Bubble expansion rates in viscous compressible liquid

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A new equation has been derived for radial expansion of a bubble in viscous liquid, taking into account the compressibility of liquid. This model is important in understanding the time-evolution of the bubble growth in a cavitating flow. We solved the equation of motion of viscous liquid and obtained the velocity field and the pressure distribution in liquid. It is found that the expansion rate of a bubble in viscous compressible liquid is governed by the friction force due to viscosity as well as the surface tension, the inertial force, and the gas pressure. Further, the bubble expansion rate is brought by the same expression to that in an incompressible liquid only in the case that the liquid is in a hydrostatic state before the bubble grows.
Key words: Bubble, expansion, compressible, viscous, cavitation.

1. Introduction

Bubble dynamics is an important problem in many fields of science and technology. When liquid is heated or decompressed, bubbles are formed as a result of the phase change from liquid to gas or the exsolution of volatile components in liquid. In these cases, gas bubbles appear in liquid. Such two-phase flow is usually called cavitating flow: For example, the magma flow in conduit or the rapid flow over a submerged body are observed as the cavitating flow. The coexistence of gas bubbles with liquid significantly changes the feature of the liquid flow. It is well known that, for the same mass, matter occupies much larger volume in the gas phase than in the liquid phase. So suddenly increasing the volume ratio of the gas phase can accelerate the flow efficiently even if the mass ratio of the gas is small (Papale, 2001). In another case, the drag coefficient in cavitating flow generally depends on the volume ratio of the bubbles (Batchelor, 1967). To understand the behavior of these cavitating flows, it is essential to know how the bubble growth proceeds in liquid.

The growth of bubbles proceeds in two stages. In the early stage, it is important to solely grow to size as a result of the expansion or the diffusive flux of the volatile component from surrounding liquid, since the distance between bubbles is very long. On the other hand, since bubbles grow enough and the distance between bubbles shortens, the coalescence of a collision between bubbles becomes a prominent process of bubble growth. The transition from the early stage to the latter stage in the dynamics of bubbles significantly depends on the behavior of the bubbles, i.e., the growth rate, in the early stage. Therefore, in this work we focus on bubble expansion in the early stage. The radial motion of an expanding spherical bubble immersed in unbounded liquid was examined by many authors (e.g., Plesset and Prosperetti, 1977; Prosperetti and Lezzi, 1986). The expansion rate of a gas bubble is controlled by the balance between the gas pressure and the liquid pressure, the surface tension, and the viscous force at the bubble surface. The liquid pressure depends on the velocity field in the liquid surrounding a bubble. Therefore, in order to derive the bubble expansion rate, information on the liquid motion is required.

The analysis of a problem in cavitating flow was originally made by Rayleigh (1917). He solved the problem of the collapsing spherical cavity in an incompressible liquid. He made the same assumption that the liquid is incompressible and flow around the cavity is spherically symmetric with only a radial component. After this study, several studies made the assumptions in investigating the oscillations of a gas bubble as being the same as Rayleigh's model (Plesset, 1949; Lauterborn, 1976; Plesset and Prosperetti, 1977). However, the assumptions of incompressibility and spherically symmetric flow on the expanding bubble lead to a serious theoretical disadvantage. The spherically symmetric radial and incompressible liquid should bring the velocity field around a bubble to be proportional to r^{-2}, where r is the distance from the bubble center. On the other hand, such a velocity field brings the total moment integrated from the bubble surface to infinity per unit solid angle as being equal to infinite. Then, an additional infinite moment is needed to accelerate the bubble expansion rate. To say conversely, we deduce that bubbles cannot virtually expand in an incompressible liquid. This theoretical inconsistency principally originates from the assumption of incompressibility. To present an appropriate model of bubble expansion, the compressibility of liquid should be considered.

Several authors developed the radial motion of a gas bubble in an ideal liquid, taking into account the liquid compressibility and examined the rapid expansion of bubbles such as explosion (Trilling, 1952; Keller and Kolodner, 1956; Prosperetti *et al.*, 1988). In the field of volcanology, theoretical models of bubble expansion due to decompression were developed in several previous studies (Sparks, 1978; Toramaru, 1995; Proussevitch and Sahagian, 1998). However, these models assume that magmas are incompressible and use the analytical expression of the bubble expansion rate on the basis of Rayleigh's solution overlooking the theoretical inconsistency mentioned above. Moreover, magma is a highly viscous liquid. The high viscosity of liquid such as magma would have some effects on the bubble expansion. Some studies (Keller and Miksis, 1980; Prosperetti and Lezzi, 1986) derived the analytical expression of the radial expansion of bubble, taking into account the viscous term in the equation describing the dynamical balance at the bubble surface. However, they assumed that the Reynolds number is larger than unity and neglected the viscous terms in the equation of liquid motion. Therefore, it is unclear whether their models produce well the motion of bubble in a highly viscous liquid. Up to now, there have been no available studies which examine bubble expansion in a viscous compressible liquid. So, we should develop an analytical expression of the expansion rate of bubbles in this study.

The aim of this paper is to derive the analytical expression for the radial dynamics of a spherical bubble embedded in viscous liquid, taking into account the liquid compressibility. To do so, we will solve a set of equations of motion. This paper is organized as follows: In Section 2, we describe the basic equations and transform the equations of motion into a simple form. We derive an analytical model on the velocity and the pressure of viscous compressible liquid and provide an expression describing the radial motion of a gas bubble in Section 3. The conclusion of our study will be presented in Section 4.

2. Mathematical Formulation

In this section, we describe the basic equations and boundary conditions governing the velocity and the pressure in liquid spreading infinitely. As assumed by several studies (e.g., Keller and Miksis, 1980), the liquid is assumed to be isothermal. Further, we consider a single sphere of gas with the radius R in the liquid. Setting such a situation is approximately valid in the case that the distance between bubbles is much larger than bubble radius R. From this condition we naturally choose spherical coordinates and make the assumption that the velocity and the pressure in liquid are spherically symmetric. For the same reason, we assume that the velocity field around the bubble has only a radial component.

