cdot10^2$ km and $L_\phi = \pi s_0/m = 1.55\cdot10^3(5/m)$ km where $s_0 = 0.707R_1$. For $m = 5$ we have $k_\theta = 3.9\cdot10^{-6}$ m^{-1}, $k_\phi = 2\cdot10^{-6}$ m^{-1}, $k_\tau = 4.4\cdot10^{-6}$ m^{-1}. Assuming also $N = 2\Omega$, hence $(N/2\Omega_r)^2 = 2$, and $B_r = 5$ G (=0.45 cm s^{-1}), we obtain $\omega_\beta = 7.8\cdot10^{-8}$ s^{-1} and $\omega_B = 2.8\cdot10^{-8}$ s^{-1}; the corresponding periods are $2\pi/\omega_\beta = 2.6$ yr and $2\pi/\omega_B = 7.1$ yr. We use the smallness of the ratio $\omega_B{}^2/\omega_\beta{}^2 = 0.13(5/m)^2$ to simplify calculations.

Equation (34) has elementary solutions proportional to $\sin\alpha$ and $\cos\alpha$ where $\alpha = k_r(r - R_1)$, and $k_r{}^2(\varpi)$ are the roots of the biquadratic

$$(\varpi + i\eta k_r^2)(\omega_\beta k_H^2 + \varpi k_r^2) - \omega_B^2 k_r^2 = 0. \tag{36}$$

The free modes eigenfrequencies can now be found. There are two different branches of $k_r^2(\varpi)$ and four independent elementary solutions. Substitution of them into four boundary conditions gives a system of homogeneous equations for amplitudes of the elementary solutions. A solvability condition of this system determines the (complex) values of eigenfrequencies, ϖ. Finding the solution is greatly simplified by the fact that both boundary conditions (27a) and (27b) for v_r, and both conditions (33a) and (33b) for b_θ can be satisfied by one term, $\sin\alpha$, with the same value of $k_r = nk_H$, where $n = 1, 2, 3, ...$; the term $\cos\alpha$ describes v_θ. There are two modes of solution: the weakly decaying magnetic Rossby mode, ϖ_{MR}, and the strongly decaying magnetic diffusion mode, ϖ_{MD}. Both modes have the same form:

$$v_\theta = v_a \cos\alpha_n, \tag{37a}$$

$$v_\phi = -iv_a(k_\theta/k_\phi)\cos\alpha_n, \tag{37b}$$

$$v_r = v_{ra}\sin\alpha_n, \tag{37c}$$

$$b_\theta = b_a \sin\alpha_n, \tag{38a}$$

$$b_\phi = -ib_a(k_\theta/k_\phi)\sin\alpha_n, \tag{38b}$$

$$c = ic_a \sin\alpha_n, \tag{38c}$$

where $\alpha_n = nk_H(r - R_1)$. The relations $v_{ra}/v_a = -\lambda_{r\theta} nk_H$ and $g_1 c_a/v_a = (2\Omega_r/k_\phi)nk_H$ follow from (26a) and (24a). The expressions (37a), (37b), (38a), and (38b) satisfy the continuity Eqs. (21a) and (21b). The amplitude ratio b_a/v_a is given by (25b): $b_a/v_a = (k_r B_r)(i\varpi - \eta k_r^2)^{-1}$.

Biquadratic Eq. (36) can be solved for ϖ in a common way but to keep the algebra as simple as possible we use (for $n = 1$) the small parameter $\omega_B^2/\omega_\beta^2$, and obtain the complex frequencies of the free modes in a very simple approximate form:

$$\varpi_{MR} = -\omega_\beta - \omega_B^2\omega_\beta^{-1} - i\gamma_{MR}, \tag{39a}$$

$$\gamma_{MR} = \tau_\eta^{-1}\omega_B^2\omega_\beta^{-2}, \tag{39b}$$

$$\varpi_{MD} = \omega_B^2\omega_\beta^{-1} - i\gamma_{MD}, \tag{40a}$$

$$\gamma_{MD} = \tau_\eta^{-1}, \tag{40b}$$

where

$$\tau_\eta^{-1} = \eta k_H^2, \tag{41a}$$

$$\tau_\eta \approx 10 \text{ yr}. \tag{41b}$$

The simple expressions (39a) and (40a) are valid only for $n = 1$. For $n > 1$ one should replace ω_β by $\omega_\beta n^{-2}$ and τ_η by $\tau_\eta n^{-2}$; the ratio ϖ_{MD}/ϖ_{MR} then would increase by the factor n^4. In this case the modes are not separated. The rate of decay also strongly increases with n. Only the $n = 1$ modes which have a smaller decay rate are considered below.

