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Implicit iteration scheme for phi-hemicontractive operators in arbitrary Banach spaces

Arif Rafiq

Abstract

The purpose of this paper is to characterize the conditions for the convergence of the implicit Mann iterative scheme with error term to the unique fixed point of ϕ-hemicontractive mappings in a nonempty convex subset of an arbitrary Banach space.

Keywords: Implicit iterative scheme, ϕ-hemicontractive mappings, Banach spaces

MSC (2000): primary: 47H10, 47H17; secondary: 54H25

Introduction

Let K be a nonempty subset of an arbitrary Banach space X and X^* be its dual space. Let $T : D(T) \subseteq X \to X$ be a mapping. The symbols $D(T)$, $R(T)$, and $F(T)$ stand for the domain, the range, and the set of fixed points of T, respectively (for a single-valued map $T : X \to X, x \in X$ is called a fixed point of T if $T(x) = x$). We denote by J the normalized duality mapping from X to 2^{X^*} defined by

$$J(x) = \left\{ f^* \in X^* : \langle x, f^* \rangle = \|x\|^2 = \left\|f^*\right\|^2 \right\}.$$

Definition 1. The mapping T is called *Lipshitzian* if there exists $L > 0$ such that

$$\left\|Tx - Ty\right\| \leqslant L \left\|x - y\right\|,$$

for all $x, y \in K$. If $L = 1$, then T is called *nonexpansive*, and if $0 \leqslant L < 1$, T is called *contraction*.

Definition 2. [1-4]

(i) T is said to be strongly pseudocontractive if there exists $t > 1$ such that for each $x, y \in D(T)$, there exists $j(x - y) \in J(x - y)$ satisfying

$$\operatorname{Re} \langle Tx - Ty, j(x - y)\rangle \leq \frac{1}{t}\|x - y\|^2.$$

(ii) T is said to be strictly hemicontractive if $F(T) \neq \emptyset$ and there exists a $t > 1$ such that for each $x \in D(T)$ and $q \in F(T)$, there exists $j(x - y) \in J(x - y)$ satisfying

$$\operatorname{Re} \langle Tx - q, j(x - q)\rangle \leq \frac{1}{t}\|x - q\|^2.$$

(iii) T is said to be ϕ-strongly pseudocontractive if there exists a strictly increasing function $\phi : [0, \infty) \to [0, \infty)$ with $\phi(0) = 0$ such that for each $x, y \in D(T)$, there exists $j(x - y) \in J(x - y)$ satisfying

$$\operatorname{Re} \langle Tx - Ty, j(x - y)\rangle \leq \|x - y\|^2 - \phi(\|x - y\|)\|x - y\|.$$

(iv) T is said to be ϕ-hemicontractive if $F(T) \neq \emptyset$ and there exists a strictly increasing function $\phi : [0, \infty) \to [0, \infty)$ with $\phi(0) = 0$ such that for each $x \in D(T)$ and $q \in F(T)$, there exists $j(x - y) \in J(x - y)$ satisfying

$$\operatorname{Re} \langle Tx - q, j(x - q)\rangle \leq \|x - q\|^2 - \phi(\|x - q\|)\|x - q\|.$$

Clearly, each strictly hemicontractive operator is ϕ-hemicontractive.

Chidume [1] established that the Mann iteration sequence converges strongly to the unique fixed point of T in case T is a Lipschitz strongly pseudocontractive mapping from a bounded closed convex subset of L_p (orl_p) into itself. Afterwards, several authors generalized this result of Chidume in various directions [2,4-11].

