Earth Planets Space, **58**, 287–303, 2006

The generalized Bouguer anomaly

Kyozo Nozaki

Tsukuba Technical Research and Development Center, OYO Corporation, 43 Miyukigaoka, Tsukuba, Ibaraki 305-0841, Japan

(Received November 11, 2004; Revised June 28, 2005; Accepted September 20, 2005; Online published March 10, 2006)

This paper states on the new concept of the generalized Bouguer anomaly (GBA) that is defined upon the datum level of an arbitrary elevation. Discussions are particularly focused on how to realize the Bouguer anomaly that is free from the assumption of the Bouguer reduction density ρ_B, namely, the ρ_B*-free Bouguer anomaly*, and on what is meant by the ρ_B*-free Bouguer anomaly* in relation to the fundamental equation of physical geodesy. By introducing a new concept of the specific datum level so that GBA is not affected by the topographic masses, we show the equations of GBA upon the specific datum levels become free from ρ_B and/or the terrain correction. Subsequently utilizing these equations, we derive an approximate equation for estimating ρ_B. Finally, we show how to compute a Bouguer anomaly on the geoid by transforming the datum level of GBA from the specific datum level to the level of the geoid. These procedures yield a new method for obtaining the Bouguer anomaly in the classical sense (say, the Bouguer disturbance), which is free from the assumption of ρ_B. We remark that GBA upon the ρ_B*-free datum level* is the gravity disturbance and that the equation of it has a tie to the fundamental equation of physical geodesy.

Key words: Generalized Bouguer anomaly, Poincaré-Prey reduction, specific datum level of gravity reduction, free-air anomaly, Bouguer reduction density.

1. Introduction

In this paper, we present a new concept of the generalized Bouguer anomaly, which is defined upon the datum level of gravity reduction of an arbitrary elevation. The classical Bouguer anomaly has been defined upon the geoid by the difference between the observed gravity reduced to the geoid and the reference gravity upon the geoid. The reference gravity has been equated to the standard gravity (e.g. Heiland, 1946). No distinction was made between the reference gravity and the standard gravity. In the theory of *modern* physical geodesy, the normal gravity upon the reference ellipsoid was introduced as the reference gravity (Heiskanen and Moritz, 1967, p. 44). However, it is well known that the surface of the reference ellipsoid is different from that of the geoid. Therefore, the reference gravity upon the geoid is represented in terms of the normal gravity and the geoid height. Hackney and Featherstone (2003) recently discussed geodetic and geophysical 'gravity anomalies'.

In order to perform the formulation, we attempted to generalize the Bouguer anomaly upon an arbitrary elevation (the orthometric height). The reduction level laterally varies in height depending on the position of the gravity station. Also it does not coincide with a boundary of a Bouguer plate (or a Bouguer spherical cap) above the geoid. As is written in the text, we define the generalized Bouguer anomaly upon the datum level of an arbitrary elevation by the difference between the reduced observed gravity and the reference gravity that is reduced within the Earth's materials by the Poincaré-Prey reduction or, in short, the Prey reduction (Heiskanen and Moritz, 1967, p. 146, pp. 163–165) from the normal gravity at the reference ellipsoid. The main aim of such a generalization of the Bouguer anomaly is to study the subsurface structures (e.g. Nozaki, 1997). The figure of the Earth is not the subject of such a generalization of the Bouguer anomaly.

One of the most prominent features of the generalized Bouguer anomaly lies in the treatment of the reference gravity field: the use of the Prey reduction for the reference gravity. This means that the level of gravity reduction is within the Earth's mass distribution outside the reference ellipsoid as well as inside.

In Section 2, we explain the motivations of this study. In Section 3, we describe the details of the formulation of the generalized Bouguer anomaly. Also, we explain the physical properties of the new formula thus obtained. In Section 4, we define the three specific datum levels of gravity reduction: the one is the datum level so that the value of the generalized Bouguer anomaly becomes invariant for any Bouguer reduction density (the so-called 'ρ_B*-free datum level*'), and the other is the datum level so that the sum of the terrain and Bouguer corrections becomes zero for any Bouguer reduction density. Also, we derive the generalized Bouguer anomaly at each specific datum level. Particularly, it will be shown that the generalized Bouguer anomaly upon the ρ_B*-free datum level*, namely, the ρ_B*-free Bouguer anomaly*, is free from the Bouguer reduction density and is equal to the 'gravity disturbance' as defined in the physical geodesy. It will be also shown that the equation of the ρ_B*-free Bouguer anomaly* is the same as the fundamental equation of physical geodesy, which defines the

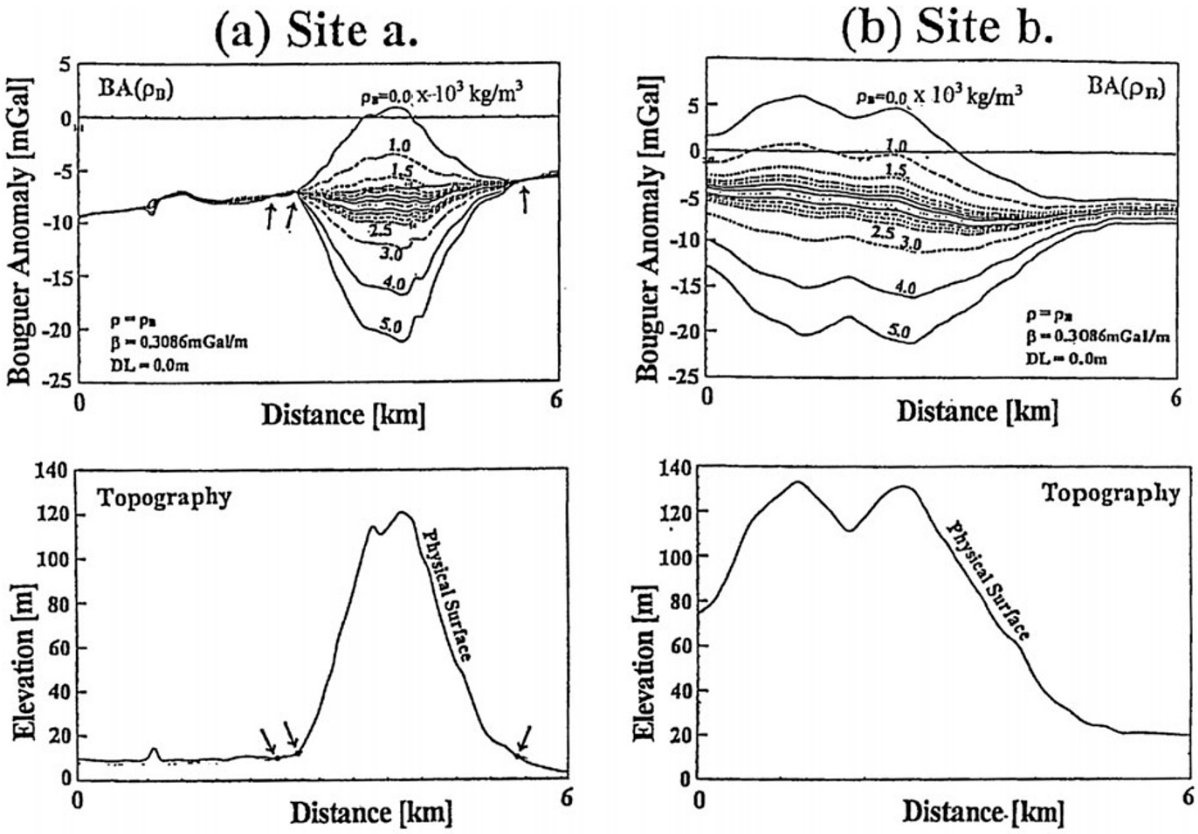

Fig. 1. Examples of the variation of the Bouguer anomaly distributions for Bouguer reduction densities. In the range of the Bouguer reduction densities ρ_Bs from 1.5×10^3 kg/m^3 to 2.5×10^3 kg/m^3, the interval is 0.1×10^3 kg/m^3. β denotes the free-air gradient, DL the elevation of the datum level of gravity reduction. Arrows indicate Bouguer anomaly invariant (BA-invariant) points. Upper panels: Bouguer anomaly profiles, lower panels: topography profiles. The Site (a) profile shows an example in which BA-invariant points occur. The site (b) profile shows an example in which BA-invariant point does not occur.

'gravity anomaly' (Heiskanen and Moritz, 1967). In Section 5, we describe the details of the derivation of an approximate equation to be satisfied by the Bouguer reduction density and the anomalous vertical gradient of the gravity. This approximation can be used for the estimation of the Bouguer reduction density. In Section 6, using such an estimated Bouguer reduction density, we describe a method to obtain the generalized Bouguer anomaly distribution upon the geoid from that upon the *ρ_B-free datum level*. We show that it is nothing but the Bouguer anomaly in the classical sense (say, the Bouguer disturbance) which has been used to study the subsurface structures.

2. Motivations of the Approach

The most important motivation in this study is the 'existence of the Bouguer anomaly invariant point'. Figure 1 shows variation of Bouguer anomaly distributions due to the variation of the Bouguer reduction density ρ_B. The range of ρ_B variation is between 0.0 kg/m^3 to 5,000 kg/m^3. The Bouguer anomaly is defined upon the geoid as is done in a classical textbook (e.g. Heiland, 1946). Namely, the elevation of the datum level of gravity reduction is taken at the geoid. The adopted free-air gradient is taken as 0.3086 mGal/m (10^{-5} m/s^2/m).

On Fig. 1(a), one can notice three points that are indicated by arrows. The Bouguer anomalies (BAs) for these points are independent of the variation of the Bouguer reduction densities ρ_B. For convenience, we call each of these points a 'BA-invariant point'. On Fig. 1(b), no such BA-invariant points occur. What does such a BA-invariant point mean? At such a point, the Bouguer anomaly is free from the surrounding topographic masses.

Why do BA-invariant points exist? The reason is the elevation of the datum level of the gravity reduction. In this case, it is the geoid. If one changes the elevation of the datum level upwards and downwards, the location of the BA-invariant points would also change. In other words, any gravity station can become a BA-invariant point for each gravity data by adjusting the elevation of the datum level.

If gravity anomalies are mapped by using only such BA-invariant points, it could be the most useful one for studying subsurface structures, because the gravity anomalies are independent of the Bouguer reduction density ρ_B. From this point of view, the elevation of the datum level of the gravity reduction could have freedom to be selected.

3. Formulation of the Generalized Bouguer Anomaly

3.1 Height system and definition

In this paper, we will use the orthometric height system. Firstly, we make some comments on the height system. As mentioned above, the main goal of this paper is the generalization of the *classical* Bouguer anomaly, which has been referred to the geoid. In the land gravity survey, the data set are given by the observed gravity with position and elevation.

Although the exact determination of the orthometric height requires the complete knowledge of the actual gravity field within the topographic masses, this type of the height system is the most familiar one in the *classical* Bouguer anomaly. Therefore, in deriving the concept of generalized Bouguer anomaly, we will use this height system. In Fig. 2(a), we show the correspondence between the height system used in this paper based on the orthometric height and the standard height system based on the normal height.

Here, we define the generalized Bouguer anomaly of the observed gravity g_p. Let the elevation of an arbitrary datum level be H_p (orthometric height from the geoid). The elevation of the geoid is zero in this height system. The elevation of the datum level of gravity reduction is H_d. The elevation of the surface of the reference ellipsoid is H_0. The vertical gradient of gravity (VGG) anomaly $\partial\Delta g/\partial r$ is included. The truncated spherical shell system of gravity correction is included by a truncation angle ψ (see Fig. 3). Then, since the 'anomaly' can be defined by the difference between the observed value and the reference value, we define the generalized Bouguer anomaly ($\Delta g_{p,H_d}$) by the difference between the observed gravity reduced onto the datum level at an elevation H_d and the reference gravity reduced onto the same datum level at H_d. Namely, the generalized Bouguer anomaly is defined by the form

$$\Delta g_{p,H_d} := g_{p,H_d} - \gamma_{H_d}, \tag{1}$$

where, g_{p,H_d} denotes the reduced observed gravity, and γ_{H_d} denotes the reference gravity. The detailed equations are described in Section 3.2. In Fig. 4, we show a schematic view of the observed gravity reduced onto an arbitrary datum level of elevation H_d and the reference gravity within the mass distribution.

This approach of defining the generalized Bouguer anomaly at the same datum level looks classical. However, in the followings, we will find a new relation between the generalized Bouguer anomaly and the 'gravity anomaly' defined in the physical geodesy (Heiskanen and Moritz, 1967). In spite of the difference at the same datum level, we use the notation Δ instead of δ. This is because we do not see the terminology 'Bouguer disturbance' in the literature.

3.2 Formulation

In this section, we discuss the generalized Bouguer anomaly based on the defining equation, Eq. (1). In the formulation, we express the generalized Bouguer anomaly $\Delta g_{p,H_d}$ for the cases of $H_d < H_p$ and $H_d > H_p$ separately to make their physical meanings clear, even though the both expressions are equivalent.