The main term in (39) is a well known frequency of the Rossby waves, $-\omega_\beta$, which always propagate to the west with the non-dispersive phase velocity $V_\beta = \omega_\beta/k_\phi \approx 3.6$ cm s^{-1}. This velocity is much greater than fluid velocities in the core, and the latter may be ignored while considering magnetic Rossby waves. The Rossby frequency, ω_β, is non-dispercive because we neglected the inertia term $d_t(\nabla_\tau \times \boldsymbol{v}_\tau)_r$ in (18). Magnetic correction, $\omega_B^2\omega_\beta^{-1}$, to the frequency (39a), is an order of magnitude smaller than ω_β, and it is highly dispersive. The decay rate, γ_{MR}, is much smaller than even this magnetic correction, therefore we can calculate the group velocity of the magnetic Rossby waves in a usual way as $\partial\omega_{MR}/\partial k_\phi$. The energy of the waves propagate to the west:

$$\partial\omega_{MR}/\partial k_\phi = V_\phi - V_\beta[1 + \omega_B^2\omega_\beta^{-2}k_\tau^{-2}(k_\phi^2 - k_\theta^2)]. \tag{42}$$

It is interesting to estimate separately each term in the energy density of free magnetic Rossby waves, $\varepsilon_\Sigma = \varepsilon_v + \varepsilon_b + \varepsilon_\alpha$. The absolute values are not given by the linear theory, only the proportions can be calculated. We ignore the small radial components and the decay rate γ_{MR}. According to (25b) we have the ratio $b_a/v_a \approx ik_H B_r/\omega_\beta$. An Archimedean (buoyancy) part of energy density is proportional to $(g_1 c_a/N)^2 = (2\Omega_r/N)^2(k_H/k_\phi)^2 v_a^2$. For the energy components averaged over r, x_θ, x_ϕ the following proportion (which gives $\varepsilon_v < \varepsilon_b < \varepsilon_\alpha$) can be established:

$$\varepsilon_v:\varepsilon_b:\varepsilon_\alpha = \omega_\beta^2(N/2\Omega_r)^2(k_\tau^2/k_H^2):\omega_B^2:\omega_\beta^2(k_\theta^2/k_\phi^2). \tag{43}$$

4. Topographic Core-Mantle Coupling

Topographic coupling relies on the non-spherical form of the CMB, which can be described by the equation

$$r = R_1 - h(\theta, \phi), \tag{44}$$

where $h(\theta, \phi)$ gives the topography of the SOC's bottom ($h > 0$ for hills, and $h < 0$ for valleys). It is certain that $h << R_1$, e.g. $h \sim 0.3$ km corresponds to $h/R_1 \sim 10^{-4}$, but the exact value of the function $h(\theta, \phi)$ is unknown. The topographic torque, $\boldsymbol{M}$, exerted by the core on the mantle due to interaction between the core fluid flow and the CMB topography is

$$\boldsymbol{M} = \int \boldsymbol{r} \times \mathbf{1}_n \rho_0 p dA, \tag{45}$$

where the integral is taken over the CMB. Here $\rho_0 p$ is the pressure (note that pressure divided by density, ρ_0, is denoted by p), dA is an element of the surface area, and $\mathbf{1}_n$ is the unit vector directed along the normal to CMB from the core to the mantle. Since $h/R_1 << 1$, the integral in (45) can be taken over the spherical "unperturbed" CMB, $r = R_1$. The vector of the normal can be written as $\mathbf{1}_n = \mathbf{1}_r + \nabla h$, then (45) takes the form

$$\boldsymbol{M} = \rho_0 \int p\boldsymbol{r} \times \nabla h dA \tag{46a}$$

$$= -\rho_0 \int h\boldsymbol{r} \times \nabla p dA, \tag{46b}$$

compare (Roberts, 1988). We are especially interested in the component of $\boldsymbol{M}$ parallel to Ω:

$$M_z = \rho_0 \int s p \nabla_\phi h dA \tag{47a}$$

$$= -\rho_0 \int s h \nabla_\phi p dA. \tag{47b}$$

A general topography can be expressed as a sum of the terms proportional to $\exp(i\boldsymbol{k}_\tau \cdot \boldsymbol{r}_\tau)$; effects of these terms are mutually independent in the linear approximation. The effect of uneveness with the wavelength $\sim R_1$ is of a global character; it is difficult to separate this effect from the whole core dynamics. We consider only bumps of rather short wavelength. Their influence on the global core motion can be modelled as linear and local, hence the plane model of the previous section is applicable. The topography has the amplitude h_a and can be written in the complex form:

$$h(x_\theta, x_\phi) = h_a \cos(k_\theta x_\theta) \exp(i k_\phi x_\phi), \tag{48}$$

which is similar to the horizontal dependence of magnetic Rossby waves (22a) and (22b). The topography creates a velocity perturbation at the bottom of the SOC. To consider it we use Eqs. (24)–(26), and (34). The boundary conditions for velocity at $r = R_S$ are given by (27b) and (28b); the conditions for magnetic field at $r = R_1$, $r = R_S$ are accepted in the form (33a) and (33b). The conditions for velocity at $r = R_1$ are, however, different from (27a) and (28a). We assume $\boldsymbol{V}_\tau = \mathbf{1}_\phi V_\phi$ and replace (27a) by the condition $(\boldsymbol{V}_\tau + \boldsymbol{v}) \cdot \mathbf{1}_n = 0$ in the linear approximation:

$$v_r = -V_\phi \nabla_\phi h = -ihk_\phi V_\phi \quad \text{at} \quad r = R_1. \tag{49}$$

It is just the velocity perturbation (49) which generate the perturbation of pressure $p(R_1, \theta, \phi)$ and the topographic torque (47a) and (47b). Both the velocity perturbation (49) and the integrand of (47a) are proportional to $\nabla_\phi h$, thus M_z is proportional to the amplitude of $(\nabla_\phi h)^2$. Note however that the integrals (47a), (47b) would be zero if $p(R_1, \theta, \phi)$ were proportional to $h(\theta, \phi)$.

We consider an interaction of the CMB topography with the stationary flow $\mathbf{1}_\phi V_\phi$ and substitute $\varpi = -k_\phi V_\phi$ in the above equations. The typical values $V_\phi \sim 5 \cdot 10^{-4}$ m s^{-1} and $k_\phi \sim 2 \cdot 10^{-6}$ m^{-1} give $\varpi \sim 10^{-9}$ s^{-1}. This "frequency" is much smaller then ω_B and ω_β allowing a convenient approximate method of solution of Eq. (34). Elementary solutions are sought in the form $\exp(\alpha)$ where $\alpha = \kappa(r - R_1)$, therefore, perturbations generated by topography decrease with distance from the CMB. The biquadratic equation for κ follows from (34). It is similar to (36) but with k_r^2 replaced by $-\kappa^2$. There are two solutions for κ^2, corresponding to velocity and magnetic modes, κ_v^2 and κ_b^2, which are approximately equal to

$$\kappa_v^2 = \kappa_{\beta B}^2 (1 + \varepsilon_\varpi), \tag{50a}$$

$$\kappa_b^2 = -i(\kappa_\eta^2 \kappa_\beta^2 / \kappa_{\beta B}^2)(1 - \varepsilon_\varpi), \tag{50b}$$

where

$$\varepsilon_\varpi = \kappa_\eta^2 \kappa_B^2 / \kappa_{\beta B}^4, \tag{51a}$$

$$\kappa_\eta^2 = \varpi / \eta = 2/\delta_\eta^2, \tag{51b}$$

$$\kappa_{\beta B}^2 = \kappa_\beta^2 + i\kappa_B^2, \tag{52a}$$

$$\kappa_\beta^2 = k_H^2 \omega_\beta / \varpi, \tag{52b}$$

$$\kappa_B^2 = \omega_B^2 / \eta \varpi. \tag{52c}$$

The correction, $\varepsilon_\varpi \sim \varpi^2/\omega_B^2$ is very small, for example, for $\varpi \sim 10^{-9}$ s^{-1} we have $\varepsilon_\varpi \sim 10^{-3}$. We neglect it and take approximately $\kappa_v = \kappa_{\beta B}$ and $\kappa_b = (1 - i)\delta_\eta^{-1}(\kappa_\beta / \kappa_{\beta B})$. It is assumed that $\mathrm{Re}(\kappa_v) > 0$ and $\mathrm{Re}(\kappa_b) > 0$ where $\mathrm{Re}(\ )$ denotes a real part. The quantity δ_η is a common estimate of a skin depth for the frequency ϖ. Using the same values of parameters as in Section 3 we obtain the estimates: $\delta_\eta = 63$ km, $\kappa_\beta H \sim 28$, $\kappa_B H \sim 50$.