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In 2001, Xu and Ori [12] introduced the following implicit iteration process for a finite family of nonexpansive mappings $\{T_i : i \in I\}$ (here, $I = \{1, 2, \ldots, N\}$), with $\{\alpha_n\}$ a real sequence in $(0, 1)$, and an initial point $x_0 \in K$:

$$\begin{aligned}
x_1 &= (1-\alpha_1)x_0 + \alpha_1 T_1 x_1,\\
x_2 &= (1-\alpha_2)x_1 + \alpha_2 T_2 x_2,\\
&\vdots\\
x_N &= (1-\alpha_N)x_{N-1} + \alpha_N T_N x_N,\\
x_{N+1} &= (1-\alpha_{N+1})x_N + \alpha_{N+1} T_{N+1} x_{N+1},\\
&\vdots
\end{aligned}$$

which can be written in the following compact form:

$$x_n = (1-\alpha_n)x_{n-1} + \alpha_n T_n x_n, \text{ for all } n \geq 1, \qquad \text{(XO)}$$

where $T_n = T_{n\,(\mathrm{mod}\,N)}$ (here, the mod N function takes the values in I). Xu and Ori [12] proved the weak convergence of this process to a common fixed point of the finite family of nonexpansive mappings defined in a Hilbert space. They further remarked that it is yet unclear what assumptions on the mappings and/or the parameters $\{\alpha_n\}$ are sufficient to guarantee the strong convergence of the sequence $\{x_n\}$.

In [13], Chidume et al. proved the following results:

Lemma 3. [13] *Let E be a real Banach space. Let K be a nonempty closed and convex subset of E. Let $T : K \to K$ be a strictly pseudocontractive map in the sense of Browder and Petryshyn. Let $x^* \in F(T)$. For a fixed $x_0 \in K$, define a sequence $\{x_n\}$ by*

$$x_{n+1} = (1-\alpha_n)x_n + \alpha_n T x_n,$$

where $\{\alpha_n\}$ is a real sequence in $[0, 1]$ satisfying the following conditions: (i) $\sum_{n=1}^{\infty} \alpha_n = \infty$ and (ii) $\sum_{n=1}^{\infty} \alpha_n^2 < \infty$. Then, (a) $\liminf_{n\to\infty} \|x_n - Tx_n\| = 0$, (b) $\{x_n\}$ is bounded and $\lim_{n\to\infty} \|x_n - x^*\|$ exists.

Theorem 4. [13] *Let E be a real Banach space. Let K be a nonempty closed and convex subset of E. Let $T : K \to K$ be a strictly pseudocontractive map in the sense of Browder and Petryshyn with $F(T) := \{x \in K : Tx = x\} \neq \emptyset$. For a fixed $x_0 \in K$, define a sequence $\{x_n\}$ by*

$$x_{n+1} = (1-\alpha_n)x_n + \alpha_n T x_n,$$

where $\{\alpha_n\}$ is a real sequence satisfying the following conditions: (i) $\sum \alpha_n = \infty$ and (ii) $\sum \alpha_n^2 < \infty$. If T is demicompact, then $\{x_n\}$ converges strongly to some fixed point of T in K.

In [14], Osilike proved the following results:

Theorem 5. *Let E be a real Banach space and K be a nonempty closed convex subset of E. Let $\{T_i : i \in I\}$ be N strictly pseudocontractive self-mappings of K with $F = \bigcap_{i=1}^{N} F(T_i) \neq \emptyset$. Let $\{\alpha_n\}_{n=1}^{\infty}$ be a real sequence satisfying the following conditions:*

$$\begin{aligned}
&(i) \quad 0 < \alpha_n < 1,\\
&(ii) \quad \sum_{n=1}^{\infty}(1-\alpha_n) = \infty,\\
&(iii) \quad \sum_{n=1}^{\infty}(1-\alpha_n)^2 < \infty.
\end{aligned}$$

From arbitrary $x_0 \in K$, define the sequence $\{x_n\}$ by the implicit iteration process (XO). Then, $\{x_n\}$ converges strongly to a common fixed point of the mappings $\{T_i : i \in I\}$ if and only if $\lim_{n\to\infty} \inf d(x_n, F) = 0$.