Symbols used in the formulation are summarized as follows:

g_p**:** observed gravity at a station P,

H_p**:** elevation of the gravity station,

H_d**:** elevation of the datum level of the gravity reduction,

H_0**:** elevation of the normal ellipsoid surface,

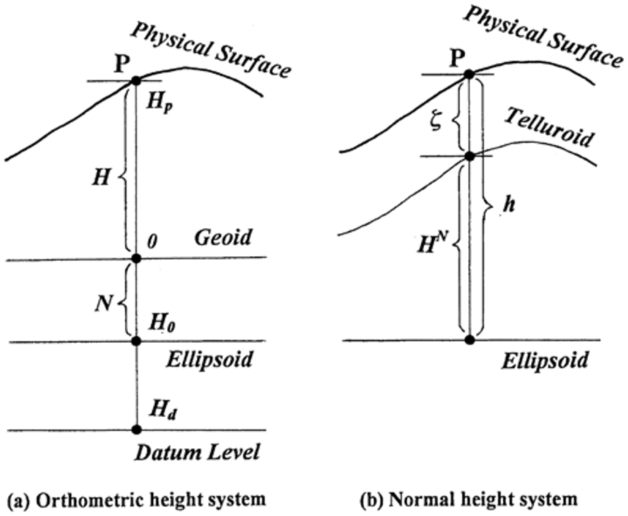

Fig. 2. Height systems. (a) The orthometric height system. H: the orthometric height, N: the geoid height. Symbols H_p: the elevation (the orthometric height) of P, H_0: the elevation of the ellipsoid, and H_d: the elevation of the datum level of gravity reduction used in the text. (b) Normal height system. H^N : the normal height (Torge, 2001, equation (3.107)), ζ: the height anomaly, h: the geometrical height (Heiskanen and Moritz, 1967) or the ellipsoidal height (Torge, 1989, equations (2.70a) and (2.71a); $h = H^N + \zeta = H + N$).

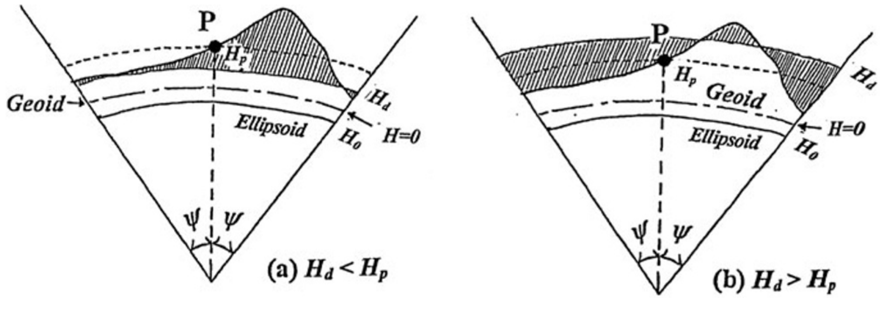

Fig. 3. Schematic illustration of the truncated spherical shell system of gravity correction. (a) For the case of $H_d < H_p$. (b) For the case of $H_d > H_p$. Hatching indicates the areas of mass-redistribution accompanied by the terrain and Bouguer corrections. H_p denotes the elevation of the gravity station P, H_d the elevation of the datum level of the gravity reduction, H_0 the elevation of the surface of the normal ellipsoid, and ψ the truncation angle of spherical gravity correction.

Table 1. Definition of ‘correction’ and ‘reduction’.

Term	Correction	Reduction
Free-air	$FC_p = \int \frac{\partial \gamma}{\partial r} dr$	$FR_p = FC_p$
Terrain	$TC_p = \rho_B TC_p(1)$	---
Bouguer	$BC_p = \int 2\pi G \rho_B H^{\pm}(1,\psi) dr$	$BR_p = TC_p + BC_p + FC_p$
Prey	$PC_p = \int 2\pi G \rho_B \left[H^{+}(1,\psi) - H^{-}(1,\psi)\right] dr$	$PR_p = PC_p + FC_p$ $= \int \left\{2\pi G \rho_B \left[H^{+}(1,\psi) - H^{-}(1,\psi)\right] + \frac{\partial \gamma}{\partial r}\right\} dr$

* Each Italic capital with subscription ‘p’ denotes the corresponding quantity concerning a gravity station P.

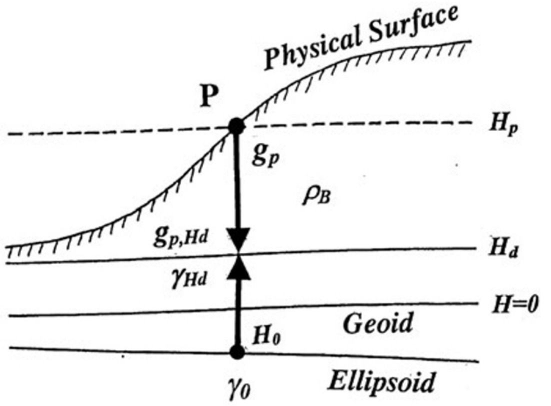

Fig. 4. A conceptual illustration explaining the definition of the generalized Bouguer anomaly upon H_d. g_{p,H_d} and γ_{H_d} denote the reduced observed gravity and the reference gravity at the datum level H_d, respectively. The generalized Bouguer anomaly ($\Delta g_{p,H_d}$) is defined by $\Delta g_{p,H_d} = g_{p,H_d} - \gamma_{H_d}$. ρ_B denotes the Bouguer reduction density. The reference gravity field is within the Earth’s mass distribution.

γ_0**:** the normal gravity (upon the normal ellipsoid),

$TC_p(1)$**:** the quantity of the terrain correction at the station P for the unit density,

$BC_p(1)$**:** the quantity of the Bouguer correction at the station P for the unit density,

ρ_B**:** Bouguer reduction density,

TC_p**:** value of the terrain correction ($= \rho_B TC_p(1)$),

BC_p**:** value of the Bouguer correction ($= \rho_B BC_p(1)$),

FA**:** gravity anomaly in the physical geodesy or free-air anomaly in the Molodensky sense,

f**:** sum of the terrain and Bouguer corrections ($= TC_p + BC_p$),

G**:** Newtonian gravitational constant,

$\partial\gamma/\partial r$**:** VGG of the normal gravity field,

$\partial\Delta g/\partial r$**:** VGG anomaly defined by the difference between the actual VGG after terrain and Bouguer corrections, and the normal VGG, which is compared at a point in the free-air space where the topographic masses of the density ρ_B are moved or removed by the terrain and Bouguer corrections,

r**:** the geocentric radial coordinate (positive upwards),

$H^{\pm}(1, \psi)$**:** sphericity factor for the spherical terrain and Bouguer corrections,

ψ**:** truncation angle of the spherical terrain and Bouguer corrections.

In the VGG anomaly $\partial\Delta g/\partial r$, the gravitational effect of the near surface density anomaly, which represents the inhomogeneity of the topographic mass-density field from ρ_B, is included together with the VGG anomaly in the free-air space. The functions $H^{+}(1, \psi)$ and $H^{-}(1, \psi)$, which govern the gravitational behaviour of a thin spherical cap with a truncation angle ψ (Nozaki, 1999), are given as

$$H^{\pm}(1, \psi) = \sqrt{\frac{1 - \cos\psi}{2}} \pm 1, \tag{2}$$

(hereafter, double signs should be taken in the same order). The derivation and the physical properties of these functions are shown in Appendix A. It is clear that an identical equation

$$H^{+}(1, \psi) - H^{-}(1, \psi) \equiv 2 \tag{3}$$

holds for any truncation angle ψ. Concerning the spherical gravity corrections, see Nozaki (1981). Corresponding to the functions $H^{+}(1, \psi)$ and $H^{-}(1, \psi)$, we distinguish the notation of the Bouguer correction BC_p for $H_d < H_p$ from that for $H_d > H_p$: BC_p^{+} denotes the Bouguer correction for $H_d < H_p$ and BC_p^{-} does that for $H_d > H_p$, respectively.

In this paper, we distinguish the terminologies between ‘reduction’ and ‘correction’ (after Nettlelton, 1940, Chapter 4). For example, the Bouguer ‘reduction’ is performed by the terrain ‘correction’, Bouguer ‘correction’ and free-air ‘correction’. The terminology ‘reduction’ is used for the level-transformation of the gravity value from one elevation level to another. Such technical terms used in this paper are summarized in Table 1.

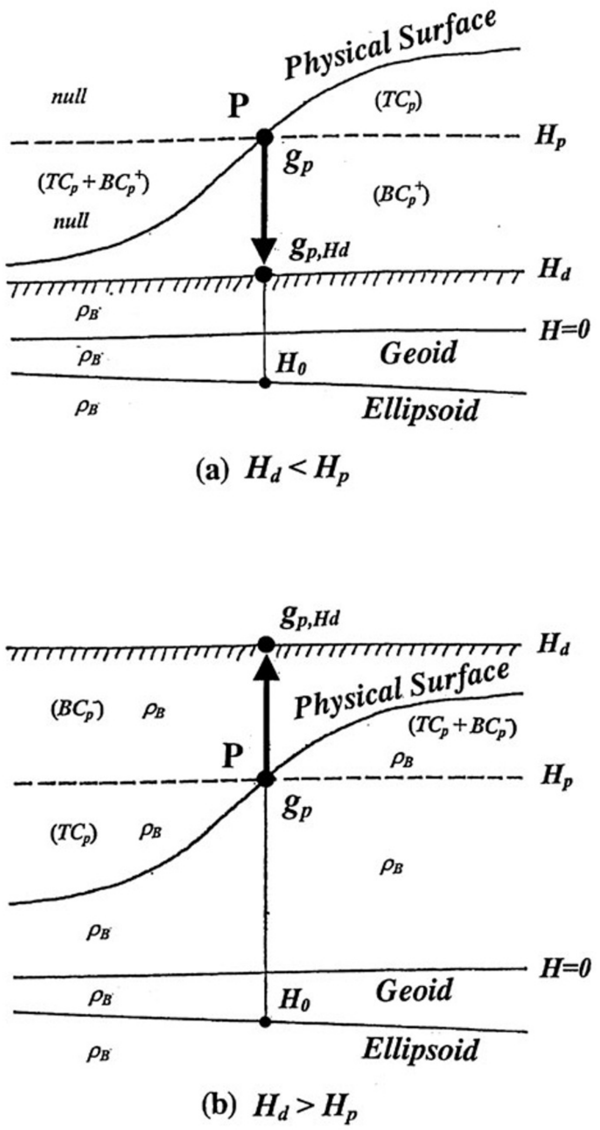

Fig. 5. Schematic illustration of the process to compute the reduced observed gravity g_{p,H_d}. (a): Bouguer-reduced observed gravity for the case of $H_d < H_p$; the observed gravity g_p is corrected by the terrain and Bouguer corrections (TC_p and BC_p^+), then, reduced by the free-air reduction over the interval $[H_p, H_d]$ as indicated by the arrow. (b): Prey-reduced observed gravity for the case of $H_d > H_p$; the observed gravity g_p is corrected by the terrain and Bouguer corrections (TC_p and BC_p^-), then, reduced by the Prey reduction over the interval $[H_p, H_d]$ as indicated by the arrow. Mass-density distribution after the reduction for each case is shown in the figure. ρ_B is the Bouguer reduction density.

3.2.1 Reduced observed gravity

(1) *The case* $H_d < H_p$

In this case, the observed gravity g_p at the elevation H_p is reduced to the Bouguer-reduced observed gravity g_{pH_d} at the elevation H_d ($< H_p$) as shown in Fig. 5(a).

When the elevation of the datum level of gravity reduction H_d is lower than that of the gravity station H_p (see Fig. 3(a)), the Bouguer correction is to remove the Earth's materials above the datum level of gravity reduction. Accordingly, we introduce the spherical Bouguer correction for $H_d < H_p$ denoted by

$$BC_p^+ = \int_{H_p}^{H_d} 2\pi G\rho_b H^+(1, \psi)dr.$$

Then, in the case of $H_d < H_p$, the Bouguer-reduced observed gravity g_{p,H_d} upon the datum level of the gravity reduction H_d can be expressed as

$$g_{p,H_d} = (g_p + f^+) + \int_{H_p}^{H_d} \left(\frac{\partial\gamma}{\partial r} + \frac{\partial\Delta g}{\partial r}\right) dr, \quad (4)$$

where, f^+ denotes the sum of the terrain and Bouguer corrections for the datum level of $H_d < H_p$:

$$\begin{aligned} f^+ &= TC_p + BC_p^+ \\ &= \rho_B TC_p(1) + \int_{H_p}^{H_d} 2\pi G\rho_B H^+(1, \psi)dr. \quad (5) \end{aligned}$$

The Bouguer reduction (see Table 1) is made up of the spherical terrain correction (TC_p), the spherical Bouguer correction (BC_p^+), and the free-air correction over the interval $[H_p, H_d]$. Notice that, in the free-air correction, the term of the VGG anomaly is added to the integrand on the right-hand side of Eq. (4). A schematic view of the Bouguer-reduced observed gravity g_{p,H_d} for the case of $H_d < H_p$ is illustrated in Fig. 5(a).