A solution of Eq. (34) together with (25), (26), and boundary conditions (27), (28), (33), and (49) can be written as

$$v_r = v_{ra}[f_v(r) + \varepsilon_\varpi f_{bs}(r)] \approx v_{ra} f_v, \tag{53a}$$

$$v_{ra} = -ih_a k_\phi V_\phi, \tag{53b}$$

$$v_\theta = v_a[f_v(r) + (\varepsilon_\varpi \kappa_{\beta B}/\kappa_b) f_{bc}(r)], \tag{54a}$$

$$v_a = -ih_a k_\phi \kappa_{\beta B}^{-1}(N^2/2\Omega_r), \tag{54b}$$

$$b_\theta = b_a[f_v(r) - f_{bs}(r)], \tag{55a}$$

$$b_a = ih_a k_\phi \kappa_{\beta B}^{-2}(B_r/\eta)(N^2/2\Omega_r), \tag{55b}$$

$$f_v(r) = \exp[\kappa_v(r - R_1)], \tag{56a}$$

$$f_{bs}(r) = \sinh[\kappa_b(r - R_S)]/\sinh(\kappa_b H), \tag{56b}$$

$$f_{bc}(r) = \cosh[\kappa_b(r - R_S)]/\sinh(\kappa_b H). \tag{56c}$$

A simple exponential function is used in (56a) instead of a hyperbolic sinh because $\kappa_v H \approx \kappa_{\beta B} H >> 1$, so that $f_v(R_S) \approx 0$.

The tangential velocity is much greater than the radial velocity. For example, $V_\phi \sim 5 \cdot 10^{-4}$ m s^{-1} corresponds to $v_a/v_{ra} = N^2(2\Omega_r V_\phi \kappa_{\beta B})^{-1} \sim 10^3$. The velocity mode strongly dominates in the radial velocity; the magnetic term in (53a) is of order of $\varepsilon_\varpi \sim 10^{-3}$, and we ignore it. The velocity mode dominates also in v_θ though not so strongly as in v_r. Using (50b) and (51a) we obtain $\varepsilon_\varpi \kappa_{\beta B}/\kappa_b \sim \kappa_\eta/\kappa_\beta \sim 6 \cdot 10^{-2}$. The magnetic perturbation, b_θ, expressed in the velocity units, exceeds the velocity perturbation, v_θ; the amplitude ratio is $b_a/v_a = -B_r(\eta\kappa_{\beta B})^{-1} \sim 4$.

We are now prepared to calculate the topographic coupling. It is convenient to introduce the topographic stress, $\pi^h{}_{r\phi}$, as an average, $\langle ... \rangle$, of the tangential force per unit area: $\pi^h{}_{r\phi} = -\rho_0 \langle h \nabla_\phi p \rangle$; here the integrand of (47b) is used. The expression $\nabla_\phi p = -2\Omega_r v_\theta$ can be obtained from (17) where v_θ is given by (54a). Here we retain only the velocity mode as a reasonable approximation. Transforming all quantities to the real form, using $h(x_\theta, x_\phi)$ given by (48), and averaging $\mathrm{Re}(h)\mathrm{Re}(v_\theta)$ over x_θ, x_ϕ we obtain

$$\pi^h{}_{r\phi} = \rho_0(Nh_a/2)^2(k_\phi/|\kappa_{\beta B}|)\sin\varphi_{\beta B}, \quad (57a)$$

$$|\kappa_{\beta B}|^2 = (\kappa_B{}^4 + \kappa_\beta{}^4)^{1/2}, \quad (58a)$$

$$\sin(2\varphi_{\beta B}) = \kappa_B{}^2/|\kappa_{\beta B}|^2. \quad (58b)$$

Here $\sin\varphi_{\beta B}$ is a numerical factor, ~0.5, determined by a phase difference between h and p; it has the same sign as V_ϕ. For example, $\sin\varphi_{\beta B} = 0.64$, 0.59, 0.49, and 0.38 for $\kappa_\beta{}^2/\kappa_B{}^2 = 0.2$, 0.3, 0.6, and 1. For the parameters used above ($m = 5$ etc.) and $\sin\varphi_{\beta B} \sim 0.5$, (57a) gives