In [15], Su and Li proved the following results:

Theorem 6. [15] *Let E be a real Banach space and K be a nonempty closed and convex subset of E. Let $\{T_i\}_{i=1}^{N}$ be N strictly pseudocontractive self-maps of K in the sense of Browder and Petryshyn such that $F = \bigcap_{i=1}^{N} F(T_i) \neq \phi$, where $F(T_i) = \{x \in K : T_i x = x\}$. For a fixed $x_0 \in K$, define a sequence $\{x_n\}_{n=1}^{\infty}$ by*

$$\begin{aligned}
x_n &= \alpha_n x_n + (1-\alpha_n)Ty_n,\\
y_n &= \beta_n x_n + (1-\beta_n)Ty_n,
\end{aligned}$$

where $T_n = T_{nmodN}$ and $\{\alpha_n\}_{n=1}^{\infty}, \{\beta_n\}_{n=1}^{\infty}$ be real sequen– ces in $[0, 1]$ satisfying the following conditions: (i) $\sum_{n=1}^{\infty}(1-\alpha_n) = \infty$, (ii) $\sum_{n=1}^{\infty}(1-\alpha_n)^2 < \infty$, (iii) $\sum_{n=1}^{\infty}(1-\beta_n) < \infty$, and (iv) $(1-\alpha_n)(1-\beta_n)L^2 < 1$. Then, (a) $\liminf_{n\to\infty} \|x_n - T_n x_n\| = 0$ and (b) $\lim_{n\to\infty} \|x_n - x^*\|$ exist for all $x^* \in F$.

Remark 7. *(i) One can easily see that for $\alpha_n = 1 - \frac{1}{n^{\frac{1}{2}}}$, $\sum(1-\alpha_n)^2 = \infty$ .Hence, the results of Osilike [14] and Su and Li [15] are to be improved.*
(ii) Proofs of Chidume et al. [13] main results based on ϕ^{-1} : Let us define $\phi : [0, \infty) \to [0, \infty)$ by $\phi(\alpha) = \frac{3^\alpha - 1}{3^\alpha + 1}$, then it can be easily seen that (i) ϕ is increasing and (ii) $\phi(0) = 0$, but $\lim_{\alpha\to\infty} \phi(\alpha) = 1$ and $\phi^{-1}(2)$ make no sense.

The purpose of this paper is to characterize the conditions for the convergence of the implicit iterative scheme with error term in the sense of [16-18] to the unique fixed point of ϕ-hemicontractive mappings in a nonempty convex subset of an arbitrary Banach space. Our results extend and improve most of the results in recent literature [7,12-14,19-22].

Preliminaries

The following results are now well known:

Lemma 8. [23] *For all $x, y \in X$ and $j(x+y) \in J(x+y)$,*

$$\|x+y\|^2 \le \|x\|^2 + 2\mathrm{Re}\,\langle y, j(x+y)\rangle.$$

Main results

Now, we prove our main results.

Theorem 9. *Let K be a nonempty closed convex subset of an arbitrary Banach space X and let $T: K \to K$ be a uniformly continuous and ϕ-hemicontractive mapping. Suppose that $\{u_n\}_{n=1}^{\infty}$ is a bounded sequence in K and $\{a'_n\}_{n=1}^{\infty}$, $\{b'_n\}_{n=1}^{\infty}$, and $\{c'_n\}_{n=1}^{\infty}$ are sequences in $[0,1]$ satisfying conditions (i) $a'_n + b'_n + c'_n = 1$, (ii) $\lim_{n\to\infty} b'_n = 0$, (iii) $c'_n = o(b'_n)$, and (iv) $\sum_{n=1}^{\infty} b'_n = \infty$. For a sequence $\{v_n\}_{n=1}^{\infty}$ in K, suppose that $\{x_n\}_{n=1}^{\infty}$ is the sequence generated from an arbitrary $x_0 \in K$ by*

$$x_n = a'_n x_{n-1} + b'_n T v_n + c'_n u_n, n \ge 1, \tag{3.1}$$

and satisfying $\lim_{n\to\infty}\|v_n - x_n\| = 0$. Then, the following conditions are equivalent:

(a) $\{x_n\}_{n=1}^{\infty}$ converges strongly to the unique fixed point q of T,
(b) $\{Tx_n\}_{n=1}^{\infty}$ is bounded.

Proof. From (iii), we have $c'_n = t_n b'_n$, where $t_n \to 0$ as $n \to \infty$.

Since T is ϕ-hemicontractive, it follows that $F(T)$ is a singleton. Let $F(T) = \{q\}$ for some $q \in K$.