Taking the assumptions of the spherical symmetry and the radial motion, the motion of liquid is governed by the equation of continuity (Landau and Lifshitz, 1987)

$$\frac{\partial \rho}{\partial t} + \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \rho v\right) = 0 \tag{1}$$

and the equation of motion

$$\rho\left[\frac{\partial v}{\partial t} + v\frac{\partial v}{\partial r}\right] = -\frac{\partial P}{\partial r} + \left(\zeta + \frac{4}{3}\eta\right)\left(\Delta v - \frac{2v}{r^2}\right), \tag{2}$$

where $\rho(r,t)$, $P(r,t)$, and $v(r,t)$ are the density, the pressure, and the radial velocity, respectively, in the liquid. Since we consider compressible liquid, two different viscosities are presented. In Eq. (2), η and ζ are the viscosity and the second viscosity, respectively. To obtain a relation between the density and the pressure, we need the equation of state. It is assumed to be

$$P = P_\infty + c^2(\rho - \rho_\infty), \tag{3}$$

where c is the sound velocity, P_∞ is the hydrostatic pressure in the liquid, and ρ_∞ is the liquid density at infinity. Then, the sound velocity is assumed to be constant owing to that the liquid is isothermal.

The boundary conditions on the velocity and the pressure are imposed on the bubble's surface, i.e., the position r equals R, as

$$P_{\text{gas}} = P(R,t) + \frac{2\gamma}{R} - \frac{4\eta}{3}\left(\frac{\partial v(R,t)}{\partial r} - \frac{v(R,t)}{R}\right) - \zeta\left(\frac{\partial v(R,t)}{\partial r} + \frac{2v(R,t)}{R}\right) \tag{4}$$

and, at infinity,

$$\begin{cases} v = 0, \\ P = P_\infty, \end{cases} \tag{5}$$

where P_{gas} is the pressure within a bubble and γ is the surface tension of an interface between gas and liquid. Equation (4) represents the balance of forces acting on the interface between a gas bubble and liquid. Since the expansion rate of the bubble, $\dot{R}$, is equivalent to the liquid velocity at the bubble's surface, another boundary condition on the velocity of liquid at its interface is given by

$$\dot{R} = v(R,t). \tag{6}$$

As for initial distributions of the velocity and the pressure in the liquid, we specify them in this paper as arbitrary functions of radial coordinate r to express our result in a general form.

We assume that the change in the density $\rho(r,t)$ and the liquid velocity $v(r,t)$ are very small perturbations. This assumption is valid as long as the expansion rate of bubbles is much smaller than the sound velocity of the liquid. The density is divided into the two parts of the unperturbed term ρ_∞ and the perturbed term $\delta\rho(r,t)$ as

$$\rho(r,t) = \rho_\infty + \delta\rho(r,t). \tag{7}$$

Substituting Eq. (7) into Eqs. (1) and (2), equations for the perturbation are obtained as

$$\frac{\partial \delta\rho}{\partial t} + \frac{\rho_\infty}{r^2}\frac{\partial}{\partial r}\left(r^2 v\right) = 0 \tag{8}$$

and

$$\rho_\infty\frac{\partial v}{\partial t} = -\frac{\partial P}{\partial r} + \left(\zeta + \frac{4}{3}\eta\right)\left(\Delta v - \frac{2v}{r^2}\right), \tag{9}$$

respectively. In the derivation of these equations, we linearize them. Using Eq. (3) to represent pressure gradients as gradients of density and substituting Eq. (8) into the time derivative of Eq. (9), we obtain an equation of $v(r,t)$ as

$$\frac{\partial^2 v}{\partial t^2} = \left(c^2 + \nu\frac{\partial}{\partial t}\right)\frac{\partial}{\partial r}\left[\frac{1}{r^2}\frac{\partial}{\partial r}(r^2 v)\right], \tag{10}$$

where ν represents the extended kinematic viscosity defined by

$$\nu = \frac{1}{\rho_\infty}\left(\zeta + \frac{4}{3}\eta\right). \tag{11}$$

For later convenience, we introduce a new function $\chi(r, t)$ defined by

$$v(r, t) = -\frac{\partial}{\partial r}\left\{\frac{\chi(r, t)}{r}\right\}. \tag{12}$$

Substituting Eq. (12) into Eq. (10), it is rewritten as

$$\frac{\partial^2 \chi}{\partial t^2} = \left(c^2 + \nu\frac{\partial}{\partial t}\right)\frac{\partial^2 \chi}{\partial r^2}. \tag{13}$$

This is the basic equation of our problem. From the condition on the velocity at infinity in Eq. (5), $\chi(r, t)/r$ must tend to a constant value at infinity. Now we can subtract an arbitrary constant from $\chi(r, t)/r$ without any change in the velocity field defined in Eq. (13). So, for simplicity, we assume that $\chi(r, t)/r$ tends to zero at infinity. In other words, $\chi(r, t)$ is assumed to tend to a constant value at the infinite position of r.

In the case of an ideal liquid, the above equation becomes the wave equation. From this limiting behavior, it is clear that the basic equation represents the behavior of liquid as a compressible one. The boundary condition imposed on the inner boundary is expressed by a function of time as

$$\chi(R, t) = \chi_0(t). \tag{14}$$

The function $\chi_0(t)$ is determined by the inner boundary condition of the liquid velocity, that is, Eq. (6). It will be given by Eq. (34). For the initial condition we impose two conditions on the function $\chi(r, t)$. These conditions correspond to the conditions of the velocity and the pressure. As mentioned previously, we express them in a general form as

$$\chi(r, 0) = \phi(r) \tag{15}$$

and

$$\frac{\partial \chi(r, 0)}{\partial t} = \psi(r), \tag{16}$$

where $\phi(r)$ and $\psi(r)$ are functions of r. If the liquid is initially in the hydrostatic state, the functions $\phi(r)$ and $\psi(r)$ should be zero. Moreover, substituting Eq. (12) into Eq. (9) and integrating it, we obtain

$$P = P_\infty + \frac{\rho_\infty}{r}\frac{\partial \chi}{\partial t} - \left(\zeta + \frac{4}{3}\eta\right)\frac{1}{r}\frac{\partial^2 \chi}{\partial r^2}. \tag{17}$$

Note that the above equation corresponds to Bernoulli's equation.