$$\pi^h{}_{r\phi} \sim 8.5 \cdot 10^{-2} h_a{}^2 (V_\phi/V_{\phi 0})^{1/2}, \quad (57b)$$

where $\pi^h{}_{r\phi}$ is expressed in N m^{-2}, h_a in km, and $V_{\phi 0} = 5 \cdot 10^{-2}$ cm s^{-1}.

To interpret (57a) we note that to cross the hill h_a the fluid stream near the bottom of the SOC is moving up the slope $k_\phi h_a$ against the retarding (parallel to the slope) component of gravitational force of order of $\rho_0 g_r(h_a\partial_r C)k_\phi\, h_a \sim \rho_0 k_\phi\, h_a{}^2 N^2$. The estimate $c \sim h_a\partial_r C$ is used here. The retarding force acts in the moving fluid layer of thickness of $\sim\kappa_v{}^{-1} \sim |\kappa_{\beta B}|^{-1}$, thus creating the stress of order of (57a). The retardation is partially compensated when the stream goes down the slope, but owing to the lack of the "for-and-aft" symmetry of the solution, the non-compensation is left which is measured by the coefficient $\sin\varphi_{\beta B}$.

It is interesting to calculate the Joule heating associated with the magnetic field perturbation produced by the bumps. We write $(\nabla \times \boldsymbol{b})^2 \approx (\partial_r b_\theta)^2 + (\partial_r b_\phi)^2$ into (15) neglecting tangential derivatives. Using (21b) we replace $|\partial_r b_\phi|^2$ by $|\partial_r b_\theta|^2 k_\theta{}^2/k_\phi{}^2$. The magnetic mode can be neglected in $|\partial_r b_\theta|$ because $\kappa_b << \kappa_v = \kappa_{\beta B}$, therefore the integral $\int Q_J dr$ is proportional to $\int |f_v|^2 dr = [2\mathrm{Re}(\kappa_{\beta B})]^{-1}$. A simple transformation gives

$$\int Q_J dr = \rho_0(Nh_a/2)^2(k_\phi/|\kappa_{\beta B}|)V_\phi \sin\varphi_{\beta B}. \quad (59)$$

Comparing this expression with (57a) we obtain a simple energy balance relation

$$\int Q_J dr = \pi^h{}_{r\phi} V_\phi. \quad (60)$$

It is interesting to compare (57a) with the stresses due to the core viscosity and with magnetic stresses due to the finite magnetic conductivity of the mantle, σ_M. The former is $\pi^v{}_{r\phi} = \rho_0 \nu V_\phi/\delta_v$, where $\delta_v = (\nu/\Omega_r)^{1/2}$ is the Ekman layer depth. Assuming $\nu \sim 10^{-6}$ m^{2}s^{-1} we obtain

$$\pi^v{}_{r\phi} = \rho_0(\nu\Omega_r)^{1/2} V_\phi \sim 5 \cdot 10^{-5}(V_\phi/V_{\phi 0}). \quad (61)$$

The latter can be easily estimated if we assume that the mantle conductivity is concentrated in the narrow layer near the CMB. The electric current of density $j_\theta = \sigma_M V_\phi B_r$ is flowing in the layer of small thickness L_M, and generates the stress $j_\theta B_r L_M$, so that

$$\pi^B{}_{r\phi} = \sigma_M L_M B_r{}^2 V_\phi \sim 3.8 \cdot 10^{-3}(V_\phi/V_{\phi 0}). \quad (62)$$

Here the estimate $\sigma_M L_M \sim 3 \cdot 10^7$ S of the lower mantle conductivity is taken according to the results by Peyronneau and Poirier (1989). The viscous stress is about two orders of magnitude smaller than the magnetic one. The topographic stress (57b) is of the same order of magnitude as the magnetic stress if the amplitude of the uneveness is $h_a \sim 0.2$ km.