Suppose that $\lim_{n\to\infty} x_n = q$, then the uniform continuity of T yields that

$$\lim_{n\to\infty} Tx_n = q.$$

Therefore, $\{Tx_n\}_{n=1}^{\infty}$ is bounded.

Note that $\lim_{n\to\infty}\|v_n - x_n\| = 0$ and the continuity of T imply that

$$\lim_{n\to\infty}\|Tv_n - Tx_n\| = 0. \tag{3.2}$$

Put

$$M_1 = \|x_0 - q\| + \sup_{n\ge1}\|Tx_n - q\| + \sup_{n\ge1}\|u_n - q\| + \sup_{n\ge1}\|Tv_n - Tx_n\|. \tag{3.3}$$

It is clear that $||x_0 - q|| \le M_1$. Let $||x_{n-1} - q|| \le M_1$. Next, we will prove that $||x_n - q|| \le M_1$.

Consider

$$\begin{aligned}
\|x_n - q\| &= \left\|a'_n x_{n-1} + b'_n T v_n + c'_n u_n - q\right\| \\
&= \left\|a'_n (x_{n-1} - q) + b'_n (Tv_n - q) + c'_n (u_n - q)\right\| \\
&\le \left(1 - b'_n\right)\|x_{n-1} - q\| + b'_n\|Tv_n - q\| + c'_n\|u_n - q\| \\
&\le \left(1 - b'_n\right) M_1 + b'_n(\|Tv_n - Tx_n\| + \|Tx_n - q\|) \\
&\quad + c'_n\|u_n - q\| \\
&= \left(1 - b'_n\right)\Bigg[\|x_0 - q\| + \sup_{n\ge1}\|Tx_n - q\| \\
&\qquad + \sup_{n\ge1}\|u_n - q\| + \sup_{n\ge1}\|Tv_n - Tx_n\|\Bigg] \\
&\quad + b'_n(\|Tv_n - Tx_n\| + \|Tx_n - q\|) + c'_n\|u_n - q\| \\
&\le \|x_0 - q\| \\
&\quad + \left(\left(1 - b'_n\right)\sup_{n\ge1}\|Tx_n - q\| + b'_n\|Tx_n - q\|\right) \\
&\quad + \left(\left(1 - b'_n\right)\sup_{n\ge1}\|u_n - q\| + b'_n\|u_n - q\|\right) \\
&\quad + \left(\left(1 - b'_n\right)\sup_{n\ge1}\|Tv_n - Tx_n\| + b'_n\|Tv_n - Tx_n\|\right) \\
&\le \|x_0 - q\| \\
&\quad + \left(\left(1 - b'_n\right)\sup_{n\ge1}\|Tx_n - q\| + b'_n\sup_{n\ge1}\|Tx_n - q\|\right) \\
&\quad + \left(\left(1 - b'_n\right)\sup_{n\ge1}\|u_n - q\| + b'_n\sup_{n\ge1}\|u_n - q\|\right) \\
&\quad + \left(\left(1 - b'_n\right)\sup_{n\ge1}\|Tv_n - Tx_n\| + b'_n\sup_{n\ge1}\|Tv_n - Tx_n\|\right) \\
&= \|x_0 - q\| + \sup_{n\ge1}\|Tx_n - q\| + \sup_{n\ge1}\|u_n - q\| \\
&\quad + \sup_{n\ge1}\|Tv_n - Tx_n\| \\
&= M_1.
\end{aligned}$$

So, from the above discussion, we can conclude that the sequence $\{x_n - q\}_{n\ge1}$ is bounded. Thus, there is a constant $M > 0$ satisfying

$$M = \sup_{n\ge1}\|x_n - q\| + \sup_{n\ge1}\|Tx_n - q\| + \sup_{n\ge1}\|u_n - q\| + \sup_{n\ge1}\|Tv_n - Tx_n\|. \tag{3.4}$$