(2) *The case* $H_d > H_p$

In this case, as shown in Fig. 5(b), the observed gravity g_p at the elevation H_p is reduced to the elevation H_d ($> H_p$) by the Prey reduction (e.g. Heiskanen and Moritz, 1967) after terrain and Bouguer corrections. When the elevation of the datum level of the gravity reduction H_d is higher than that of the gravity station H_p, the Prey reduction should be applied over the interval $[H_p, H_d]$ to the observed gravity g_p (see also Fig. 3(b)). This is because we have to fill up the open space above the Earth's surface H_p by the Bouguer correction with the Earth's materials whose density is ρ_B. Accordingly, we introduce the spherical Bouguer correction for $H_d > H_p$ denoted by

$$BC_p^- = \int_{H_p}^{H_d} 2\pi G\rho_B H^-(1, \psi)dr.$$

Thus, in the case of $H_d > H_p$, we have the Prey-reduced observed gravity, g_{p,H_d},

$$\begin{aligned} g_{p,H_d} = (g_p + f^-) &+ \int_{H_p}^{H_d} \Bigg(2\pi G\rho_B[H^+(1, \psi) - H^-(1, \psi)] \\ &+ \frac{\partial\gamma}{\partial r} + \frac{\partial\Delta g}{\partial r}\Bigg)dr, \quad (6) \end{aligned}$$

where, f^- denotes the sum of terrain and Bouguer corrections for the datum level of $H_d > H_p$:

$$\begin{aligned} f^- &= TC_p + BC_p^- \\ &= \rho_B TC_p(1) + \int_{H_p}^{H_d} 2\pi G\rho_B H^-(1, \psi)dr. \quad (7) \end{aligned}$$

Notice that the term of the Prey correction as well as that of the VGG anomaly are added to the integrand on the right-hand side of Eq. (6). A schematic view of the Prey-reduced observed gravity after terrain and Bouguer corrections g_{p,H_d} for the case of $H_d > H_p$ is illustrated in Fig. 5(b). Although

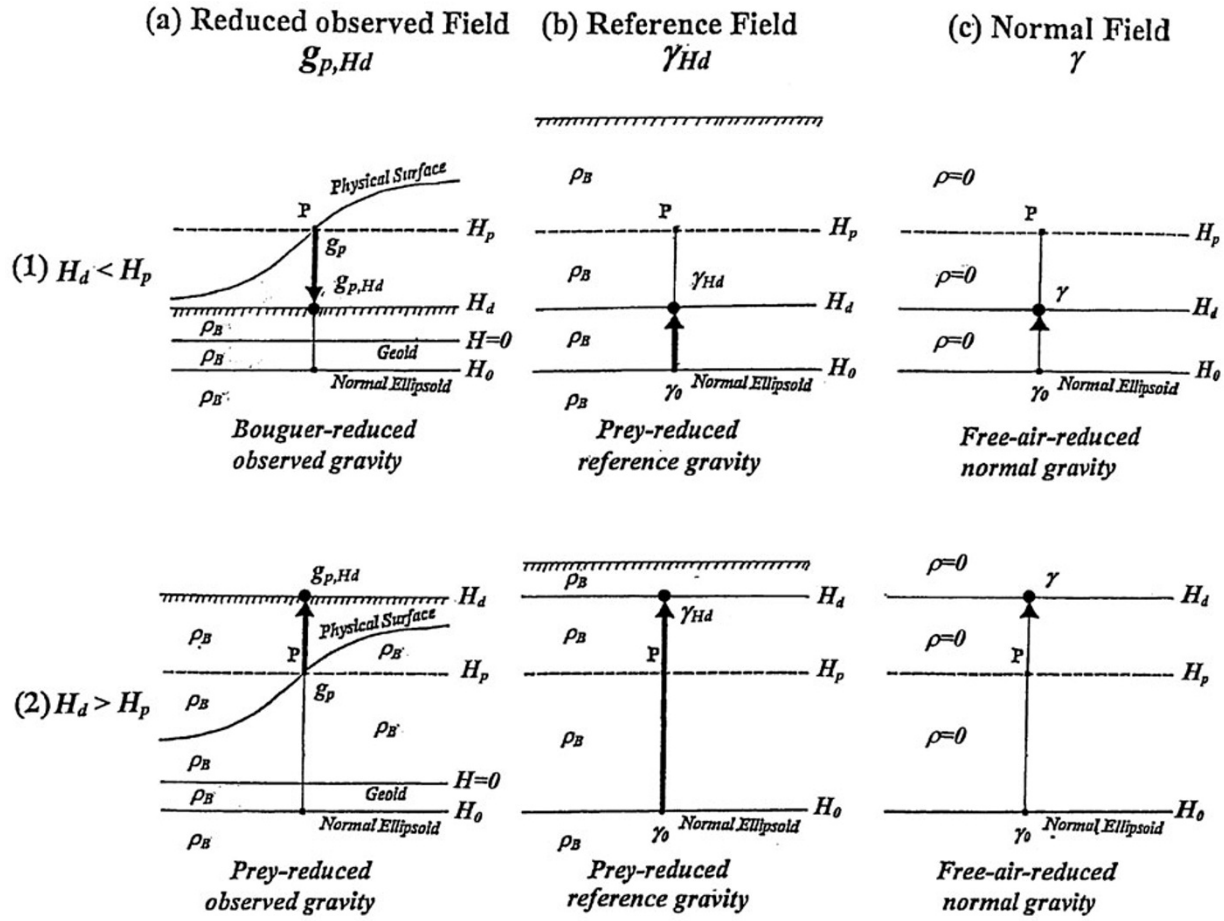

Fig. 6. Correspondence of the reductions.; (a) the reduced observed gravity, (b) the Prey-reduced reference gravity, and (c) the normal gravity. Upper panels: the case $H_d < H_p$; lower panels: the case $H_d > H_p$. (a) The observed gravity at the station level H_p is reduced onto the datum level H_d. (b) The reduction of the reference gravity is done within the earth's materials whose density is ρ_B (Prey-reduction). (c) The reduction in the normal gravity field is done in the open or null space. The generalized Bouguer anomaly is defined by the difference between the Bouguer- or Prey-reduced observed gravity, and the Prey-reduced reference gravity.

the term $2\pi G\rho_B[H^+(1,\psi) - H^-(1,\psi)]$ on the right-hand side of Eq. (6) is $4\pi G\rho_B$ regardless of ψ (see Eq. (3)), we shall retain this form in the following sections to show explicitly the gravitational contribution of the Bouguer spherical cap. We notice here, the difference between Eqs. (5) and (7) is that the factor of the integrand on the right-hand side of Eq. (5) is $H^+(1,\psi)$, while in Eq. (7), it is $H^-(1,\psi)$.

3.2.2 Prey-reduced reference gravity The reference gravity γ_{H_d}, at the datum level H_d, is defined in this paper by the equation

$$\gamma_{H_d} = \gamma_0 + \int_{H_0}^{H_d} \left\{ 2\pi G\rho_B[H^+(1,\psi) - H^-(1,\psi)] + \frac{\partial\gamma}{\partial r} \right\} dr. \quad (8)$$

The reference gravity γ_{H_d} is reduced by the Prey reduction from the normal gravity γ_0, e.g. from the level H_0 to the level H_d. Here we applied the Prey reduction over the interval $[H_0, H_d]$, instead of the Bouguer or free-air reduction. This is because the reduction of the reference gravity from the level H_0 to another level H_d should be done within the Earth's materials whose mass-density is ρ_B. Thus, one can apply Eq. (8) both for the cases $H_d > H_p$ and $H_d < H_p$, and even the case $H_d < H_0$. We call the newly introduced reference gravity the Prey-reduced reference gravity. In Fig. 6, the correspondence between the Prey-reduced reference gravity γ_{H_d}, the reduced observed gravity g_{p,H_d}, and the normal gravity is schematically illustrated. The detailed explanation of the reference field is added in Appendix C.

3.2.3 Formula of the generalized Bouguer anomaly
(1) *The case* $H_d < H_p$

When the elevation of the datum level H_d is lower than that of the gravity station H_p, the formula of the Bouguer-reduced observed gravity is given by Eq. (4). Substituting Eqs. (4) and (8) into Eq. (1) and arranging the terms with respect to the Bouguer reduction density ρ_B, we obtain the formula of the generalized Bouguer anomaly $\Delta g_{p,H_d}$, for the case of $H_d < H_p$, as

$$\Delta g_{p,H_d} = g_p - \gamma_0 - \int_{H_0}^{H_p} \frac{\partial\gamma}{\partial r} dr + \int_{H_p}^{H_d} \frac{\partial \Delta g}{\partial r} dr + \rho_B \left\{ TC_p(1) + \int_{H_p}^{H_d} 2\pi G H^+(1,\psi) dr - \int_{H_0}^{H_d} 2\pi G[H^+(1,\psi) - H^-(1,\psi)] dr \right\}. \quad (9)$$

The first and second terms in the braces of Eq. (9) are the contribution of the terrain and Bouguer corrections to the observed gravity for $H_d < H_p$, while the third term in the braces is essentially that of the Prey correction to the reference gravity (see Eqs. (5) and (4)).

(2) *The case* $H_d > H_p$

When the elevation of the datum level H_d is higher than that of the gravity station H_p, the formula of the Prey-reduced observed gravity is given by Eq. (6). Substituting Eqs. (6) and (8) into Eq. (1) and arranging the terms with respect to the Bouguer reduction density ρ_B, we obtain the formula of the generalized Bouguer anomaly $\Delta g_{p,H_d}$, for the case of $H_d > H_p$, as

$$\Delta g_{p,H_d} = g_p - \gamma_0 - \int_{H_0}^{H_p} \frac{\partial \gamma}{\partial r} dr + \int_{H_p}^{H_d} \frac{\partial \Delta g}{\partial r} dr + \rho_B \left\{ TC_p(1) + \int_{H_p}^{H_d} 2\pi G H^-(1,\psi) dr - \int_{H_0}^{H_p} 2\pi G [H^+(1,\psi) - H^-(1,\psi)] dr \right\}. \tag{10}$$

In the braces of Eq. (10), the first and second terms are the contribution of the terrain and Bouguer corrections to the observed gravity for $H_d > H_p$, while the third term is essentially that of the Prey correction to the reference gravity. Notice, that the interval of integration of the Prey correction is not $[H_0, H_d]$ but $[H_0, H_p]$. This is because, when $H_d > H_p$, the term of the Prey reduction over the interval $[H_p, H_d]$ of the reduced observed gravity in Eq. (6) is canceled out by subtracting that of the reference gravity in Eq. (8). At the same time, this corresponds to the fact that the mass-density above the station height H_p is zero for the case of $H_d > H_p$.

3.3 Remarks about the generalized Bouguer anomaly

3.3.1 Unified expression of the formula of the generalized Bouguer anomaly The unified expression of the generalized Bouguer anomaly $\Delta g_{p,H_d}$ can be written in the same form both for $H_d < H_p$ and for $H_d > H_p$. By arranging Eqs. (4) and (5) for the case of $H_d < H_p$, and Eqs. (6) and (7) for the case of $H_d > H_p$, it can be shown that we have

$$g_{p,H_d} = g_p + \rho_B TC_p(1) + \int_{H_p}^{H_d} 2\pi G \rho_B H^+(1,\psi) dr + \int_{H_p}^{H_d} \left(\frac{\partial \gamma}{\partial r} + \frac{\partial \Delta g}{\partial r} \right) dr. \tag{11}$$

This means that Eqs. (9) and (10) are equivalent each other.

3.3.2 Effect of the datum level change on the generalized Bouguer anomaly When we regard the elevation of the datum level H_d as an independent variable, the changing rate of the generalized Bouguer anomaly $\Delta g_{p,H_d}$ with respect to the elevation of the datum level H_d can be expressed by the equation

$$\frac{\partial \Delta g_{p,H_d}}{\partial H_d} = 2\pi G \rho_B H^-(1,\psi) + \frac{\partial \Delta g}{\partial r}. \tag{12}$$

This is directly derived from any one of Eqs. (9) and (10). In this paper, we will call this rate $\partial \Delta g_{p,H_d} / \partial H_d$ the 'reduction rate'. Equation (12) implies that, when we ignore the term $\partial \Delta g / \partial r$ as is usually the case, the upward transformation of the datum level H_d brings the decrease of the generalized Bouguer anomaly $\Delta g_{p,H_d}$ at the reduction rate of $2\pi G \rho_B H^-(1,\psi)$, and *vice versa*. Notice that the reduction rate of Eq. (12) does not include the terms of the normal gravity field.

4. Generalized Bouguer Anomaly at Some Specific Datum Level of the Gravity Reduction

In this section, we define the specific datum levels of the gravity reduction so that the generalized Bouguer anomalies are not affected by the topographic effects. Also we discuss the physical properties of the generalized Bouguer anomalies upon the specific datum levels.

4.1 Specific datum levels of the gravity reduction

4.1.1 Specific datum level H_{d0} The condition of the specific datum level H_{d0} of gravity reduction, upon which the generalized Bouguer anomaly $\Delta g_{p,H_d}$ becomes independent of any ρ_B (see Section 2 and Fig. 1), is given by the equation

$$\frac{\partial \Delta g_{p,H_d}}{\partial \rho_B} = 0. \tag{13}$$

By this condition, one can obtain the defining equation of H_{d0} from Eq. (9) or equivalently from Eq. (10) as

$$H_{d0} = \frac{2(H_p - H_0)}{H^-(1,\psi)} + H_p - \frac{TC_p(1)}{2\pi G H^-(1,\psi)}. \tag{14}$$

In the following we shall call this specific datum level of gravity reduction H_{d0} '*ρ_B-free specific datum level*' or in short '*ρ_B-free datum level*'.

The meaning of the *ρ_B-free datum level* H_{d0} can be understood as follows. The condition expressed by Eq. (13) corresponds to that the sum of the terms in the braces on the right-hand side of Eq. (9) or Eq. (10) is zero, i.e. independent of the Bouguer reduction density ρ_B.