3. Analytical Result

3.1 Solution

Since bubbles expand or shrink, the bubble radius changes with time. This implies that the location of the inner boundary depends on time. Then, it is difficult to solve Eq. (13) analytically with no assumptions or simplifications. So, we need further approximations.

Because the expansion speed of a bubble surface is usually much smaller than the sound velocity, the bubble radius hardly changes during the transmission time of sound waves. This means that the velocity and the pressure in liquid are immediately adjusted to increasing bubble radius. For this restricted case, we can neglect the time-dependence of the bubble radius in deriving an analytical solution to Eq. (13). However, we take into account the time-dependence of the bubble radius in the derivation of the bubble expansion rate.

We analytically solve Eq. (13) under the conditions described by Eqs. (14)–(16). Since Eq. (13) is a linear equation, we divide $\chi(r, t)$ into two functions, $\chi_1(r, t)$ and $\chi_2(r, t)$, namely $\chi(r, t) = \chi_1(r, t) + \chi_2(r, t)$. Each function is a solution of Eq. (13). The inner boundary conditions on $\chi_1(r, t)$ and $\chi_2(r, t)$ are set to be

$$\chi_1(R, t) = 0 \tag{18}$$

and

$$\chi_2(R, t) = \chi_0(t), \tag{19}$$

respectively. At the outer boundary, $\chi_1(r, t)$ and $\chi_2(r, t)$ are required to tend to constant at an infinite position of r. The initial conditions on $\chi_1(r, t)$ and $\chi_2(r, t)$ are set to be

$$\begin{cases} \chi_1(r, 0) &= \phi(r), \\ \dfrac{\partial \chi_1(r, 0)}{\partial t} &= \psi(r) - 2c\chi_0(0)\delta(r - R), \end{cases} \tag{20}$$

and

$$\begin{cases} \chi_2(r, 0) &= 0, \\ \dfrac{\partial \chi_2(r, 0)}{\partial t} &= 2c\chi_0(0)\delta(r - R), \end{cases} \tag{21}$$

respectively.

First we solve Eq. (13) to obtain $\chi_1(r, t)$. After the usual procedure of separation of variables r, t, and owing to the outer boundary condition, we obtain the function $\chi_1(r, t)$ written in terms of a complete set of normal modes as

$$\begin{aligned}\chi_1(r, t) = &\int_0^\infty \mathrm{d}k\,\{\sin[k(r - R)] + A(k)\cos[k(r - R)]\} \\ &\times \left[B(k)\exp(\omega_+ t) + C(k)\exp(\omega_- t)\right],\end{aligned} \tag{22}$$

where $A(k)$, $B(k)$, and $C(k)$ are the coefficients. The frequencies $\omega_+(k)$ and $\omega_-(k)$ are given by

$$\omega_\pm(k) = -\frac{\nu k^2}{2} \pm ick\sqrt{1 - \frac{\nu^2 k^2}{4c^2}}, \tag{23}$$

where the double-signs on both sides correspond to each other. The coefficients in Eq. (22) should be determined by the boundary and initial conditions. The coefficient $A(k)$ is required to be zero from the inner boundary condition, $\chi_1(R, t) = 0$. Using the orthogonality of the sine function, we obtain $B(k)$ and $C(k)$ from the initial conditions as

$$B(k) = -\frac{2}{\pi}\int_R^\infty \frac{\{\omega_-(k)\phi(\xi) - \psi(\xi)\}\sin[k(\xi - R)]}{\omega_+(k) - \omega_-(k)}\mathrm{d}\xi \tag{24}$$

and

$$C(k) = \frac{2}{\pi}\int_R^\infty \frac{\{\omega_+(k)\phi(\xi) - \psi(\xi)\}\sin[k(\xi - R)]}{\omega_+(k) - \omega_-(k)}\mathrm{d}\xi, \tag{25}$$

respectively. Note that $B(k)$ and $C(k)$ depend on k and R. Hence, $\chi_1(r,t)$ is given by

$$\chi_1(r,t) = \int_0^\infty \mathrm{d}k \sin[k(r-R)]\{B(k)\exp(\omega_+ t) + C(k)\exp(\omega_- t)\}. \tag{26}$$

Secondly, let us consider about $\chi_2(r,t)$. To obtain $\chi_2(r,t)$, we use Duhamel's principle (e.g., John, 1982). We introduce another function $U(r-R, t-\tau)$, which is the solution of Eq. (13) and is satisfied with the following inner boundary and initial conditions:

$$U(0, t-\tau) = 1, \tag{27}$$

$$U(r-R, 0) = 0, \tag{28}$$

and

$$\frac{\partial U(r-R,0)}{\partial t} = 2c\delta(r-R). \tag{29}$$

In addition, we impose a condition that it does not become infinite at $r=\infty$ on $U(r-R,t-\tau)$. Using $U(r-R,t-\tau)$, $\chi_2(r,t)$ is expressed as

$$\chi_2(r,t) = \int_0^t \mathrm{d}\tau \frac{\partial U(r-R,t-\tau)}{\partial t}\chi_0(\tau). \tag{30}$$