Obviously, $M < \infty$. Consider

$$\begin{aligned}
\|Tv_n - q\| &\le \|Tv_n - Tx_n\| + \|Tx_n - q\| \\
&\le \sup_{n\ge1}\|Tv_n - Tx_n\| + \sup_{n\ge1}\|Tx_n - q\| \\
&\le M.
\end{aligned} \tag{3.5}$$

By virtue of Lemma 4 and (3.1), we infer that

$$\begin{aligned}\|x_n-q\|^2 &= \left\|a'_n x_{n-1}+b'_n Tv_n+c'_n u_n-q\right\|^2\\ &= \left\|a'_n(x_{n-1}-q)+b'_n(Tv_n-q)+c'_n(u_n-q)\right\|^2\\ &\le \left(1-b'_n\right)^2\|x_{n-1}-q\|^2+2b'_n\mathrm{Re}\,\langle Tv_n-q,j(x_n-q)\rangle\\ &\quad +2c'_n\mathrm{Re}\,\langle u_n-q,j(x_n-q)\rangle\\ &\le \left(1-b'_n\right)^2\|x_{n-1}-q\|^2+2b'_n\mathrm{Re}\,\langle Tv_n-Tx_n,j(x_n-q)\rangle\\ &\quad +2b'_n\mathrm{Re}\,\langle Tx_n-q,j(x_n-q)\rangle+2c'_n\|u_n-q\|\|x_n-q\|\\ &\le \left(1-b'_n\right)^2\|x_{n-1}-q\|^2+2b'_n\|Tv_n-Tx_n\|\|x_n-q\|\\ &\quad +2b'_n\|x_n-q\|^2-2b'_n\phi(\|x_n-q\|)\|x_n-q\|+2M^2c'_n\\ &= \left(1-b'_n\right)^2\|x_{n-1}-q\|^2+2Mb'_nw_n+2b'_n\|x_n-q\|^2\\ &\quad -2b'_n\phi(\|x_n-q\|)\|x_n-q\|+2M^2c'_n,\end{aligned}\tag{3.6}$$

where

$$w_n=\|Tv_n-Tx_n\|.\tag{3.7}$$

Consider

$$\begin{aligned}\|x_n-q\|^2 &= \left\|a'_n x_{n-1}+b'_n Tv_n+c'_n u_n-q\right\|^2\\ &= \left\|a'_n(x_{n-1}-q)+b'_n(Tv_n-q)+c'_n(u_n-q)\right\|^2\\ &\le a'_n\|x_{n-1}-q\|^2+b'_n\|Tv_n-q\|^2+c'_n\|u_n-q\|^2\\ &\le \|x_{n-1}-q\|^2+M^2\left(b'_n+c'_n\right),\end{aligned}\tag{3.8}$$

where the first inequality holds by the convexity of $\|.\|^2$.

Substituting (3.8) in (3.6), we get

$$\begin{aligned}\|x_n-q\|^2 &\le \left[\left(1-b'_n\right)^2+2b'_n\right]\|x_{n-1}-q\|^2\\ &\quad +2Mb'_n\left(w_n+M\left(b'_n+2t_n\right)\right)\\ &\quad -2b'_n\phi(\|x_n-q\|)\|x_n-q\|\\ &= \left(1+b'^2_n\right)\|x_{n-1}-q\|^2\\ &\quad +2Mb'_n\left(w_n+Mb'_n+2t_n\right))\\ &\quad -2b'_n\phi(\|x_n-q\|)\|x_n-q\|\\ &\le \|x_{n-1}-q\|^2\\ &\quad +Mb'_n\left(3Mb'_n+2(w_n+2Mt_n)\right)\\ &\quad -2b'_n\phi(\|x_n-q\|)\|x_n-q\|\\ &= \|x_{n-1}-q\|^2+b'_nl_n\\ &\quad -2b'_n\phi(\|x_n-q\|)\|x_n-q\|,\end{aligned}\tag{3.9}$$

where

$$l_n=M\left(3Mb'_n+2\left(w_n+2Mt_n\right)\right)\to 0,\tag{3.10}$$

as $n\to\infty$.