4.1.2 Specific datum levels H_{d1} and H_{d2} Another condition for defining the specific datum levels H_{d1} and H_{d2}, upon which the topographic gravitational effects are eliminated, is given by Eqs. (5) and (7):

$$TC_p + BC_p^{\pm} = \rho_B TC_p(1) + \int_{H_p}^{H_d} 2\pi G \rho_B H^{\pm}(1,\psi) dr = 0. \tag{15}$$

This is the condition that the sum of the terrain correction TC_p and the Bouguer correction BC_p is always zero regardless of ρ_B. This condition leads to the definition of the additional specific datum levels (H_{d1} and H_{d2}):

$$H_{d1} = H_p - \frac{TC_p(1)}{2\pi G H^+(1,\psi)} \tag{16}$$

and

$$H_{d2} = H_p - \frac{TC_p(1)}{2\pi G H^-(1,\psi)}. \tag{17}$$

Since the terrain correction $TC_p(1)$ is almost everywhere positive for a small truncation angle ψ (say $\psi < 3$ degrees), the elevation of the specific datum level H_{d1} is almost everywhere lower than that of the gravity station H_p

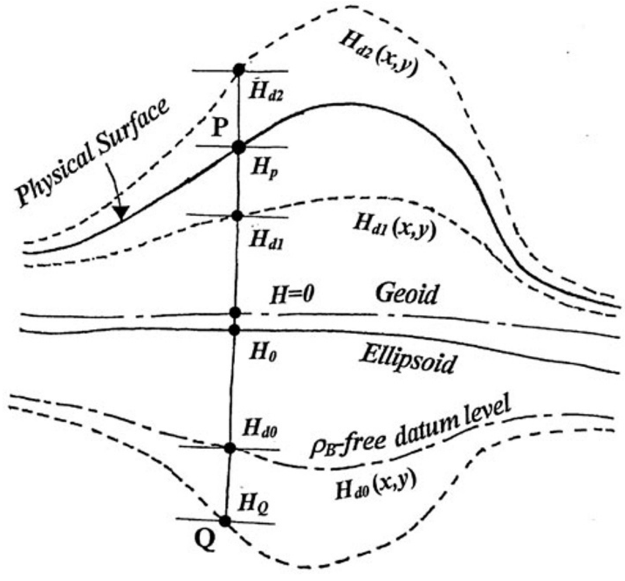

Fig. 7. Geometric relation between the specific datum levels H_{d0}, H_{d1} and H_{d2}. Each specific datum level spreads the surface with undulation as a function of the horizontal coordinates x and y: $H_{d0}(x, y)$, $H_{d1}(x, y)$ or $H_{d2}(x, y)$. For the flat Earth approximation, the specific datum levels H_{d1} and H_{d2} are located at the mirror-imaged positions with respect to the elevation of the gravity station H_p (see Eq. (23)); and so the specific datum levels H_{d1} and H_{d0} with respect to the elevation of the normal ellipsoid H_0 (see Eq. (22)). The elevation of the point Q (H_Q) is $H_Q = 2(H_p - H_0)/H^-(1, \psi) + H_p$.

($H_{d1} < H_p$), and the elevation of the specific datum level H_{d2} is almost everywhere higher than that of the gravity station H_p ($H_{d2} > H_p$).

Notice that H_{d1} and H_{d2} can be computed from the known quantity of the terrain correction for the unit density $TC_p(1)$ for each gravity station. Also H_{d0} is computable if H_0 is given. Remember that H_0 is the elevation of the reference ellipsoid in the orthometric height system of this paper, and its magnitude is equal to the geoid height N (i.e. $H_0 = -N$).

4.1.3 Relation between the specific datum levels Since the specific datum levels H_{d0}, H_{d1} and H_{d2} can be defined for each gravity station, these specific datum levels form surfaces as functions of the horizontal position (x, y):

$$H_{d0} = H_{d0}(x, y), \quad H_{d1} = H_{d1}(x, y) \quad \text{and} \quad H_{d2} = H_{d2}(x, y).$$

Each of the three surfaces of $H_{d0}(x, y)$, $H_{d1}(x, y)$ and $H_{d2}(x, y)$ has undulation like the Molodensky telluroid (Heiskanen and Moritz, 1967).

The geometric relation among the surfaces of the specific datum levels $H_{d0}(x, y)$, $H_{d1}(x, y)$ and $H_{d2}(x, y)$ is shown in Fig. 7. From Eqs. (14) and (16), the specific datum levels H_{d0} and H_{d1} are related to H_0 in the following way

$$H_0 = \frac{-H^-(1, \psi)H_{d0} + H^+(1, \psi)H_{d1}}{2}. \tag{18}$$

Also, from Eqs. (16) and (17), the specific datum levels H_{d1} and H_{d2} have a relation against H_p as

$$H_p = \frac{H^+(1, \psi)H_{d1} - H^-(1, \psi)H_{d2}}{2}. \tag{19}$$

In the same manner, from Eqs. (14) and (17), the specific datum levels H_{d0} and H_{d2} have a relation

$$H_{d0} = \frac{2(H_p - H_0)}{H^-(1, \psi)} + H_{d2}. \tag{20}$$

It is interesting that the relation between H_{d0} for H_0 and H_{d2} for H_p is reciprocal:

$$H_{d2} = \frac{2(H_0 - H_p)}{H^-(1, \psi)} + H_{d0}. \tag{21}$$

From Eqs. (18) and (19), we have the relation equivalent to Eq. (20) or Eq. (21):

$$2(H_p - H_0) = -H^-(1, \psi)(H_{d2} - H_{d0}).$$

Particularly in a flat Earth approximation of the gravity correction, i.e.,

$$H^+(1, \psi) \to +1 \quad \text{and} \quad H^-(1, \psi) \to -1$$
$$[\text{for } \psi \sim 0, \text{ refer to Eq. (2)}],$$

the specific datum levels H_{d0} and H_{d1} locate at the same distance of lower and upper positions with respect to H_0, respectively (see Eq. (18)), resulting in

$$H_0 = \frac{H_{d0} + H_{d1}}{2}, \quad (\text{for } \psi \sim 0). \tag{22}$$

Also from Eq. (19), H_p takes the algebraic mean value of H_{d1} and H_{d2}:

$$H_p = \frac{H_{d1} + H_{d2}}{2}, \quad (\text{for } \psi \sim 0). \tag{23}$$

Furthermore, if the topography is very gentle and hence the value of the terrain correction $TC_p(1)$ is negligibly small, H_{d0} degenerates into the elevation H_Q of the point Q, that is,

$$H_Q = H_p - 2(H_p - H_0)/H^-(1, \psi)$$

as shown in Fig. 7 (see Eq. (14)). Also, Eq. (22) represents that H_{d0} and H_{d1} are at the mirror-image position of the reference ellipsoid level H_0. Also, Eq. (23) represents that H_{d1} and H_{d2} are at the mirror-image position of H_p, and degenerate into H_p when $TC_p(1)$ is negligibly small, (see Eqs. (16) and (17)).

On the other hand, particularly in the spherical shell system of gravity correction, i.e.,

$$H^+(1, \psi) \to +2 \quad \text{and} \quad H^-(1, \psi) \to -0$$
$$[\text{for } \psi = \pi, \text{ see Eq. (2)}],$$

the datum levels H_{d0} and H_{d2} take the infinite values (see Eqs. (14) and (17)), and lose their physical meanings, while H_{d1} still takes a definite value of

$$H_{d1} = H_p - \frac{TC_p(1)}{4\pi G}, \quad (\text{for } \psi = \pi) \tag{24}$$

(see Eq. (16) for $\psi = \pi$). At first sight, this seems to be inconsistent with Eq. (18). However, substituting Eq. (14) into Eq. (18), we get

$$H^+(1, \psi)H_{d1} = 2H_0 + 2(H_p - H_0) + H^-(1, \psi)H_p - \frac{TC_p(1)}{2\pi G}, \quad (\text{forr } \psi = \pi).$$

This relation is consistent with Eq. (24) since $H^+(1, \psi) = 2$ and $H^-(1, \psi) = 0$ for $\psi = \pi$.

4.2 Introduction of the new notation FA

Let FA denote

$$FA = \left(g_p + \int_{H_p}^{0} \frac{\partial \gamma}{\partial r} dr \right) - \gamma_0, \qquad (25\text{-}1)$$

where,

g_p: observed gravity at the elevation H_p,

$\int_{H_p}^{0} \frac{\partial \gamma}{\partial r} dr$: free-air correction from the level of elevation H_p to the level of the geoid,

γ_0: normal gravity upon the normal ellipsoid.

Note that the interval of integration $[H_p, 0]$ in Eq. (25-1) and the interval $[0, H_0]$ in the interval $[H_0, H_p]$ in Eq. (9) or (10) are complementary to each other with respect to the whole interval of the free-air correction. The integration over the interval $[H_0, 0]$ plays an important role in the geodetic interpretation of FA (as is described in Section 4.5).

Ignoring the VGG anomaly, FA is the approximation of the free-air anomaly Δg_F (e.g. Heiskanen and Moritz, 1967, equation (3-62), p. 146). Alternatively, FA can be rewritten as

$$FA = g_p - \left(\gamma_0 + \int_{0}^{H_p} \frac{\partial \gamma}{\partial r} dr \right), \qquad (25\text{-}2)$$

and also as

$$FA \approx g_p - \left(\gamma_0 + \int_{-\varsigma}^{H_p - \varsigma} \frac{\partial \gamma}{\partial r} dr \right), \qquad (26)$$

where ζ is the height anomaly. This can be easily understood by changing the interval of integration $[0, H_p]$ in Eq. (25-2) to $[H_0, H_p + H_0]$, and assuming N (geoid height) $=$ ζ (height anomaly), and H (orthometric height) $= H^N$ (normal height):

$$\int_{0}^{H_p} \frac{\partial \gamma}{\partial r} dr \Rightarrow \int_{H_0}^{H_p + H_0} \frac{\partial \gamma}{\partial r} dr = \int_{-N}^{H_p - N} \frac{\partial \gamma}{\partial r} dr \approx \int_{-\varsigma}^{H_p - \varsigma} \frac{\partial \gamma}{\partial r} dr. \qquad (27)$$

In this case, it is not necessarily required that $\partial \gamma / \partial r$ is constant. Importantly, in this case, the integration (free-air correction) over the interval $[H_p - \zeta, H_p]$, which is complementary to the interval of integration $[-\zeta, H_p - \zeta]$ in Eq. (26) for the whole interval $[-\zeta, H_p] \approx [H_0, H_p]$ in Eq. (9) or (10), plays essentially the same role as that over the interval $[H_0, 0]$ as mentioned above.

Thus, FA of Eq. (26) represents the new gravity anomaly (Heiskanen and Moritz, 1967, equation (8-7), p. 293), or the point free-air anomaly (e.g. Torge, 1989, equation (3-7a), p. 54), that is the difference between the measured gravity at the ground and the normal gravity at the telluroid. In this sense, FA is the free-air anomalies in the Molodensky's sense, although FA of Eq. (25-1) was firstly defined for the free-air corrected observed gravity on the geoid as the gravity anomaly (Heiskanen and Moritz, 1967, equation (2-139), p. 83).

Using FA, hereafter, we will proceed to formulate the *ρ_B-free generalized Bouguer anomaly*.

4.3 Representation of the generalized Bouguer anomaly at the specific datum level

The generalized Bouguer anomaly upon an arbitrary datum level of H_d is given by any one of Eqs. (9) and (10). By substituting FA as defined by Eq. (25) into Eq. (9), we have

$$\Delta g_{p,H_d} = FA - \int_{H_0}^{0} \frac{\partial \gamma}{\partial r} dr + \int_{H_p}^{H_d} \frac{\partial \Delta g}{\partial r} dr + \rho_B \left\{ TC_p(1) + \int_{H_p}^{H_d} 2\pi G H^+(1, \psi) dr - \int_{H_0}^{H_d} 2\pi G [H^+(1, \psi) - H^-(1, \psi)] dr \right\}. \qquad (28)$$

The first and second terms in the braces of Eq. (28) are essentially the terrain and Bouguer corrections for the observed gravity, while the third term in the braces is essentially the Prey correction for the reference gravity. As was previously mentioned, one can derive the same results by using Eq. (10) instead of Eq. (9).

Substituting H_{d0}, H_{d1} and H_{d2} (Eqs. (14), (16) and (17)) into H_d in Eq. (28), we have, respectively, the representation formulae of the generalized Bouguer anomalies at the specific datum levels of gravity reduction H_{d0}, H_{d1} and H_{d2} as follows:

$$\Delta g_{p,H_{d0}} = FA - \int_{H_0}^{0} \frac{\partial \gamma}{\partial r} dr + \int_{H_p}^{H_{d0}} \frac{\partial \Delta g}{\partial r} dr + \rho_B \left\{ TC_p(1) + \int_{H_p}^{H_{d0}} 2\pi G H^+(1, \psi) dr - \int_{H_0}^{H_{d0}} 2\pi G [H^+(1, \psi) - H^-(1, \psi)] dr \right\}, \quad (\text{for } H_{d0} < H_p), \qquad (29)$$

$$\Delta g_{p,H_{d1}} = FA - \int_{H_0}^{0} \frac{\partial \gamma}{\partial r} dr + \int_{H_p}^{H_{d1}} \frac{\partial \Delta g}{\partial r} dr + \rho_B \left[TC_p(1) + \int_{H_p}^{H_{d1}} 2\pi G H^+(1, \psi) dr \right] - \rho_B \int_{H_0}^{H_{d1}} 2\pi G [H^+(1, \psi) - H^-(1, \psi)] dr, \quad (\text{for } H_{d1} < H_p), \qquad (30)$$

and

$$\Delta g_{p,H_{d2}} = FA - \int_{H_0}^{0} \frac{\partial \gamma}{\partial r} dr + \int_{H_p}^{H_{d2}} \frac{\partial \Delta g}{\partial r} dr + \rho_B \left[TC_p(1) + \int_{H_p}^{H_{d2}} 2\pi G H^-(1, \psi) dr \right] - \rho_B \int_{H_0}^{H_p} 2\pi G [H^+(1, \psi) - H^-(1, \psi)] dr \quad (\text{for } H_{d2} > H_p), \qquad (31)$$

For the sake of the later calculations, in Eqs. (29)–(31), we retain the terms that vanish under the conditions of the specific datum levels.