Therefore, it is sufficient to obtain $U(r-R,t-\tau)$ in spite of $\chi_2(r,t)$ for our problem. Through the same manner taken in the derivation of $\chi_1(r,t)$, we can suppose the following expression as the solution of $U(r-R,t-\tau)$:

$$\begin{aligned} U(r-R,t-\tau) &= \frac{2c}{\pi}\int_0^\infty \mathrm{d}k \cos[k(r-R)] \\ &\times \left\{\frac{\exp[\omega_+(t-\tau)] - \exp[\omega_-(t-\tau)]}{\omega_+ - \omega_-}\right\}. \end{aligned} \tag{31}$$

This function is satisfied with the boundary and initial conditions (see details in the Appendix). Substituting Eq. (31) into Eq. (30), $\chi_2(r,t)$ is obtained as

$$\begin{aligned} \chi_2(r,t) &= \frac{2c}{\pi}\int_0^t \mathrm{d}\tau\,\chi_0(\tau)\int_0^\infty \mathrm{d}k\cos[k(r-R)] \\ &\times \left\{\frac{\omega_+\exp[\omega_+(t-\tau)] - \omega_-\exp[\omega_-(t-\tau)]}{\omega_+ - \omega_-}\right\}. \end{aligned} \tag{32}$$

From Eqs. (26) and (32), the solution to Eq. (13) is given by

$$\begin{aligned} \chi(r,t) &= \int_0^\infty \mathrm{d}k\sin[k(r-R)]\{B(k)\exp(\omega_+ t) + C(k)\exp(\omega_- t)\} \\ &+ \frac{2c}{\pi}\int_0^t \mathrm{d}\tau\,\chi_0(\tau)\int_0^\infty \mathrm{d}k\cos[k(r-R)] \\ &\times \left\{\frac{\omega_+\exp[\omega_+(t-\tau)] - \omega_-\exp[\omega_-(t-\tau)]}{\omega_+ - \omega_-}\right\}. \end{aligned} \tag{33}$$

In the case of an ideal liquid (i.e., $\nu = 0$), Eq. (33) becomes the solution of a sound wave propagating in liquid. We obtain the velocity field in liquid through Eq. (12) and the pressure distribution through Eq. (17).

Finally, we consider the expression of $\chi_0(t)$. The liquid velocity at the bubble surface is equivalent to the bubble expansion rate. Thus, substituting Eq. (33) into Eq. (12) and setting $r=R$, $\chi_0(t)$ is given by

$$\chi_0(t) = R^2\dot{R}(t) + R\int_0^\infty \mathrm{d}k\,\{B(k)\exp(\omega_+ t) + C(k)\exp(\omega_- t)\}\,k. \tag{34}$$

3.2 Equation of radial motion of a gas bubble

The pressure difference between the inside and outside of a bubble leads to the expansion or shrink of the bubble. Using the velocity distribution obtained in the previous subsection, we derive the radial motion of a gas bubble in viscous compressible liquid.

As seen from Eq. (4), the gas pressure balances with sum of forces acting on a boundary sphere such as the liquid pressure, the viscosity, and the surface tension. Substituting Eq. (17) into Eq. (4), Eq. (4) is rewritten as

$$P_{\mathrm{gas}} = P_\infty + \frac{2\gamma}{R} + 4\eta\frac{\dot{R}}{R} + \frac{\rho_\infty}{R}\frac{\partial\chi(R,t)}{\partial t}. \tag{35}$$

We call $\partial\chi(R,t)/\partial t$ an inertial term and rewrite it, using the analytical solution of $\chi(r,t)$. Noting that the bubble radius is a function of time and differentiating Eq. (33) with respect to time t, we obtain

$$\begin{aligned} \frac{\partial\chi(R,t)}{\partial t} &= -\dot{R}\int_0^\infty \mathrm{d}k\,\{B(k)\exp(\omega_+ t) + C(k)\exp(\omega_- t)\}\,k \\ &+ \frac{2c}{\pi}\int_0^\infty \mathrm{d}k\,\chi_0(t) \\ &+ \frac{2c}{\pi}\int_0^t \mathrm{d}\tau\,\chi_0(\tau)\int_0^\infty \mathrm{d}k \\ &\times \left\{\frac{\omega_+^2\exp[\omega_+(t-\tau)] - \omega_-^2\exp[\omega_-(t-\tau)]}{\omega_+(k) - \omega_-(k)}\right\}. \end{aligned} \tag{36}$$

On the other hand, substituting Eq. (33) into Eq. (6), the expansion rate is given by

$$\begin{aligned} \dot{R} &= -\frac{1}{R}\int_0^\infty \mathrm{d}k\,\{B(k)\exp(\omega_+ t) + C(k)\exp(\omega_- t)\}\,k \\ &+ \frac{2c}{\pi R^2}\int_0^t \mathrm{d}\tau\,\chi_0(\tau)\int_0^\infty \mathrm{d}k \\ &\times \left\{\frac{\omega_+\exp[\omega_+(t-\tau)] - \omega_-\exp[\omega_-(t-\tau)]}{\omega_+(k) - \omega_-(k)}\right\}. \end{aligned} \tag{37}$$

Comparing Eqs. (36) and (37), we obtain the following relationship:

$$\frac{\partial\chi(R,t)}{\partial t} = \frac{\mathrm{d}}{\mathrm{d}t}\left(R^2\dot{R}\right)$$

$$+ R\dot{R} \int_0^\infty \mathrm{d}k \left\{ \frac{\partial B(k)}{\partial R} \exp(\omega_+ t) + \frac{\partial C(k)}{\partial R} \exp(\omega_- t) \right\} k$$
$$+ R \int_0^\infty \mathrm{d}k \left\{ \omega_+ B(k) \exp(\omega_+ t) + \omega_- C(k) \exp(\omega_- t) \right\} k. \tag{38}$$