The perturbation produced by the CMB topography is considered here in the linear approximation. It is assumed $\boldsymbol{V}\cdot\nabla = V_\phi \nabla_\phi$ which is valid if $k_\theta v_\theta/k_\phi V_\phi << 1$. This ratio can be estimated using (54b) and the above numerical estimates. This gives

$$k_\theta v_a/k_\phi V_\phi \sim h_a(V_{\phi 0}/V_\phi)^{1/2}, \quad (63)$$

where h_a is expressed in km. For example, we have $\pi^h{}_{r\phi} \sim \pi^B{}_{r\phi}$ for $h_a \sim 0.2$ km, and the linearization is still valid in this case.

5. Concluding Remarks

The MAC-waves, which are characterized by the near-equilibrium between magnetic, Archimedean and Coriolis forces, take the form of the magnetic Rossby waves in the specific conditions of the SOC. The magnetic Rossby waves (MR-waves) are "natural" motions in the SOC associated with periods of order of years to decades. In this paper we consider only free MR-waves, and MR-perturbations generated by the topography of the CMB, but various other kinds of motion of the same type may be expected.

Dynamics of the geomagnetic "jerks" with characteristic periods of a few years, and dynamics of various processes leading to decade geomagnetic variations are probably determined by MR-waves excited in the SOC. MR-waves are essential part of mechanisms of the geomagnetic secular variations, the length of day variation, and the oscillation of the Earth's pole position. In this paper only small-scale free MR-waves ($m \geq 5$) with corresponding free periods, ~3 yr, are considered using a simple plane model. An interaction of fluid flow in the SOC with the mantle topography also is considered for the bumps of the same small scale. To investigate large-scale MR-waves and an interaction with large-scale topography one should make cumbersome numerical calculations.

Braginsky (1987a) demonstrated that a steady oscillation with the period about 30 yr can be isolated from the data of the length of day variations, along with the decaying 65 yr oscillation. Yokoyama (1993) found the 30 yr period in variations of the geomagnetic Gauss coefficients. The period about 30 yr revealed also in the variation of the Earth's rotation pole position, see e.g. Lambeck (1980), Hulot *et al.* (1996). A simple extrapolation of (35a) and (39a) shows that the period ~30 yr may be expected for a global-scale MR-waves, with $m \sim 1, 2$. The axisymmetric MAC-oscillations in the SOC and torsional oscillations in the bulk of the core also can have periods ~30 yr (Braginsky, 1993). Further investigations are necessary to understand the complicated mechanism of 30 yr variations, including the global-scale MR-waves, their generation, and their interaction with MAC-oscillations and with the bulk of the core. Results of such investigations will be helpful in understanding the nature of the secular variations, and in improving our knowledge of the SOC parameters.

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References

Anufriev, A. P. and S. I. Braginsky, Effect of irregularities of the boundary of the earth's core on the velocity of the liquid and on the magnetic field, *Geomagn. Aeron.*, **15**, 754–757, 1975.

Anufriev, A. P. and S. I. Braginsky, Influence of irregularities of the boundary of the earth's core on fluid velocity and the magnetic field, II, *Geomagn. Aeron.*, **17**, 78–82, 1977a.

Anufriev, A. P. and S. I. Braginsky, Effect of irregularities of the boundary of the earth's core on the speed of the fluid and on the magnetic field, III, *Geomagn. Aeron.*, **17**, 492–496, 1977b.

Bergman, M. I., Magnetic Rossby waves in a stably stratified layer near the surface of the Earth's outer core, *Geophys. Astrophys. Fluid Dyn.*, **68**, 151–176, 1993.

Braginsky, S. I., Short-period geomagnetic secular variation, *Geophys. Astrophys. Fluid Dyn.*, **30**, 1–78, 1984.

Braginsky, S. I., Generation of a 65-year oscillation in the Earth's core, *Izv. Acad. Sci. USSR, Earth Physics*, **23**, No. 9, 745–748, 1987a.

Braginsky, S. I., Waves in a stably stratified layer on the surface of the terrestrial core, *Geomagn. Aeron.*, **27**, 410–414, 1987b.

Braginsky, S. I., MAC-oscillations of the hidden ocean of the core, *J. Geomag. Geoelectr.*, **45**, 1517–1538, 1993.

Braginsky, S. I., Dynamics of the stably stratified ocean at the top of the core, *Phys. Earth Planet. Inter.*, 1998 (submitted).