Notice that, in Eq. (29), the sum of the terrain and Bouguer corrections (the first and second terms in the braces) is canceled out by the term of the Prey correction (the third term in the braces), resulting in all the terms concerning ρ_B in the braces on the right-hand side vanish by setting the datum level at H_{d0}. This is because the specific datum level H_{d0} is so defined as to satisfy Eq. (13). Also, in Eqs. (30) and (31), the sum of the terrain and Bouguer corrections, which corresponds to the first and the second terms in the braces on the right-hand sides, vanish by setting the datum levels at H_{d1} and H_{d2}, respectively. This is because the specific datum levels H_{d1} and H_{d2} are so defined as to satisfy Eq. (15). Notice, that the interval of integration of the fifth term on the right-hand side of Eq. (31) is not $[H_0, H_{d2}]$ but $[H_0, H_p]$, because the mass-density ρ_B is zero over the interval $[H_p, H_{d2}]$.

When these vanishing terms in Eqs. (29), (30) and (31) are set to zero, we have the final equations

$$\Delta g_{p,H_{d0}} = FA - \int_{H_0}^{0} \frac{\partial \gamma}{\partial r} dr + \int_{H_p}^{H_{d0}} \frac{\partial \Delta g}{\partial r} dr, \quad (\text{for } H_{d0} < H_p), \tag{32}$$

$$\Delta g_{p,H_{d1}} = FA - \int_{H_0}^{0} \frac{\partial \gamma}{\partial r} dr - \int_{H_0}^{H_{d1}} 2\pi G\rho_B [H^+(1,\psi) - H^-(1,\psi)] dr + \int_{H_p}^{H_{d1}} \frac{\partial \Delta g}{\partial r} dr, \quad (\text{for } H_{d1} < H_p), \tag{33}$$

$$\Delta g_{p,H_{d2}} = FA - \int_{H_0}^{0} \frac{\partial \gamma}{\partial r} dr - \int_{H_0}^{H_p} 2\pi G\rho_B [H^+(1,\psi) - H^-(1,\psi)] dr + \int_{H_p}^{H_{d2}} \frac{\partial \Delta g}{\partial r} dr, \quad (\text{for } H_{d2} > H_p), \tag{34}$$

respectively.

Equation (32) is a representation of the condition that the generalized Bouguer anomaly (left-hand side of Eq. (9) or Eq. (29)) is independent of the Bouguer reduction density ρ_B. Equations (33) and (34) are representations of the condition that the sum of the terrain and Bouguer corrections is zero regardless of ρ_B. Namely, by setting the datum level at the *ρ_B-free datum level H_{d0}*, the generalized Bouguer anomaly $\Delta g_{p,H_{d0}}$ results in being free from the Bouguer reduction density ρ_B. In this paper, we call this generalized Bouguer anomaly $\Delta g_{p,H_{d0}}$ the *ρ_B-free Bouguer anomaly*. Also, by setting the datum levels H_{d1} and H_{d2}, the generalized Bouguer anomalies $\Delta g_{p,H_{d1}}$ and $\Delta g_{p,H_{d2}}$ result in being free from the terrain and Bouguer corrections. The integrand of $2\pi G\rho_B[H^+(1,\psi) - H^-(1,\psi)]$ in Eq. (33), as well as that in Eq. (34), corresponds to the Prey correction for the reference field.

Here we shall pay special attention to that Eqs. (32)–(34) yield the relation between FA and the generalized Bouguer anomaly at the specific datum level ($\Delta g_{p,H_{d0}}$, $\Delta g_{p,H_{d1}}$, or $\Delta g_{p,H_{d2}}$), respectively.

4.4 The meaning of the generalized Bouguer anomaly at the *ρ_B-free datum level* H_{d0}

In the simple case when the term of the VGG anomaly $\partial\Delta g/\partial r$ is sufficiently small, Eq. (32) of the generalized Bouguer anomaly at the *ρ_B-free datum level H_{d0}* ($\Delta g_{p,H_{d0}}$) is approximated to

$$\Delta g_{p,H_{d0}} = FA - \int_{H_0}^{0} \frac{\partial \gamma}{\partial r} dr. \tag{35}$$

Rewriting Eq. (35) by using Eq. (25), we obtain the following approximate representations

$$\begin{aligned}
\Delta g_{p,H_{d0}} &= g_p + \int_{H_p}^{H_0} \frac{\partial \gamma}{\partial r} dr - \gamma_0 : \\
&\quad \text{gravity disturbance on the ellipsoid} \\
&= \left(g_p + \int_{H_p}^{0} \frac{\partial \gamma}{\partial r} dr \right) - \left(\gamma_0 + \int_{H_0}^{0} \frac{\partial \gamma}{\partial r} dr \right) : \\
&\quad \text{gravity disturbance on the geoid} \\
&= \left(g_p + \int_{H_p}^{H_{dc}} \frac{\partial \gamma}{\partial r} dr \right) - \left(\gamma_0 + \int_{H_0}^{H_{dc}} \frac{\partial \gamma}{\partial r} dr \right) : \\
&\quad \text{gravity disturbance at any datum level } H_{dc}.
\end{aligned} \tag{36}$$

Thus, we conclude that the generalized Bouguer anomaly at the *ρ_B-free datum level H_{d0}*, that is, the *ρ_B-free Bouguer anomaly* $\Delta g_{p,H_{d0}}$, is the gravity disturbance. Also, Eq. (36) represents that the gravity disturbance is invariant for the level transformation in the free-air space.

Equation (35) represents the relation between the gravity disturbance ($\Delta g_{p,H_{d0}}$) and FA (the Molodensky's free-air anomaly). Although the details will be described in the next section, this fact suggests that Eq. (35) has a tie to the fundamental equation of physical geodesy. Also, Eq. (36) implies that the gravity disturbance in the free-air space can be defined not only at the elevation of the geoid but also at any level H_{dc}. The gravity disturbance ($\Delta g_{p,H_{d0}}$) is not defined by the right-hand side of Eq. (36) but results in the right-hand side of Eq. (36). Such a view of the gravity disturbance ($\Delta g_{p,H_{d0}}$) is schematically illustrated in Fig. 8. Particularly, when $\Delta g_{p,H_{d0}}$ is upward-continued in the free-air space to the station level at P, it is interesting that the gravity disturbance ($\Delta g_{p,H_{d0}}$ at P) does not change the value even though the removed or moved topographic masses are completely restored (cf. Eqs. (29) and (32)).

From Eqs. (32), (33) and (34), the relations of $\Delta g_{p,H_{d1}}$ and $\Delta g_{p,H_{d2}}$ to the *ρ_B-free Bouguer anomaly* $\Delta g_{p,H_{d0}}$ are given as

$$\Delta g_{p,H_{d1}} = \Delta g_{p,H_{d0}} - \int_{H_0}^{H_{d1}} 2\pi G\rho_B [H^+(1,\psi) - H^-(1,\psi)] dr + \int_{H_{d0}}^{H_{d1}} \frac{\partial \Delta g}{\partial r} dr \tag{37}$$

and

$$\Delta g_{p,H_{d2}} = \Delta g_{p,H_{d0}} - \int_{H_0}^{H_p} 2\pi G\rho_B[H^+(1,\psi) - H^-(1,\psi)]dr + \int_{H_{d0}}^{H_{d2}} \frac{\partial \Delta g}{\partial r} dr, \tag{38}$$

respectively.

4.5 Relation to the fundamental equation of physical geodesy

Since $H_0 = -N$, it is shown below that Eq. (35) has a tie to the fundamental equation of physical geodesy (Heiskanen and Moritz, 1967, equation (2-148), p. 86).

Regarding VGG of the normal gravity field as constant, Eq. (35) yields

$$FA = \Delta g_{p,H_{d0}} + (-H_0)\frac{\partial \gamma}{\partial r}. \tag{39}$$

On the other hand, the fundamental equation of physical geodesy is written as

$$\Delta g = -\frac{\partial T}{\partial r} + \frac{T}{\gamma}\frac{\partial \gamma}{\partial r}, \tag{40}$$

where, Δg denotes the (geodetic) gravity anomaly, and T denotes the gravity disturbing potential. Then, using the relation

$$\delta g = -\frac{\partial T}{\partial r}$$

and the Bruns' formula

$$T = N\gamma,$$

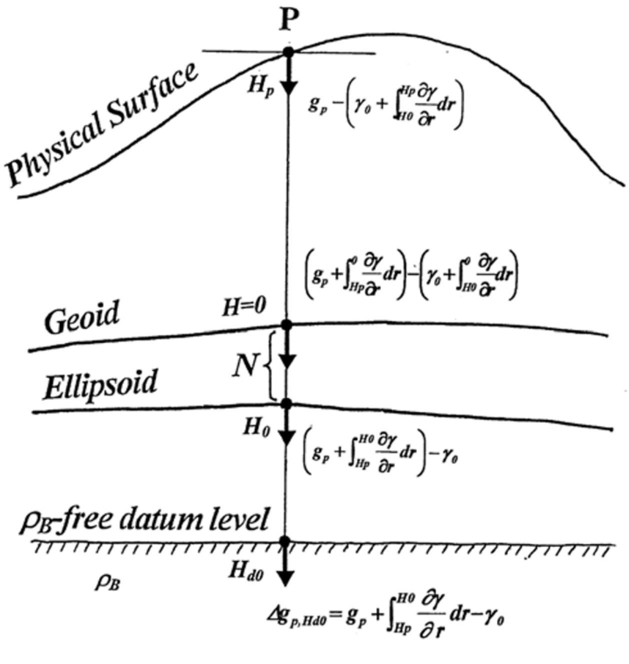

Fig. 8. Schematic illustration that represents the equivalence of the generalized Bouguer anomaly at the ρ_B*-free datum level* H_{d0} to the gravity disturbance at any datum level. The VGG anomaly $(\partial \Delta g/\partial r)$ is neglected here. N denotes the geoid height ($N = -H_0$).

Eq. (40) is written as

$$\Delta g = \delta g + N\frac{\partial \gamma}{\partial r}, \tag{41}$$

which is equivalent to the fundamental equation of physical geodesy. Comparing Eqs. (39) and (41), it is clear that these two equations are similar to each other, identifying FA with Δg, $\Delta g_{p,H_{d0}}$ with δg, and $-H_0$ with N.

5. Estimation of the Bouguer Reduction Density

We will show in this section that the Bouguer reduction density ρ_B is estimated by the plot of FA against the specific datum levels, and also H_0 is estimated on the same plot.

5.1 Derivation of the equation for estimating the Bouguer reduction density

As was explained previously, each of the specific datum levels H_{d0}, H_{d1} and H_{d2}, upon which the generalized Bouguer anomalies $\Delta g_{p,H_{d0}}$, $\Delta g_{p,H_{d1}}$, $\Delta g_{p,H_{d2}}$ are defined, forms a surface as a function of the horizontal coordinates (x, y): $H_{d0} = H_{d0}(x, y)$, $H_{d1} = H_{d1}(x, y)$ and $H_{d2} = H_{d2}(x, y)$.

Here, we shall notice that Eqs. (32), (33) and (34), which represent the generalized Bouguer anomalies at the specific datum levels, hold for every point of horizontal coordinates (x, y). Therefore, one can consider the differential quantities of the generalized Bouguer anomalies $\Delta g_{p,H_{d0}}$, $\Delta g_{p,H_{d1}}$, and $\Delta g_{p,H_{d2}}$ with respect to the specific datum levels H_{d0}, H_{d1} and H_{d2} in the neighbourhood of (x, y), respectively. In this case, it is necessary to differentiate not only $\Delta g_{p,H_{d0}}$, $\Delta g_{p,H_{d1}}$, $\Delta g_{p,H_{d2}}$ and FA, but also H_p and $TC_p(1)$, since they are functions of H_{d0}, H_{d1} and H_{d2} that are functions of x and y.

Differentiating Eqs. (29), (30) and (31) with respect to H_{d0}, H_{d1} and H_{d2}, respectively, we have

$$\frac{d\Delta g_{p,H_{d0}}}{dH_{d0}} = \frac{dFA}{dH_{d0}} + \left(1 - \frac{dH_p}{dH_{d0}}\right)\frac{\partial \Delta g}{\partial r}, \tag{42}$$

$$\frac{d\Delta g_{p,H_{d1}}}{dH_{d1}} = \frac{dFA}{dH_{d1}} + \left(1 - \frac{dH_p}{dH_{d1}}\right)\frac{\partial \Delta g}{\partial r} - 2\pi G\rho_B[H^+(1,\psi) - H^-(1,\psi)], \tag{43}$$

and

$$\frac{d\Delta g_{p,H_{d2}}}{dH_{d2}} = \frac{dFA}{dH_{d2}} + \left(1 - \frac{dH_p}{dH_{d2}}\right)\frac{\partial \Delta g}{\partial r} - \left(\frac{dH_p}{dH_{d2}}\right)2\pi G\rho_B[H^+(1,\psi) - H^-(1,\psi)]. \tag{44}$$

As for the derivation of these equations, refer to Appendix B.