Substituting Eq. (38) into Eq. (35), we finally have

$$\begin{aligned} P_{\text{gas}} = P_\infty &+ \frac{2\gamma}{R} + 4\eta \frac{\dot{R}}{R} + \rho_\infty \left(2\dot{R}^2 + R\ddot{R} \right) \\ &+ \rho_\infty \dot{R} \int_0^\infty \mathrm{d}k \left\{ \frac{\partial B(k)}{\partial R} \exp(\omega_+ t) \right. \\ &\left. + \frac{\partial C(k)}{\partial R} \exp(\omega_- t) \right\} k \\ &+ \rho_\infty \int_0^\infty \mathrm{d}k \left\{ \omega_+ B(k) \exp(\omega_+ t) \right. \\ &\left. + \omega_- C(k) \exp(\omega_- t) \right\} k. \end{aligned} \tag{39}$$

The same expression as Eq. (39) has been obtained except a coefficient of 2 before $\dot{R}^2$ by assuming an incompressible liquid if we set $B(k) = C(k) = 0$ (e.g., Scriven, 1959). The inconsistency of the coefficient before $\dot{R}^2$ arises because the advective term in Eq. (17) was neglected in this work.

Let us observe an example of time-evolutions of the velocity field surrounding the oscillating bubble. The radial motion of a bubble is strongly related to the internal pressure in a bubble. We assume in this paper that the bubble oscillates with a certain frequency ω_0 for simplicity. That is, the bubble radius is given by

$$R(t) = R_0 - \delta R \cos(\omega_0 t), \tag{40}$$

where R_0 is the bubble radius at the equilibrium state and δR represents a small amplitude for the change in the bubble radius. Such an oscillation as Eq. (40) cannot be maintained in a viscous liquid without external forces and the amplitude then decreases with time owing to the viscous dissipation. However, Eq. (40) would be useful in understanding the motion of liquid around the oscillating bubble. Therefore, we adopt Eq. (40) for the time-evolution of the bubble radius. Moreover, the liquid is assumed to be initially in the hydrostatic state.

We transform the length, the wave number, and time into non-dimensional forms defined by

$$\tilde{r} = \frac{r}{R_0}, \tag{41}$$

$$\tilde{k} = kR_0, \tag{42}$$

and

$$\tilde{t} = \frac{ct}{R_0}, \tag{43}$$

respectively. Then, $\chi(r, t)$ and $R(t)$ are rewritten by

$$\begin{aligned} &\tilde{\chi}(\tilde{r}, \tilde{t}) \\ &= \frac{2}{\pi} \int_0^{\tilde{t}} \mathrm{d}\tilde{\tau}\, \tilde{\chi}_0(\tilde{\tau}) \int_0^\infty \mathrm{d}\tilde{k} \cos[\tilde{k}(\tilde{r} - 1)] \\ &\times \left\{ \frac{\tilde{\omega}_+ \exp\left[\dfrac{\tilde{\omega}_+(\tilde{t} - \tilde{\tau})}{R\mathrm{e}}\right] - \tilde{\omega}_- \exp\left[\dfrac{\tilde{\omega}_-(\tilde{t} - \tilde{\tau})}{R\mathrm{e}}\right]}{\tilde{\omega}_+ - \tilde{\omega}_-} \right\} \end{aligned} \tag{44}$$

and

$$\tilde{R}(\tilde{t}) = 1 - \frac{M}{Q} \cos(Q\tilde{t}), \tag{45}$$

respectively. In the above equations, $R\mathrm{e}$, M, and Q are the Reynolds number, the Mach number, and the ratio of the frequency ω_0 to c/R_0 defined by

$$R\mathrm{e} = \frac{R_0 c}{\nu}, \tag{46}$$

$$M = \frac{\delta R \omega_0}{c}, \tag{47}$$

and

$$Q = \frac{R_0 \omega_0}{c}, \tag{48}$$

respectively. Further, in Eq. (44), $\tilde{\omega}_\pm$ and $\tilde{\chi}$ are normalized by ν/R_0^2 and $R_0^2 c$, respectively. Note that the velocity $\delta R \omega_0/\sqrt{2}$ corresponds to the root-mean-square of the bubble oscillating rate. Using Eqs. (12) and (44), we can obtain the velocity normalized by the sound speed.

The velocity fields surrounding an oscillating bubble are shown in Fig. 1 for $R\mathrm{e} = 0.25$, $M = 1 \times 10^{-6}$, and $Q = 1 \times 10^{-5}$. Panels (a) and (b) correspond to the radial distribution of the velocity at $\tilde{t} = 50$ and 1×10^2, respectively. The solid line represents the velocity fields in incompressible liquid, while the dotted line represents those in compressible liquid. Since the bubble expands in both cases, the velocity at the surface of bubble increases with time. Moreover, the ratios of the velocity in the compressible case to that in the incompressible case are shown in Figs. 2(a) and (b). The values of the parameters are the same

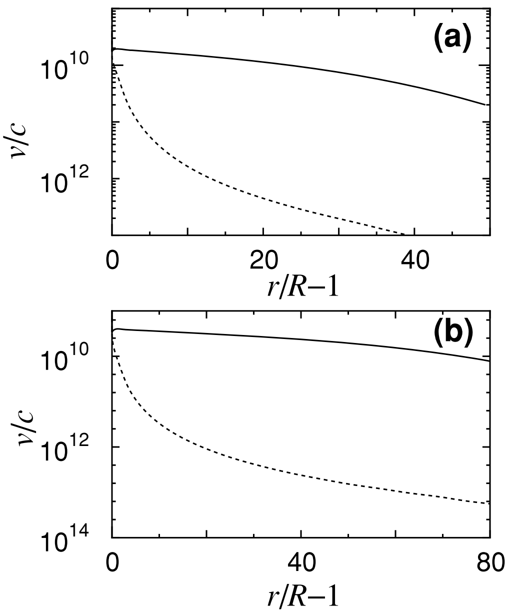

Fig. 1. Velocity fields in liquid at $\tilde{t} = 50$ (a) and 1×10^2 (b) for $R\mathrm{e} = 0.25$, $M = 1 \times 10^{-6}$, and $Q = 1 \times 10^{-5}$. The values of parameters are as follows: $c = 1 \times 10^3$ cm/s, $R_0 = 1 \times 10^{-4}$ cm, $\delta R = 1 \times 10^{-5}$ cm, and $\omega_0 = 100$ s^{-1}. The solid and dotted lines correspond to the radial distribution of the velocity in the cases of incompressible and compressible liquid, respectively. The distributions in the both cases are in harmony at the bubble surface. This coincidence is guaranteed by the boundary condition, i.e., Eq. (6).