Let $\delta=\inf\{\|x_{n+1}-q\|:n\ge 0\}$. We claim that $\delta=0$. Otherwise $\delta>0$. Thus, (3.10) implies that there exists a positive integer $N_1>N_0$ such that $l_n<\phi(\delta)\delta$ for each $n\ge N_1$. In view of (3.9), we conclude that

$$\|x_{n+1}-q\|^2\le\|x_n-q\|^2-\phi(\delta)\delta b'_n,\quad n\ge N_1,$$

which implies that

$$\phi(\delta)\delta\sum_{n=N_1}^{\infty}b'_n\le\|x_{N_1}-q\|^2,\tag{3.11}$$

which contradicts (*iv*). Therefore, $\delta=0$. Thus, there exists a subsequence $\{x_{n_i+1}\}_{n=0}^{\infty}$ of $\{x_{n+1}\}_{n=0}^{\infty}$ such that

$$\lim_{i\to\infty}x_{n_i+1}=q.\tag{3.12}$$

Let $\epsilon>0$ be a fixed number. By virtue of (3.10) and (3.12), we can select a positive integer $i_0>N_1$ such that

$$\left\|x_{n_{i_0}+1}-q\right\|<\epsilon,\ l_n<\phi(\epsilon)\epsilon,\ n\ge n_{i_0}.\tag{3.13}$$

Let $p=n_{i_0}$. By induction, we show that

$$\|x_{p+m}-q\|<\epsilon,\ m\ge 1.\tag{3.14}$$

Observe that (3.13) means that (3.14) is true for $m=1$. Suppose that (3.14) is true for some $m\ge 1$. If $\|x_{p+m+1}-q\|\ge\epsilon$, by (3.9) and (3.13), we know that

$$\begin{aligned}\epsilon^2 &\le \|x_{p+m+1}-q\|^2\\ &\le \|x_{p+m}-q\|^2+\frac{b'_{p+m}l_{p+m}}{1-2b'_{p+m}}\\ &\quad -\frac{2b'_{p+m}}{1-2b'_{p+m}}\phi(\|x_{p+m+1}-q\|)\|x_{p+m+1}-q\|\\ &< \epsilon^2+\frac{b'_{p+m}\phi(\epsilon)\epsilon}{1-2b'_{p+m}}-\frac{2b'_{p+m}\phi(\epsilon)\epsilon}{1-2b'_{p+m}}\\ &< \epsilon^2,\end{aligned}$$

which is impossible . Hence, $\|x_{p+m+1}-q\|<\epsilon$. That is, (3.14) holds for all $m\ge 1$. Thus, (3.14) ensures that $\lim_{n\to\infty}x_n=q$. This completes the proof. □

Using the method of proofs in Theorem 6, we have the following result:

Theorem 10. *Let $X, K, T, \{u_n\}_{n=1}^{\infty}, \{v_n\}_{n=1}^{\infty}$, and $\{x_n\}_{n=1}^{\infty}$ be as in Theorem 9. Suppose that $\{a'_n\}_{n=1}^{\infty}, \{b'_n\}_{n=1}^{\infty}$, and $\{c'_n\}_{n=1}^{\infty}$ are sequences in $[0,1]$ satisfying conditions (i), (ii), (iv), and*

$$\sum_{n=1}^{\infty}c'_n<\infty.$$

Then, the conclusion of Theorem 9 holds.

Corollary 11. *Let K be a nonempty closed convex subset of an arbitrary Banach space X and let $T:K\to K$ be a uniformly continuous and ϕ-hemicontractive mapping. Suppose that $\{u_n\}_{n=1}^{\infty}$ is a bounded sequence in K, and*

$\{a'_n\}_{n=1}^{\infty}$, $\{b'_n\}_{n=1}^{\infty}$, and $\{c'_n\}_{n=1}^{\infty}$ are sequences in $[0,1]$ satisfying conditions (i) $a'_n + b'_n + c'_n = 1$, (ii) $\lim_{n\to\infty} b'_n = 0$, (iii) $c'_n = 0(b'_n)$, and (iv) $\sum_{n=1}^{\infty} b'_n = \infty$. Suppose that $\{x_n\}_{n=1}^{\infty}$ is the sequence generated from an arbitrary $x_0 \in K$ by

$$x_n = a'_n x_{n-1} + b'_n T x_n + c'_n u_n, n \geq 1.$$

Then, the following conditions are equivalent:

(a) $\{x_n\}_{n=1}^{\infty}$ converges strongly to the unique fixed point q of T,
(b) $\{Tx_n\}_{n=1}^{\infty}$ is bounded.