In the above calculation, dH_p/dH_{d0}, dH_p/dH_{d1} and dH_p/dH_{d2} are taken into account because H_{d0}, H_{d1} and H_{d2} are not independent variables but functions of H_p. Besides, the elevation of a gravity station H_p is a function of the horizontal coordinates (x, y). Therefore, the total differential of the generalized Bouguer anomaly at the specific datum level ($d\Delta g_{p,H_{di}}$, for $i = 0, 1, 2$) can be written as

$$d\Delta g_{p,H_{di}} = \frac{\partial \Delta g_{p,H_{di}}}{\partial x} dx + \frac{\partial \Delta g_{p,H_{di}}}{\partial y} dy + \frac{\partial \Delta g_{p,H_{di}}}{\partial H_{di}} dH_{di}, \quad (\text{for } i = 0, 1, 2) \quad (45)$$

as a function of the horizontal coordinates (x, y) and H_{di}, $(i = 0, 1, 2)$. When we assume that each $\Delta g_{p,H_{di}}$, $(i = 0, 1, 2)$ takes a constant value in the neighborhood of (x, y), i.e. the lateral variation of $\Delta g_{p,H_{di}}$ is sufficiently small, Eq. (45) yields

$$\frac{d\Delta g_{p,H_{di}}}{dH_{di}} \approx \frac{\partial \Delta g_{p,H_{di}}}{\partial H_{di}}, \quad (\text{for } i = 0, 1, 2). \quad (46)$$

Since the partial differential coefficients satisfy the equation

$$\frac{\partial \Delta g_{p,H_{d0}}}{\partial H_{d0}} = \frac{\partial \Delta g_{p,H_{d1}}}{\partial H_{d1}} = \frac{\partial \Delta g_{p,H_{d2}}}{\partial H_{d2}},$$

Eq. (46) leads directly to the following approximate relation:

$$\frac{d\Delta g_{p,H_{d0}}}{dH_{d0}} = \frac{d\Delta g_{p,H_{d1}}}{dH_{d1}} = \frac{d\Delta g_{p,H_{d2}}}{dH_{d2}}. \quad (47)$$

Thus, subtracting Eq. (42) from Eq. (43) and using Eq. (47), and assuming constant VGG anomaly, $(\partial \Delta g/\partial r) = \Delta\beta$, we have

$$2\pi G\rho_B[H^+(1, \psi) - H^-(1, \psi)] + \left(\frac{dH_p}{dH_{d1}} - \frac{dH_p}{dH_{d0}}\right)\Delta\beta = \frac{dFA}{dH_{d1}} - \frac{dFA}{dH_{d0}}. \quad (48)$$

From Eqs. (43) and (44), one can also derive the result equivalent to Eq. (48). We shall notice again that all quantities in Eq. (48) but for ρ_B and $\Delta\beta$ are known. Differential coefficients in Eq. (48) have the following relations:

$$\frac{dH_p}{dH_{d1}} = \frac{H^+(1, \psi)}{H^-(1, \psi)} \frac{dH_p}{dH_{d0}}, \quad (49)$$

and

$$\frac{dFA}{dH_{d1}} = \frac{H^+(1, \psi)}{H^-(1, \psi)} \frac{dFA}{dH_{d0}}. \quad (50)$$

These relations of Eqs. (49) and (50) can be derived from Eq. (18).

When the VGG anomaly $\Delta\beta$ is sufficiently small, Eq. (48) gives the following equation

$$\rho_B \approx \left(\frac{dFA}{dH_{d1}} - \frac{dFA}{dH_{d0}}\right) \Big/ 4\pi G. \quad (51\text{-}1)$$

Thus, we have the final approximate equation, Eq. (51-1), for estimating the Bouguer reduction density ρ_B. Equation (51-1) means that ρ_B is calculated from the gradients of FA with respect to the specific datum levels. Concrete method for evaluating the gradients dFA/dH_{d0} and dFA/dH_{d1} will be described in the next section. The effect of the VGG anomaly ($\Delta\beta$) on the Bouguer reduction density estimation can be evaluated by Eq. (48).

5.2 FA vs. H_{d0}, H_{d1} and H_{d2} diagram

Each free-air anomaly FA in Eqs. (32), (33) and (34), which is a computable quantity, is a function of each specific datum level H_{d0}, H_{d1} and H_{d2}, respectively. Here, we plot the free-air anomaly (FA) against the datum level H_d (e.g. $H_d = H_{d0}$, H_{d1} or H_{d2}) for every gravity station. Then, we obtain generally a set of plots as schematically shown in Fig. 9. In this paper, we call the diagram of these plots 'FA vs. H_d diagram'.

The characteristics found in the FA vs. H_d diagram are as follows:

(1) One can measure the gradient of the regression lines on the FA vs. H_d diagram. Namely, the gradient dFA/dH_{d0} of the regression line for the FA vs. H_{d0} plot. Similarly, the gradients dFA/dH_{d1} and dFA/dH_{d2} of the regression lines for the FA vs. H_{d1}, and H_{d2} plots respectively.

(2) There exists, in general, an intersection point C of the regression lines H_{d1}-line and H_{d2}-line. The intersection point C does not generally coincide with the origin (0, 0). Notice, that the position of the intersection point C is definite because the H_{d1} and the H_{d2} are given by Eqs. (16) and (17).

(3) Also, one can draw a regression line H_{d0}-line on the FA vs. H_{d0} plot so that it passes through the intersection point C ($H_d = \Delta H_d$, $FA = \Delta\gamma_0$). Notice that H_{d0}-line in Fig. 9 has a degree of freedom for parallel translation along the axis H_d, because the equation of H_{d0} (Eq. (14)) contains H_0 as an unknown parameter.

5.2.1 Evaluation of the gradients dFA/dH_{d0} and dFA/dH_{d1}

Based on the characteristics (1) in the above section, Eq. (51-1) shows that one can estimate the Bouguer reduction density ρ_B from the gradients dFA/dH_{d0} of the H_{d0}-line and dFA/dH_{d1} of the H_{d1}-line. Particularly when $\psi \sim 0$, $\Delta\beta = 0$ and $H_0 = 0$, which is the case for the flat Earth, the relation between the gradient dFA/dH_{d1} of the H_{d1}-line and the gradient dFA/dH_{d0} of the H_{d0}-line is

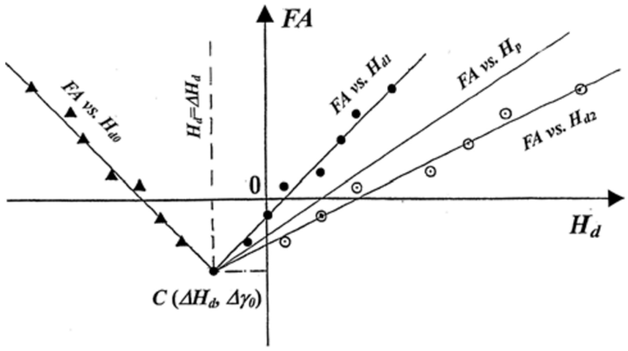

Fig. 9. Typical FA vs. H_d diagram. The intersection point C (ΔH_d, $\Delta\gamma_0$) is definite because the regression lines H_{d1}-line for FA vs. H_{d1} plot and H_{d2}-line for FA vs. H_{d2} plot are definite. One can draw the regression line H_{d0}-line for FA vs. H_{d0} plot so that it passes through the definite intersection point C. In case of the flat Earth approximation of the gravity corrections, FA vs. H_{d0} plot and FA vs. H_{d1} plot become symmetric with respect to the vertical line $H_d = \Delta H_d$.

$dFA/dH_{d1} = -dFA/dH_{d0}$. Hence, Eq. (51-1) yields

$$\rho_B \approx \left(\frac{dFA}{dH_{d1}}\right) \bigg/ 2\pi G \tag{51-2}$$

which is essentially equivalent to the result given by Hagiwara *et al.* (1986) for estimating the Bouguer reduction density. This is a kind of the Nettleton's method (Nettleton, 1939) for density determination.

5.2.2 Estimation of H_0 On the FA vs. H_d diagram (Fig. 9), let the position of the intersection point C between the regression lines H_{d1}-line and H_{d2}-line be $H_d = \Delta H_d$, and $FA = \Delta\gamma_0$. Then, the intersecting condition $H_{d1} = H_{d2}$ (i.e. Eq. (16) =Eq. (17)) leads to

$$TC_p(1) = 0 \quad \text{and} \quad H_p = \Delta H_d \quad \text{(at the point C)}. \tag{52}$$

Next, based on the above characteristics (3), we can adjust the H_{d0}-line so as to pass through the intersection point C. Then, the intersecting condition $H_{d0} = H_{d1}$, (i.e. Eq. (14) = Eq. (16)), leads to

$$H_0 = H_p. \tag{53}$$

Finally we have, from Eqs. (53) and (52), at the intersection point C

$$H_0 = \Delta H_d. \tag{54}$$

Of course, the geoid height N is given as $N = -H_0 = -\Delta H_d$.

In addition to the above results, it can be shown that the fundamental equation of physical geodesy, equivalently Eq. (41), is satisfied at the intersection point C ($H_d = \Delta H_d$, $FA = \Delta\gamma_0$). Substituting $FA = \Delta\gamma_0$ and $H_0 = H_{d0} = \Delta H_d$ into Eq. (35), we have

$$\Delta g_{p,H_{d0}} = \Delta\gamma_0 - \int_{\Delta H_d}^{0} \frac{\partial\gamma}{\partial r} dr.$$

Because of the correspondence between Eqs. (39)–(41) (i.e. $\Delta g_{p,H_{d0}} = \delta g$, and $\Delta\gamma_0 = FA = \Delta g$, and $\Delta H_d = H_{d0} = H_0 = -N$), this equation yields Eq. (41), which is equivalent to the fundamental equation of physical geodesy (Heiskanen and Moritz, 1967). This equation shows that $\Delta g_{p,H_{d0}}$ can be computed from the known quantities ΔH_d and $\Delta\gamma_0$.

6. Derivation of the Bouguer Anomaly at the Geoid from the ρ_B-*free* *Bouguer Anomaly* at H_{d0}

The Bouguer anomaly has been used for estimating subsurface structure. The primary purpose of defining the generalized Bouguer anomaly is to obtain the Bouguer anomaly which is free from the density assumption used in the Bouguer correction as well as in the terrain correction.

So far, we have found that the ρ_B-*free Bouguer anomaly* $\Delta g_{p,H_{d0}}$ is realized upon the ρ_B-*free specific datum level* of gravity reduction H_{d0}, which is not restricted to the geoid surface. Furthermore, the ρ_B-*free Bouguer anomaly* at H_{d0}, $\Delta g_{p,H_{d0}}$, is nothing but the gravity disturbance in the theory of physical geodesy. Thus, in order to estimate subsurface density structure, we have to reduce the generalized Bouguer anomaly from the ρ_B-*free specific datum level* of H_{d0} to the geoid surface. When such a reduction is done, one can study the subsurface density structure by Tsuboi's double Fourier method (Tsuboi, 1938; Tsuboi and Fuchida, 1938).

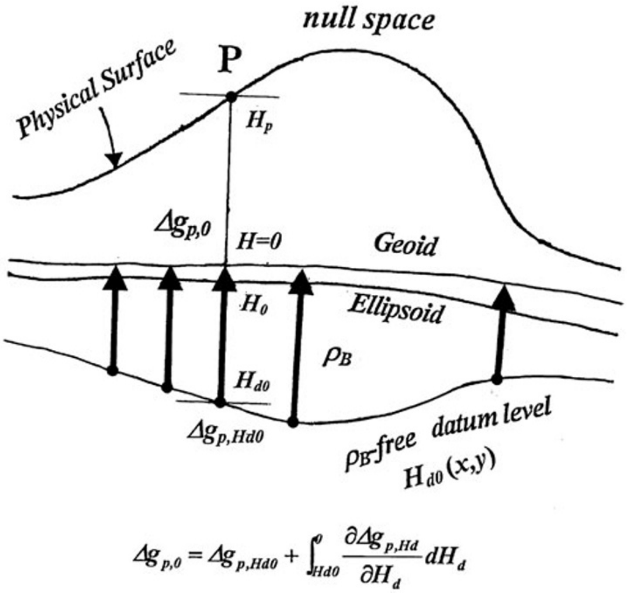

Fig. 10. Schematic illustration of computing the Bouguer anomaly on the geoid surface ($\Delta g_{p,0}$). For each gravity station, $\Delta g_{p,0}$ can be computed by the level transformation from the generalized Bouguer anomaly at the ρ_B-*free datum level* H_{do} ($\Delta g_{p,H_{d0}}$). Each arrow indicates the amount of level transformation from the ρ_B-*free datum level* to the level of the geoid. $\Delta g_{p,0}$ is equivalent to the classical Bouguer anomaly.