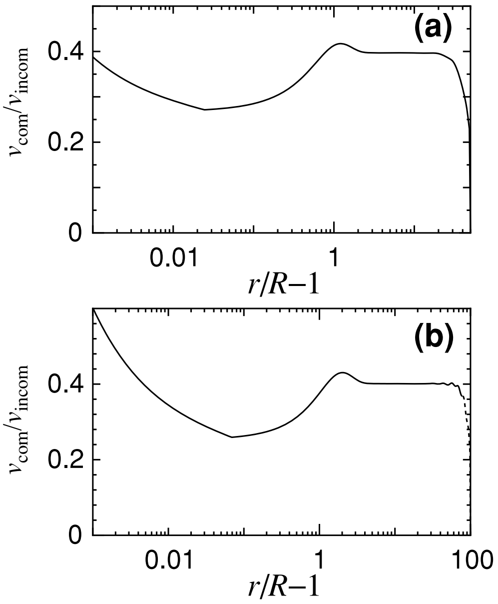

Fig. 2. The ratio of velocity field in the compressible case v_{com} to that in the incompressible case v_{incom} at $\tilde{t} = 50$ (a) and 1×10^2 (b) for $R_e = 0.25$, $M = 1 \times 10^{-6}$, and $Q = 1 \times 10^{-5}$. The values of the parameters are the same as in Fig. 1. In panel (b), the dotted line represents the numerical result with some uncertainties. Note that the x-axis is the log-plot.

as in Fig. 1. As seen in Fig. 2, the velocity of compressible liquid decreases more rapidly than that in incompressible liquid and oscillates. This oscillation implies that the outgoing wave takes place owing to the liquid compressibility. It is noted that, at a far region, it is difficult to perform the numerical calculation of the integration with respect to k in Eq. (44). In Fig. 2, the dotted line corresponds to the numerical result with large error.

4. Conclusion

We have presented a new model for the expansion rate of a single bubble within viscous liquid. Our result gives the general solution to this problem. Thus, the analytical expressions can be applied to some cases of cavitating flow. These expressions show the damping sound wave. The analytical expansion rate of a bubble shows that the bubble expansion rate in a viscous liquid is the same as that in an incompressible liquid, only in the case that the liquid is initially in the hydrostatic state. If the liquid has a certain distribution of velocity and pressure at the beginning, an additional effect of the viscous force on the expansion rate should appear. This force would significantly prevent the radial motion of a bubble in a highly viscous liquid.

In this study, we assumed that a single bubble is immersed in an unbounded liquid. This assumption is valid as long as the distance between bubbles is much larger than the typical size of bubbles. However, in real systems, a large number of bubbles are formed in liquid due to the bubble nucleation and they grow with time. So interactions between bubbles should be needed to be considered. Such cases are observed in cavitating flow at the stage when the growth of bubbles considerably proceeds. In these cases, the assumption that the outer boundary is set to be at infinity is not valid because some bubbles in the vicinity of a bubble exist. We must set the outer boundaries at the finite position of r to obtain the bubble expansion rate. It is then expected that the expansion rate of a bubble would diverge from the result presented in this paper. Such an effect is important for the time-evolution of the bubble growth in cavitating flow at the latter stage. The radial motion of a bubble in such a case should be a topic for future investigation.

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Appendix. Validity of the expression of Eq. (31)

We first derive Eq. (31). Through the same manner taken in the derivation of $\chi_1(r, t)$, we can suppose the following integration as the solution of $U(r - R, t - \tau)$:

$$\begin{aligned} U(r-R, t-\tau) &= \frac{2c}{\pi}\int_0^\infty \mathrm{d}k\left\{\cos[k(r-R)] + A'(k)\sin[k(r-R)]\right\} \\ &\times \left\{\frac{\exp[\omega_+(t-\tau)] - \exp[\omega_-(t-\tau)]}{\omega_+(k) - \omega_-(k)}\right\}. \end{aligned} \tag{A.1}$$

As seen later, note that considering the above form as the solution of $U(r - R, t - \tau)$ is valid from the uniqueness of the solution to Eq. (13).

As easily seen from Eq. (A.1), the function $U(r - R, t - \tau)$ becomes zero at $t = \tau$. First we derive the coefficient $A'(k)$ in Eq. (A.1) from Eq. (29). Differentiating Eq. (A.1) with respect to t and putting to be $t = \tau$, we have

$$\frac{1}{\pi}\int_0^\infty \mathrm{d}k\left\{\cos[k(r-R)] + A'(k)\sin[k(r-R)]\right\} = \delta(r-R). \tag{A.2}$$

Multiplying $\sin[l(r - R)]$ by Eq. (A.2) and integrating it with respect to r from R to infinity, the coefficient $A'(k)$ is expressed as

$$\begin{aligned} \pi A'(l) = &-\int_0^\infty \mathrm{d}k\int_0^\infty \mathrm{d}\xi\,\sin[(k+l)\xi] \\ &+\int_0^\infty \mathrm{d}k\int_0^\infty \mathrm{d}\xi\,\sin[(k-l)\xi], \end{aligned} \tag{A.3}$$

where ξ is a dummy variable corresponding to $r - R$.