Braginsky, S. I. and J. L. Le Mouël, Two-scale model of a geomagnetic field variation, *Geophys. J. Int.*, **112**, 147–158, 1993.

Brekhovskikh, L. and V. Goncharov, *Mechanics of Continua and Wave Dynamics*, 342 pp., Spriner-Verlag, Berlin, 1985.

Cushman-Roisin, B., *Introduction to Geophysical Fluid Dynamics*, 320 pp., Prentice Hall, London, 1994.

Dziewonski, A. M. and Don L. Anderson, Preliminary reference Earth model, *Phys. Earth Planet. Inter.*, **25**, 297–356, 1981.

Fearn, D. R. and D. E. Loper, Compositional convection and stratification in Earth's core, *Nature*, **289**, 393–394, 1981.

Frank, S., Ascending droplets in the Earth's core, *Phys. Earth Planet. Inter.*, **27**, 249–254, 1982.

Garnero, E. J., D. V. Helmberger, and S. P. Grand, Constraining outermost core velocity with SmKS waves, *Geophys. Res. Lett.*, **20**, 2463–2466, 1993.

Gill, A. E., *Atmosphere-Ocean Dynamics*, 662 pp., Academic Press, London, 1982.

Gubbins, D., C. J. Thompson, and K. A. Whaler, Stable regions in the Earth's liquid core, *Geophys. J. R. Astron. Soc.*, **68**, 241–251, 1982.

Hide, R., Interaction between the earth's liquid core and solid mantle, *Nature*, **222**, 1055–1056, 1969.

Hulot, G., M. Le Huy, and J.-L. Le Mouël, Influence of core flows on the decade variations of the polar motion, *Geophys. Astrophys. Fluid Dyn.*, **82**, 35–67, 1996.

Kuang, W. and J. Bloxham, On the effect of boundary topography on flow in the Earth's core, *Geophys. Astrophys. Fluid Dyn.*, **72**, 162–195, 1993.

Lambeck, K., *The Earth's Variable Rotation*, 449 pp., Cambridge Univ. Press, London, 1980.

Landau, L. D. and E. M. Lifshitz, *Fluid Mechanics*, 2nd ed., 539 pp., Pergamon, Oxford, 1987.

Lay, T. and C. J. Young, The stably-stratified outermost core revisited, *Geophys. Res. Lett.*, **17**, 2001–2004, 1990.

Lister, J. R. and B. A. Buffett, How much of the Earth's core would be stably stratified if Nu < 1?, *Proceedings of the 4th SEDI Symposium on Earth's Deep Interior*, Whistler Mountain, British Columbia, Canada, 1994.

Loper, D. E. and T. Lay, The core-mantle boundary region, *J. Geophys. Res.*, **111**, B4, 6397–6420, 1995.

Moffatt, H. K., Topographic coupling at the core-mantle interface, *Geophys. Astrophys. Fluid Dyn.*, **9**, 279–288, 1978.

Pedlosky, J., *Geophysical Fluid Dynamics*, 710 pp., Springer-Verlag, Berlin, 1987.

Peyronneau, J. and J. P. Poirier, Electrical conductivity of the Earth's lower mantle, *Nature*, **342**, 537–539, 1989.

Roberts, P. H., On topographic core-mantle coupling, *Geophys. Astrophys. Fluid Dyn.*, **44**, 181–187, 1988.

Shearer, M. J. and P. H. Roberts, The hidden ocean at the top of Earth's core, *Dyn. Atmos. Oceans*, **27**, 631–647, 1997.

Souriau, A. and G. Poupinet, A study of the outermost liquid core using differential travel times of the SKS, SKKS and S3KS phases, *Phys. Earth Planet. Inter.*, **68**, 183–199, 1991.

Sylvander, M. and A. Souriau, Mapping S-velocity heterogeneous in the *D* region, from SmKS differential travel times, *Phys. Earth Planet. Inter.*, **94**, 1–21, 1996.

Whaler, K. A., Does the whole of the Earth's core convect?, *Nature*, **287**, 528–530, 1980.

Yokoyama, Y., Thirty year variation in the Earth rotation and the geomagnetic Gauss coefficients, *Geophys. Res. Lett.*, **20**, 2957–2960, 1993.

Yukutake, T., A stratified core motion inferred from geomagnetic secular variations, *Phys. Earth Planet. Inter.*, **24**, 253–258, 1981.

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