Now we will show how the ρ_B-*free Bouguer anomaly* $\Delta g_{p,H_{d0}}$ is reduced to the geoid. The generalized Bouguer anomaly at the geoid surface $\Delta g_{p,0}$ is calculated by the level transformation of $\Delta g_{p,H_{d0}}$ from the datum level H_{d0} to the level of the geoid ($H_d = 0$). Then, we have

$$\Delta g_{p,0} = \Delta g_{p,H_{d0}} + \int_{H_{d0}}^{0} \frac{\partial \Delta g_{p,H_d}}{\partial H_d} dH_d. \tag{55}$$

Therefore, by using the reduction rate of Eq. (12), we can rewrite Eq. (55) as

$$\Delta g_{p,0} = \Delta g_{p,H_{d0}} + \int_{H_{d0}}^{0} \left[2\pi G\rho_B H^-(1,\psi) + \frac{\partial \Delta g}{\partial r}\right] dH_d. \tag{56}$$

When we ignore the VGG anomaly, Eq. (56) yields

$$\Delta g_{p,0} \approx \Delta g_{p,H_{d0}} - 2\pi G\rho_B H^-(1,\psi) H_{d0}. \tag{57}$$

This is the final approximate representation of $\Delta g_{p,0}$ that is represented upon an equi-potential surface of the geoid. Such a reduction of the generalized Bouguer anomaly $\Delta g_{p,H_{d0}}$ from the specific datum level H_{d0} to the level of the geoid is schematically illustrated in Fig. 10. It should be noted that we require the values of ρ_B and H_{d0} (or H_0) for computing $\Delta g_{p,0}$.

Alternatively, substituting Eq. (35) into Eq. (57), we have

$$\Delta g_{p,0} \approx FA - \int_{H_0}^{0} \frac{\partial\gamma}{\partial r} dr - 2\pi G\rho_B H^-(1,\psi) H_{d0}. \tag{58}$$

Equation (58) gives the relation between the gravity disturbance on the geoid ($\Delta g_{p,0}$), that is, the gravity anomaly in the Molodensky sense (FA), and the Bouguer reduction density ρ_B. Furthermore, substituting Eq. (51-1) into the density ρ_B in Eq. (58), we have another expression of the generalized Bouguer anomaly at the geoid ($\Delta g_{p,0}$) in the form

$$\Delta g_{p,0} \approx FA - \int_{H_0}^{0} \frac{\partial \gamma}{\partial r} dr - \frac{H^{-}(1,\psi)}{2}\left(\frac{dFA}{dH_{d1}} - \frac{dFA}{dH_{d0}}\right) H_{d0}. \quad (59)$$

In Eq. (59), the Bouguer reduction density ρ_B is represented in terms of FA. Notice that all the terms including H_0 on the right-hand side of Eq. (59) are computable quantities (see Eqs. (51) and (54)). The distribution of $\Delta g_{p,0}$ calculated by Eq. (58) or (59) is nothing but the Bouguer anomaly distribution, which has been used to study the subsurface density structure (e.g. Tsuboi and Fuchida, 1938).

7. Conclusions

(1) We defined a new concept of the generalized Bouguer anomaly upon an arbitrary datum level whose elevation from the geoid is H_d (see Eq. (1)).

(2) Three specific datum levels H_{d0}, H_{d1} and H_{d2} are defined for every gravity station (see Eqs. (14), (16) and (17)). The specific datum level H_{d0}, so-called the *ρ_B-free datum level*, is defined by a condition that the generalized Bouguer anomaly is independent of the Bouguer reduction density ρ_B (see Eq. (13)). The specific datum levels of H_{d1} and H_{d2} are defined by a condition that the sum of the terrain and the Bouguer corrections is zero (see Eq. (15)).
The specific datum levels H_{d1} and H_{d2} can be computed in practice for each gravity station from the value of terrain correction for the unit density $TC_p(1)$. H_{d0} can be computed if the elevation of the reference ellipsoid H_0 is known. A method to evaluate H_0 is discussed (Eq. (54)).

(3) Three specific generalized Bouguer anomalies $\Delta g_{p,H_{d0}}$, $\Delta g_{p,H_{d1}}$ and $\Delta g_{p,H_{d2}}$ are derived for every gravity station at their specific datum levels H_{d0}, H_{d1} and H_{d2}, respectively (see Eqs. (32), (33) and (34)).
The specific generalized Bouguer anomaly $\Delta g_{p,H_{d0}}$, the *ρ_B-free Bouguer anomaly*, does not include the Bouguer reduction density ρ_B (see Eq. (32)) and is therefore free from the assumption of ρ_B. The *ρ_B-free Bouguer anomaly* $\Delta g_{p,H_{d0}}$ is essentially equal to the 'gravity disturbance' in the physical geodesy (Eq. (36)).
The specific generalized Bouguer anomalies $\Delta g_{p,H_{d1}}$ and $\Delta g_{p,H_{d2}}$, as well as $\Delta g_{p,H_{d0}}$, do not include the terrain correction explicitly and are not affected by the topographic gravitational effects (see Eqs. (33) and (34)).

(4) When the terms of the VGG anomaly are sufficiently small i.e. $\partial \Delta g/\partial r = 0$, we found that the generalized Bouguer anomaly at H_{d0} (i.e. the *ρ_B-free Bouguer anomaly* $\Delta g_{p,H_{d0}}$) is equal to FA minus free-air correction from the reference ellipsoid to the geoid (Eq. (35)). FA is defined as the difference of the free-air corrected observed gravity upon the geoid from the normal gravity γ_0 upon the reference ellipsoid (see Eq. (25)). We found that FA is equal to the 'gravity anomaly' in the Molodensky's sense.
Relation between the *ρ_B-free Bouguer anomaly* ($\Delta g_{p,H_{d0}}$) and the fundamental equation of physical geodesy is discussed (Eqs. (39), (40) and (41)).

(5) A method for estimating the Bouguer reduction density ρ_B is found. The Bouguer reduction density is given by the difference between the gradients of FA with respect to H_{d0} and H_{d1}, respectively (Eq. (51)). Also a condition equation to be satisfied by the Bouguer reduction density ρ_B and the VGG anomaly $\Delta\beta$ is derived (Eq. (48)). It can be used for evaluating the influence of $\Delta\beta$ on the ρ_B estimation.

(6) The generalized Bouguer anomaly upon the geoid ($\Delta g_{p,0}$) is obtained from the *ρ_B-free Bouguer anomaly* $\Delta g_{p,H_{d0}}$. It is done by the level transformation of the gravity value from the specific datum level H_{d0} to the geoid (see Eq. (55)), using the reduction rate of $2\pi G\rho_B H^{-}(1,\psi)$ (Eq. (12)). The generalized Bouguer anomaly distribution upon the geoid, $\Delta g_{p,0}$, will be used for estimating the subsurface structure.

Acknowledgments. The author expresses sincere thanks to Professor Emeritus Shozaburo Nagumo of the University of Tokyo, the chief technical adviser of OYO Corporation, for his encouragement and frequent discussions throughout the research. Also the author wishes to express his grateful thanks to Professor Shuhei Okubo of the University of Tokyo for his hearty guidance and critical comments, and to Professor Emeritus Yoshio Fukao of the University of Tokyo for his stimulating discussions and continuous encouragement during the research. The author's thanks are also extended to Drs. Masaru Kaidzu and Yuki Kuroishi of the Geographical Survey Institute for their helpful discussions on the basic concepts of the physical geodesy, and to Professor Emeritus Yukio Hagiwara of the University of Tokyo for valuable comments on the latest geodesy. The contribution of Dr. Gabriel Strykowski, National Survey and Cadastre, Denmark, who gave insightful and instructive comments on the paper, is gratefully acknowledged. The author also expresses his sincere thanks to Professor W. E. Featherstone, Curtin University of Technology, Australia, for his critical comments with his recent publications on the 'gravity anomaly'. Finally, the author gratefully acknowledges to Dr. Petr Holota, Research Institute of Geodesy, Topography and Cartography, Czech Republic, for his helpful comments to improve the manuscript.

Appendix A.

Effects of the Earth's sphericity and the truncation angle of the spherical shell

Figure A1 shows a schematic illustration of a thin spherical cap of axial symmetry. Let Δh be the thickness of the thin spherical cap, ρ the density, ψ the truncation angle, and $t = r/r^{\pm}$ the normalized geocentric distance of the spherical cap. Then, the gravity ($g^{\pm}$) due to the thin spherical cap at a station $P^{\pm}$ on the symmetry axis can be written as

$$g^{\pm} = G\rho r^{\pm} \int_{t}^{t+\Delta t} \int_{0}^{\psi} \int_{0}^{2\pi} \frac{t^2 \sin\psi (1 - t\cos\psi)}{(t^2 + 1 - 2t\cos\psi)^{3/2}} d\alpha d\psi dt, \quad \text{(A.1)}$$

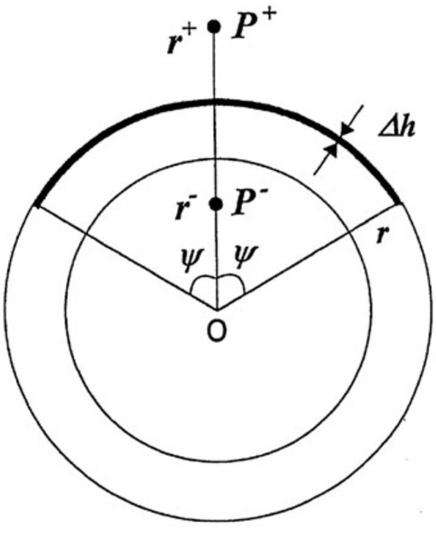

Fig. A1. Schematic illustration of a thin spherical cap with a small thickness Δh and a truncation angle ψ. r denotes the radial distance of the spherical cap. P^+ and P^- denote the computation points of gravity at the radial distances r^+ and r^-, respectively.

(double signs should be taken in the same order; the same as below), where, α is the azimuthal angle and $\Delta t = \Delta h / r^{\pm}$.

Executing the integration of Eq. (A.1) with respect to α and ψ, we have

$$g^{\pm} = 2\pi G\rho r^{\pm} \int_t^{t+\Delta t} \left(\frac{t^3 - t^2 \cos\psi}{\sqrt{t^2 + 1 - 2t\cos\psi}} \pm t^2 \right) dt. \tag{A.2}$$

By the Taylor expansion of Eq. (A.2) in the neighbourhood of $t = t$, and neglecting the higher order terms of more than or equal to $(\Delta t)^2$ under the condition of $\Delta t \ll t$, Eq. (A.2) yields

$$g^{\pm} \approx 2\pi G\rho H^{\pm}(t, \psi)\Delta h, \quad (\Delta h = r^{\pm}\Delta t) \tag{A.3}$$

(Nozaki, 1999), where,

$$H^{\pm}(t, \psi) := \frac{t^3 - t^2 \cos\psi}{\sqrt{t^2 + 1 - 2t\cos\psi}} \pm t^2. \tag{A.4}$$

Clearly, from Eq. (A.3), $H^{\pm}(t, \psi)$ defined by Eq. (A.4) is a characteristic function that governs the gravitational behaviour of the spherical cap. A graph of $H^{\pm}(t, \psi)$ is shown in Fig. A2 as a function of angular distance (or truncation angle) ψ with an argument t. Especially, when t takes a limit value $t = 1$, Eq. (A.4) yields

$$H^{\pm}(1, \psi) = \sqrt{\frac{1 - cos\psi}{2}} \pm 1. \tag{A.5}$$

$H^{\pm}(1, \psi)$ means $H^{\pm}(1 \mp \varepsilon, \psi)$, where ε denotes an infinitesimally small positive number. The first term on the right-hand side of Eq. (A.5) corresponds to the term of sphericity, and the second term does to that of an infinite plate or a Bouguer slab. When ψ becomes small enough (i.e. $\psi \approx 0$), Eq. (A.3) agrees with gravitational attraction of an infinite plate, i.e. $\pm 2\pi G\rho\Delta h$. On the other hand, when $\psi \approx \pi$, $H^+(1, \psi) = 2$ and Eq. (A.3) becomes $4\pi G\rho\Delta h$ on the outer surface of a thin spherical shell of thickness Δh. On the inner surface of the spherical shell, $H^-(1, \psi) = 0$ and Eq. (A.3) yield zero. From Eq. (A.5), clearly holds for the following identical equation:

$$H^+(1, \psi) - H^-(1, \psi) \equiv 2. \tag{A.6}$$

This relation can be confirmed on the Fig. A2 that the lines for $H^+(1-\varepsilon, \psi)$ and $H^-(1+\varepsilon, \psi)$ are parallel to each other with the distance of 2. Referring to Eq. (A.3), Eq. (A.6) implies that the gravity difference between the immediately upper and lower points on the spherical cap with a small thickness Δh is always equal to $4\pi G\rho\Delta h$ regardless of the truncation angle ψ.

Appendix B.

Differentiation of equations of the generalized Bouguer anomaly with respect to the specific datum levels H_{d0}, H_{d1} and H_{d2}

In this appendix, we demonstrate that the differentiation of Eqs. (29), (30) and (31) in the text with respect to the specific datum levels H_{d0}, H_{d1} and H_{d2} results in Eqs. (42), (43) and (44) in the text, respectively.