The first term on the right-hand side of Eq. (A.3) is rewritten as

$$\begin{aligned} &\int_0^\infty \mathrm{d}k\int_0^\infty \mathrm{d}\xi\,\sin[(k+l)\xi] \\ &= \lim_{K\to\infty}\int_0^K \mathrm{d}k \lim_{L\to\infty}\int_0^L \mathrm{d}\xi\,\sin[(k+l)\xi] \\ &= \lim_{K\to\infty}[\ln(k+l)]_0^K - \int_0^\infty \mathrm{d}k \lim_{L\to\infty}\frac{\cos[(k+l)L]}{k+l}. \end{aligned} \tag{A.4}$$

The second term on the right-hand side of Eq. (A.4) is

$$\left|\lim_{L\to\infty}\int_0^\infty \mathrm{d}k\frac{\cos[(k+l)L]}{k+l}\right|$$

$$= \left| \lim_{L\to\infty} \left[\frac{\sin(yL)}{yL} \right]_l^\infty + \lim_{L\to\infty} \frac{1}{L} \int_l^\infty \mathrm{d}y \frac{\sin(yL)}{y^2} \right|$$
$$\leq \left| \lim_{L\to\infty} \frac{\sin(lL)}{lL} \right| + \left| \lim_{L\to\infty} \frac{1}{L} \int_l^\infty \mathrm{d}y \frac{\sin(yL)}{y^2} \right|$$
$$\leq \lim_{L\to\infty} \left| \frac{1}{lL} \right| + \lim_{L\to\infty} \left| \frac{1}{L} \int_l^\infty \frac{\mathrm{d}y}{y^2} \right| \to 0. \quad (A.5)$$

Hence we obtain

$$\int_0^\infty \mathrm{d}k \int_0^\infty \mathrm{d}\xi \sin[(k+l)\xi] = \lim_{K\to\infty} [\ln(k+l)]_0^K . \quad (A.6)$$

We calculate the second term on the right-hand side of Eq. (A.3). To do so, we divide the integration as follows:

$$\int_0^\infty \mathrm{d}k \int_0^\infty \mathrm{d}\xi \sin[(k-l)\xi] = \left[\int_0^{l-\epsilon} + \int_{l-\epsilon}^{l+\epsilon} + \int_{l+\epsilon}^\infty \right] \mathrm{d}k \int_0^\infty \mathrm{d}\xi \sin[(k-l)\xi], \quad (A.7)$$

where ϵ is a smaller arbitrary positive parameter than l. Then, each term on the right-hand side of Eq. (A.7) is given by

$$\int_0^{l-\epsilon} \mathrm{d}k \int_0^\infty \mathrm{d}\xi \sin[(k-l)\xi] = \ln(\epsilon) - \ln(l) - \int_0^{l-\epsilon} \mathrm{d}k \lim_{L\to\infty} \frac{\cos[(k-l)L]}{k-l} = \ln\left(\frac{\epsilon}{l}\right), \quad (A.8)$$

$$\int_{l-\epsilon}^{l+\epsilon} \mathrm{d}k \int_0^\infty \mathrm{d}\xi \sin[(k-l)\xi] = \int_0^\infty \mathrm{d}\xi \int_{l-\epsilon}^{l+\epsilon} \mathrm{d}k \sin[(k-l)\xi] = 0, \quad (A.9)$$

and

$$\int_{l+\epsilon}^\infty \mathrm{d}k \int_0^\infty \mathrm{d}\xi \sin[(k-l)\xi] = \lim_{K\to\infty} [\ln(k-l)]_{l+\epsilon}^K - \int_{l+\epsilon}^\infty \mathrm{d}k \lim_{L\to\infty} \frac{\cos[(k-l)L]}{k-l} = \lim_{K\to\infty} [\ln(k-l)]_{l+\epsilon}^K . \quad (A.10)$$

Substituting Eqs. (A.4)–(A.10) into Eq. (A.3), we obtain

$$\pi A'(l) = -\lim_{K\to\infty} \ln\left(\frac{K+l}{K-l}\right) = 0. \quad (A.11)$$

At the position of r equal to R, Eq. (31) is written by

$$U(0, t-\tau) = \frac{2c}{\pi} \int_0^\infty \mathrm{d}k \frac{\exp[\omega_+(t-\tau)] - \exp[\omega_-(t-\tau)]}{\omega_+(k) - \omega_-(k)}. \quad (A.12)$$

It is found from Eq. (23) that the frequencies $\omega_+(k)$ and $\omega_-(k)$ can be replaced with $\omega_-(-k)$ and $\omega_+(-k)$, respectively. Using the relationship between $\omega_+(k)$ and $\omega_-(k)$, Eq. (A.12) is transformed to

$$U(0, t-\tau) = \frac{2c}{\pi} \int_{-\infty}^\infty \mathrm{d}k \frac{\exp[\omega_+(t-\tau)]}{\omega_+(k) - \omega_-(k)}. \quad (A.13)$$

We regard the wave number k as the complex variable and evaluate the integration of Eq. (A.13) in the complex plane. Equation (A.13) has singularities at $k = 0,\ 2c/\nu$, and $-2c/\nu$. We consider the integral path, C_0, shown in Fig. A.1. Integrating Eq. (A.13) along the integral path, we obtain from Cauchy's integral theorem

$$\oint_{C_0} \mathrm{d}k \frac{\exp[\omega_+(t-\tau)]}{\omega_+(k) - \omega_-(k)} = 0. \quad (A.14)$$