Corollary 12. *Let $X, K, T, \{u_n\}_{n=1}^{\infty}$, and $\{x_n\}_{n=1}^{\infty}$ be as in Corollary 11. Suppose that $\{a'_n\}_{n=1}^{\infty}, \{b'_n\}_{n=1}^{\infty}$, and $\{c'_n\}_{n=1}^{\infty}$ are sequences in $[0,1]$ satisfying conditions (i), (ii), (iv) and*

$$\sum_{n=1}^{\infty} c'_n < \infty.$$

Then, the conclusion of Corollary 12 holds.

Corollary 13. *Let K be a nonempty closed convex subset of an arbitrary Banach space X and let $T : K \to K$ be a uniformly continuous and ϕ-hemicontractive mapping. Suppose that $\{\alpha_n\}_{n=1}^{\infty}$ be any sequence in $[0,1]$ satisfying (i) $\lim_{n\to\infty} \alpha_n = 0$ and (ii) $\sum_{n=1}^{\infty} \alpha_n = \infty$. For a sequence $\{v_n\}_{n=1}^{\infty}$ in K, suppose that $\{x_n\}_{n=1}^{\infty}$ is the sequence generated from an arbitrary $x_0 \in K$ by*

$$x_n = \alpha_n x_{n-1} + (1-\alpha_n) T v_n, n \geq 1$$

and satisfying $\lim_{n\to\infty} \|v_n - x_n\| = 0$. Then, the following conditions are equivalent:

(a) $\{x_n\}_{n=1}^{\infty}$ converges strongly to the unique fixed point q of T,
(b) $\{Tx_n\}_{n=1}^{\infty}$ is bounded .

Corollary 14. *Let K be a nonempty closed convex subset of an arbitrary Banach space X and let $T : K \to K$ be a uniformly continuous and ϕ-hemicontractive mapping. Suppose that $\{\alpha_n\}_{n=1}^{\infty}$ be any sequence in $[0,1]$ satisfying (i) $\lim_{n\to\infty} \alpha_n = 0$ and (ii) $\sum_{n=1}^{\infty} \alpha_n = \infty$. For any $x_o \in K$, define the sequence $\{x_n\}_{n=1}^{\infty}$ inductively as follows:*

$$x_n = \alpha_n x_{n-1} + (1-\alpha_n) T x_n, n \geq 1.$$

Then the following conditions are equivalent:

(a) $\{x_n\}_{n=1}^{\infty}$ converges strongly to the unique fixed point q of T,
(b) $\{Tx_n\}_{n=1}^{\infty}$ is bounded.

Remark 15. *All of the above results are also valid for Lipschitz ϕ-hemicontractive mappings.*

Multi-step implicit fixed point iterations

Let K be a nonempty closed convex subset of a real normed space X and $T_1, T_2, \ldots, T_p : K \to K (p \geq 2)$ be a family of self-mappings.

Algorithm 1. *For a given $x_0 \in K$, compute the sequence $\{x_n\}$ by the implicit iteration process of arbitrary fixed order $p \geq 2$,*

$$\begin{aligned} x_n &= a'_n x_{n-1} + b'_n T_1 y_n^1 + c'_n u_n, \\ y_n^i &= a_n^i x_{n-1} + b_n^i T_{i+1} y_n^{i+1} + c_n^i v_n^i;\ i = 1,\ 2, \ldots, p-2, \\ y_n^{p-1} &= a_n^{p-1} x_{n-1} + b_n^{p-1} T_p x_n + c_n^{p-1} v_n^{p-1},\ n \geq 0, \end{aligned} \tag{4.1}$$

which is called the multi-step implicit iteration process, where $\{a_n\}, \{b_n\}, \{c_n\}, \{a_n^i\}, \{b_n^i\}, \{c_n^i\} \subset [0,1]$; $a'_n + b'_n + c'_n = 1 = a_n^i + b_n^i + c_n^i$; and $\{u_n\}$ and $\{v_n^i\}$ are arbitrary sequences in K provided $i = 1, 2, \ldots, p-1$.