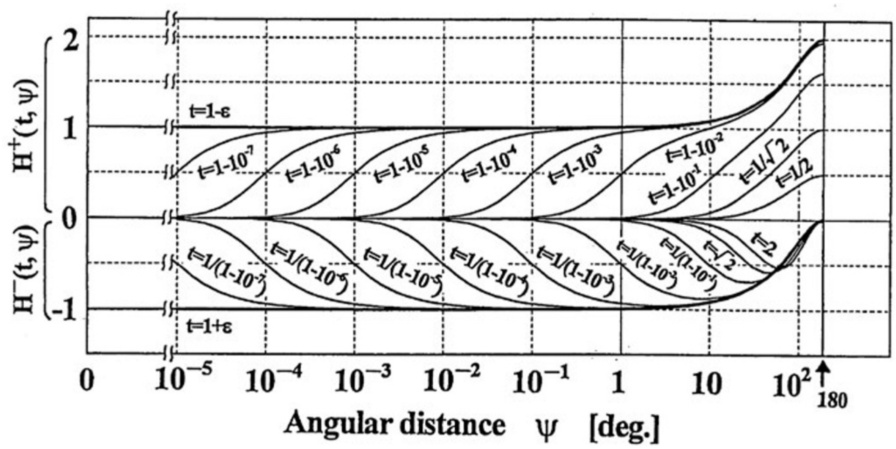

Fig. A2. A graph of the characteristic function $H^{\pm}(1, \psi)$ as a function of the truncation angle ψ with an argument t (after Nozaki, 1999). The immediately upper and lower points on the spherical cap correspond to $t = 1 - \varepsilon$ and $t = 1 + \varepsilon$, respectively, where ε is an infinitesimally small positive number. Simplified notations $H^+(1, \psi) = H^+(1 - \varepsilon, \psi)$ and $H^-(1, \psi) = H^-(1 + \varepsilon, \psi)$ are used in the text.

Here, we shall notice that Eqs. (32), (33) and (33) in the text hold corresponding to every point of horizontal coordinates (x, y). Therefore, one can consider the differential quantities in the neighbourhood of (x, y). In this case, it is necessary to differentiate not only $\Delta g_{p,H_{d0}}$, $\Delta g_{p,H_{d1}}$, $\Delta g_{p,H_{d2}}$ and FA but also H_p and $TC_p(1)$, since they are functions of H_{d0}, H_{d1} and H_{d2} that are functions of x and y.

Differentiating Eqs. (29), (30) and (31) in the text with respect to H_{d0}, H_{d1} and H_{d2}, respectively, we have

$$\begin{aligned}\frac{d\Delta g_{p,H_{d0}}}{dH_{d0}} &= \frac{dFA}{dH_{d0}} + \left(1 - \frac{dH_p}{dH_{d0}}\right)\frac{\partial \Delta g}{\partial r} \\ &\quad + \rho_B\left\{\frac{dTC_p(1)}{dH_{d0}}\right. \\ &\quad + \left(1 - \frac{dH_p}{dH_{d0}}\right)2\pi G H^+(1,\psi) \\ &\quad \left. - 2\pi G[H^+(1,\psi) - H^-(1,\psi)]\right\}, \\ &\quad (\text{for } H_{d0} < H_p), \qquad \text{(B.1)}\end{aligned}$$

$$\begin{aligned}\frac{d\Delta g_{p,H_{d1}}}{dH_{d1}} &= \frac{dFA}{dH_{d1}} + \left(1 - \frac{dH_p}{dH_{d1}}\right)\frac{\partial \Delta g}{\partial r} \\ &\quad + \rho_B\left[\frac{dTC_p(1)}{dH_{d1}}\right. \\ &\quad \left. + \left(1 - \frac{dH_p}{dH_{d1}}\right)2\pi G H^+(1,\psi)\right] \\ &\quad - 2\pi G\rho_B[H^+(1,\psi) - H^-(1,\psi)], \\ &\quad (\text{for } H_{d1} < H_p), \qquad \text{(B.2)}\end{aligned}$$

and

$$\begin{aligned}\frac{d\Delta g_{p,H_{d2}}}{dH_{d2}} &= \frac{dFA}{dH_{d2}} + \left(1 - \frac{dH_p}{dH_{d2}}\right)\frac{\partial \Delta g}{\partial r} \\ &\quad + \rho_B\left[\frac{dTC_p(1)}{dH_{d2}}\right. \\ &\quad \left. + \left(1 - \frac{dH_p}{dH_{d2}}\right)2\pi G H^-(1,\psi)\right] \\ &\quad - \left(\frac{dH_p}{dH_{d2}}\right)2\pi G\rho_B[H^+(1,\psi) \\ &\quad - H^-(1,\psi)], \quad (\text{for } H_{d2} > H_p). \qquad \text{(B.3)}\end{aligned}$$

On the other hand, differentiating Eqs. (14), (16) and (17) in the text with respect to H_{d0}, H_{d1} and H_{d2}, respectively, we have

$$\begin{aligned}\frac{dTC_p(1)}{dH_{d0}} &= \left(\frac{dH_p}{dH_{d0}}\right)2\pi G H^+(1,\psi) \\ &\quad - 2\pi G H^-(1,\psi), \qquad \text{(B.4)}\end{aligned}$$

$$\frac{dTC_p(1)}{dH_{d1}} = -\left(1 - \frac{dH_p}{dH_{d1}}\right)2\pi G H^+(1,\psi), \qquad \text{(B.5)}$$

and

$$\frac{dTC_p(1)}{dH_{d2}} = -\left(1 - \frac{dH_p}{dH_{d2}}\right)2\pi G H^-(1,\psi). \qquad \text{(B.6)}$$

Substituing Eqs. (B.4), (B.5) and (B.6) into Eqs. (B.1), (B.2) and (B.3), respectively, and arranging the terms, we have Eqs. (42), (43) and (44) in the text.

Appendix C.

A Note on the Reference Gravity

C.1 A comparison of the Prey-reduced reference gravity field and the normal gravity field

It is stated in an old text book by Garland (1965, p. 50) that an anomaly of gravity is defined by the *difference* between the observed (or reduced) gravity at some point and the *theoretical* value predicted for the same point. Therefore, for defining the difference of gravity at the geoid, the normal gravity will not be the only one *theoretical* value. Other theoretical values are possible. Selection of an appropriate value depends on the purpose of the anomaly (Garland, 1965, p. 59). The Prey-reduced reference gravity field introduced newly in this paper is one of such *theoretical* ones, and is not the normal gravity field.

C.2 The normal gravity field (a review) (see Fig. C1)

The normal gravity at the ellipsoid is defined by (Heiskanen and Moritz, 1967, equation (2-78))

$$\gamma = \gamma_0 \equiv \gamma^N_{H_0} \qquad \text{(C.1)}$$

where the right side term $\gamma^N_{H_0}$ is a new notation introduced here. The normal gravity at the normal height H^N (notation after Torge, 1989), $\gamma^N_{H^N}$, is defined by the first order approximation (Heiskanen and Moritz, 1967, equation (8-8), p. 293)

$$\gamma = \gamma(at\ H^N) \equiv \gamma^N_{H^N} \approx \gamma_0 + \left(\frac{\partial \gamma}{\partial h}\right)H^N. \qquad \text{(C.2)}$$

The normal gravity is defined upon and outside the reference ellipsoid, and not inside the reference ellipsoid.

C.3 Mass density distribution in the normal gravity field

Outside the reference ellipsoid, the space is free-air, i.e., the mass density is zero. Inside the reference ellipsoid, the mass density distribution does not need to be known (Heiskanen and Moritz, 1967, Sec. 2–7, p. 64). However, the amount of the total mass inside the reference ellipsoid is constant. The simplest mathematical model of such a

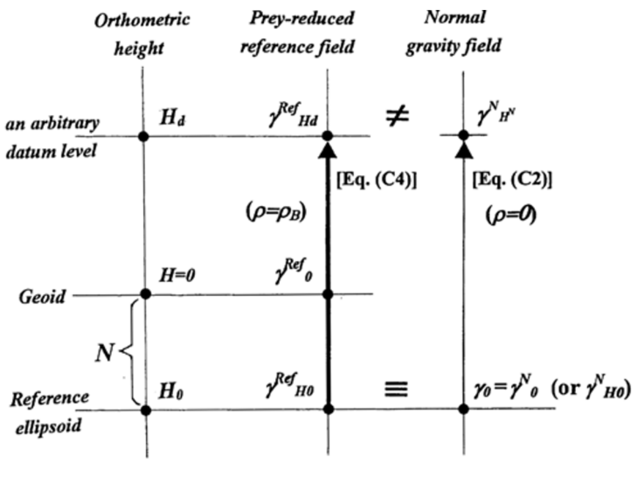

Fig. C1. Relation between the Prey-reduced reference gravity field and the normal gravity field. The reference ellipsoid connects $\gamma^{Ref}_{H_0}$ and γ^N_0 (or $\gamma^N_{H_0}$). $\gamma^{Ref}_{H_d} \equiv \gamma_{H_d}$ in the text.

mass density distribution is free-air with the centralized total mass. Another mathematical model is such that a mass density distribution inside the reference ellipsoid is constrained by the Clairaut's equation (Moritz, 1990, equations (2-114) and (4-4)).

C.4 Reference gravity field introduced in this paper (see Fig. C1)

The model earth of the reference gravity field introduced in this paper is massive, having a surface layer above and below the reference ellipsoid. The reference gravity field is defined only inside of the model earth, and not outside. The purpose of defining such a reference gravity field is to introduce the generalized Bouguer anomaly. The surface of the reference ellipsoid is common in both the normal gravity field and the reference gravity field.

The reference gravity of this paper *at the reference ellipsoid*, $\gamma_{H_0}^{Ref}$, (i.e. notation γ_{H_0} in this paper) is taken as the same as the normal gravity γ_0, as shown in Fig. C1,

$$\gamma_{H_0}^{Ref} \equiv \gamma_0 \equiv \gamma_{H_0}^{N}. \tag{C.3}$$

The reference gravity *at an arbitrary orthometric height* H_d, $\gamma_{H_d}^{Ref}$, within the near surface layer, in which the mass density is ρ_B, is defined by the next equation

$$\gamma_{H_d}^{Ref} = \gamma_0 + \int_{H_0}^{H_d} 2\pi G \rho_B [H^+(1,\psi) - H^-(1,\psi)] dr + \int_{H_0}^{H_d} \frac{\partial \gamma}{\partial r} dr. \tag{C.4}$$

On the right-hand side of Eq. (C.4), the second and the third terms are the Prey reduction. The use of the Prey reduction is due to the level transformation of the gravity value within the mass. Thus, we call the reference gravity field the Prey-reduced reference gravity field.

We remember that the normal gravity *inside* the reference ellipsoid needs not to be known. Likewise, we remark that the reference gravity *outside* the surface layer needs not to be known. The Prey-reduced reference gravity needs not to be defined *outside* the model earth. Because, the Prey-reduced reference gravity field is defined only *inside* the earth.

C.5 Mass density distribution in the reference gravity field

Mass density distribution in the Prey-reduced reference gravity field is massive inside the model earth, as stated above. The characteristics are (1) the mass density of the surface layer is ρ_B, and (2) the total mass inside the reference ellipsoid is equal to the total mass of the normal gravity field.

References

Garland, G. D., *The Earth's Shape and Gravity*, 183 pp., Pergamon Press, Oxford, 1965.

Hackney, R. I. and W. E. Featherstone, Geodetic versus geophysical perspectives of the 'gravity anomaly', *Geophys. J. Int.*, **154**, 35–43, 2003.

Hagiwara, Y., I. Murata, H. Tajima, K. Nagasawa, S. Izutuya, and S. Okubo, Gravity study of active faults—Detection of the 1931 Nishi-Saitama Earthquake Fault—, *Bull. Earthq. Res. Inst.*, **61**, 563–586, 1986 (in Japanese with English abstract).

Heiland, C. A., *Geophysical Exploration*, 2nd Printing, 1013 pp., Prentice-Hall, New York, 1946.

Heiskanen, W. A. and H. Moritz, *Physical Geodesy*, 364 pp., Freeman and Company, San Francisco, 1967.

Moritz, H., *The Figure of the Earth*, 279 pp., Wichmann, Karlsruhe, 1990.

Nettleton, L. L., Determination of density for reduction of gravimeter observations, *Geophysics*, **4**, 176–183, 1939.

Nettleton, L. L., *Geophysical Prospecting for Oil*, 444 pp., McGraw-Hill, New York, 1940.

Nozaki, K., A computer program for spherical terrain correction, *J. Geod. Soc. Japan*, **27**, 23–32, 1981 (in Japanese with English abstract and figure captions).

Nozaki, K., The latest microgravity survey and its applications, *OYO Technical Report*, No. 19, 35–60, 1997 (in Japanese with English abstract and figure captions).

Nozaki, K., A note on the basic characteristics of spherical gravity corrections: gravitational behavior of a thin spherical cap of axial symmetry, *J. Geod. Soc. Japan*, **45**, 215–226, 1999 (in Japanese with English abstract and figure captions).

Torge, W., *Gravimetry*, 465 pp., Walter de Gruyter, Berlin, 1989.

Torge, W., *Geodesy*, 3rd edition, 416 pp., Walter de Gruyter, Berlin, 2001.

Tsuboi, C., Gravity anomalies and the corresponding subterranean mass distribution, *Japan Acad. Proc.*, **14**, 170–175, 1938.

Tsuboi, C. and T. Fuchida, Relation between gravity anomalies and the corresponding subterranean mass distribution (II), *Bull. Earthq. Res. Inst., Univ. of Tokyo*, **16**, 273–284, 1938.

K. Nozaki (e-mail: nozaki-kyozo@oyonet.oyo.co.jp)