On the other hand, the residues at three singular points are given by

$$\lim_{\epsilon\to 0} \int_{|k|=\epsilon} \mathrm{d}k \frac{\exp[\omega_+(t-\tau)]}{\omega_+(k) - \omega_-(k)} = \frac{\pi}{2c} \quad (A.15)$$

and

$$\lim_{\epsilon\to 0} \int_{|k\pm 2c/\nu|=\epsilon} \mathrm{d}k \frac{\exp[\omega_+(t-\tau)]}{\omega_+(k) - \omega_-(k)} = 0, \quad (A.16)$$

respectively. Note that the above integrations were carried out on the semicircle in the upper plane. Moreover, we evaluate the value of the integration on the large semicircle C_1, which is a part of the integral path C_0. On the contour C_1, the variable k is given by $Le^{i\theta}$, where L is the radius and θ is the angle measured from the positive real axis. Assuming that L is large as compared with $2c/\nu$, the frequencies $\omega_\pm(k)$ are approximately given by

$$\omega_+ \simeq -\frac{c^2}{\nu} \quad (A.17)$$

and

$$\omega_- \simeq -\nu L^2, \quad (A.18)$$

respectively. Using Eqs. (A.17) and (A.18), the integration on the contour C_1 is estimated as

$$\lim_{L\to\infty} \left| \int_{C_1} \mathrm{d}k \frac{\exp[\omega_+(t-\tau)]}{\omega_+(k) - \omega_-(k)} \right| \leq \lim_{L\to\infty} \int_{C_1} |\mathrm{d}k| \left| \frac{\exp[\omega_+(t-\tau)]}{\omega_+(k) - \omega_-(k)} \right| = \lim_{L\to\infty} \frac{\exp\left[-\frac{c^2(t-\tau)}{\nu}\right]}{\nu} \frac{\pi L}{L^2 - (c/\nu)^2} \to 0. \quad (A.19)$$

Substituting Eqs. (A.15), (A.16) and (A.19) into Eq. (A.14), we obtain

$$\int_{-\infty}^\infty \mathrm{d}k \frac{\exp[\omega_+(t-\tau)]}{\omega_+(k) - \omega_-(k)} = \frac{\pi}{2c}. \quad (A.20)$$

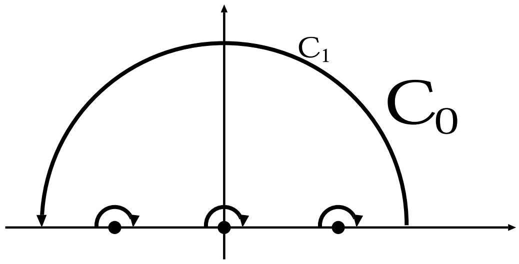

Fig. A.1. The integral path C_0 is shown. The filled circles represent the singular points ($k = 0, \pm 2c/\nu$). The contour C_1 denotes the integral path shown by the largest semicircle in the upper half-plane.

This result indicates that $U(0, t-\tau)$ becomes unity. Hence, it was proved that Eq. (31) is satisfied with the boundary and initial conditions given by Eqs. (27)–(29).

We secondly consider the uniqueness of solution to Eq. (13). To prove the uniqueness, we assume that there are two solutions to Eq. (13) with the same boundary and initial conditions. Here two solutions are denoted by $\chi^{(1)}(r,t)$ and $\chi^{(2)}(r,t)$. Now we introduce the function $X(r,t)$ defined by

$$X(r,t) = \chi^{(1)}(r,t) - \chi^{(2)}(r,t). \tag{A.21}$$

Note that the function $X(r,t)$ is also a solution to Eq. (13). Then, the boundary conditions at the position of r equal to R and ∞ imposed on $X(r,t)$ are

$$X(R,t) = 0 \tag{A.22}$$

and

$$X(\infty,t) = 0, \tag{A.23}$$

respectively. The initial conditions are also given by

$$X(r,0) = 0 \tag{A.24}$$

and

$$\frac{\partial X(r,0)}{\partial t} = 0. \tag{A.25}$$

We introduce two positive functions of time t defined as

$$F(t) = \int_R^\infty \left[c^2 \left(\frac{\partial X}{\partial r} \right)^2 + \left(\frac{\partial X}{\partial t} \right)^2 \right] \mathrm{d}r \tag{A.26}$$

and

$$G(t) = 2\nu \int_R^\infty \left(\frac{\partial^2 X}{\partial r \partial t} \right)^2 \mathrm{d}r. \tag{A.27}$$

We assume that the integral of $F(t)$ and $G(t)$ is integrable. Differentiating $F(t)$ with respect to t and performing the partial integral, we obtain

$$\frac{\mathrm{d}F(t)}{\mathrm{d}t} = 2\nu \int_R^\infty \frac{\partial X}{\partial t} \frac{\partial^3 X}{\partial t \partial r^2} \mathrm{d}r, \tag{A.28}$$

where we used Eq. (13) and the boundary and initial conditions imposed on $X(r,t)$. On the other hand, performing the partial integral for the function $G(t)$, it is rewritten as

$$G(t) = -2\nu \int_R^\infty \frac{\partial X}{\partial t} \frac{\partial^3 X}{\partial t \partial r^2} \mathrm{d}r. \tag{A.29}$$

From Eqs. (A.28) and (A.29), the time derivative of the function $F(t)$ is expressed in terms of $G(t)$ as

$$\frac{\mathrm{d}F(t)}{\mathrm{d}t} = -G(t). \tag{A.30}$$

It is shown from Eq. (A.30) that the function $F(t)$ can never increase because the function $G(t)$ is positive. Moreover, since $F(t)$ is zero at $t = 0$, it cannot exceed zero. However, the function $F(t)$ is a positive function. Thus, we conclude that the function $F(t)$ is equivalent to zero. Since each integrand in Eq. (A.26) is positive as well, we have from the condition that $F(t) = 0$

$$\frac{\partial X}{\partial t} = \frac{\partial X}{\partial r} = 0. \tag{A.31}$$

Integrating Eq. (A.31) and using the initial and boundary conditions, we consequently obtain

$$X(r,t) = 0. \tag{A.32}$$

This means that $\chi^{(1)}(r,t)$ is consistent with $\chi^{(2)}(r,t)$. It was proved that Eq. (13) has only one solution.

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