For $p = 3$, we obtain the following three-step implicit iteration process:

Algorithm 2. *For a given $x_0 \in K$, compute the sequence $\{x_n\}$ by the iteration process*

$$\begin{aligned} x_n &= a'_n x_{n-1} + b'_n T_1 y_n^1 + c'_n u_n, \\ y_n^1 &= a_n^1 x_{n-1} + b_n^1 T_2 y_n^2 + c_n^1 v_n^1, \\ y_n^2 &= a_n^2 x_{n-1} + b_n^2 T_3 x_n + c_n^2 v_n^2,\ n \geq 0, \end{aligned} \tag{4.2}$$

where $\{a'_n\}, \{b'_n\}, \{c'_n\}, \{a_n^i\}, \{b_n^i\}, \{c_n^i\} \subset [0,1]$; $a'_n + b'_n + c'_n = 1 = a_n^i + b_n^i + c_n^i$; and $\{u_n\}$ and $\{v_n^i\}$ are arbitrary sequences in K provided $i = 1, 2$.

For $p = 2$, we obtain the following two-step implicit iteration process:

Algorithm 3. *For a given $x_0 \in K$, compute the sequence $\{x_n\}$ by the iteration process*

$$\begin{aligned} x_n &= a'_n x_{n-1} + b'_n T_1 y_n^1 + c'_n u_n, \\ y_n^1 &= a_n^1 x_{n-1} + b_n^1 T_2 x_n + c_n^1 v_n^1,\ n \geq 0, \end{aligned} \tag{4.3}$$

where $\{a'_n\}, \{b'_n\}, \{c'_n\}, \{a_n^1\}, \{b_n^1\}, \{c_n^1\} \subset [0,1]$; $a'_n + b'_n + c'_n = 1 = a_n^1 + b_n^1 + c_n^1$; and $\{u_n\}$ and $\{v_n^1\}$ are arbitrary sequences in K.

If $T_1 = T, T_2 = I, b_n^1 = 1$, and $c_n^1 = 0$ in (4.3), we obtain the implicit Mann iteration process:

Algorithm 4. [2] *For any given $x_0 \in K$, compute the sequence $\{x_n\}$ by the iteration process*

$$x_n = a'_n x_{n-1} + b'_n T x_n + c'_n u_n,\ n \geq 0, \tag{4.4}$$

where $\{a'_n\}$, $\{b'_n\}$, $\{c'_n\} \subset [0,1]$; $a'_n + b'_n + c'_n = 1$; and $\{u_n\}$ is an arbitrary sequence in K.

Theorem 16. *Let K be a nonempty closed convex subset of an arbitrary Banach space X and $T_1, T_2, \ldots, T_p$ $(p \geq 2)$ be self-mappings of K. Let T_1 be a continuous ϕ-hemicontractive mapping and $R(T_2)$ is bounded. Let $\{a'_n\}$, $\{b'_n\}$, $\{c'_n\}$, $\{a^i_n\}$, $\{b^i_n\}$, $\{c^i_n\}$ be real sequences in $[0,1]$; $a'_n + b'_n + c'_n = 1 = a^i_n + b^i_n + c^i_n$, $i = 1, 2, \ldots, p-1$ satisfying (i) $\lim_{n\to\infty} b'_n = 0$, (ii) $c'_n = 0(b'_n)$, and (iii) $\sum_{n=1}^{\infty} b'_n = \infty$, $\lim_{n\to\infty} b^1_n = 0 = \lim_{n\to\infty} c^1_n$. For arbitrary $x_0 \in K$, define the sequence $\{x_n\}$ by (4.1). Then, $\{x_n\}$ converges strongly to the common fixed point of $\bigcap_{i=1}^{p} F(T_i) \neq \emptyset$.*

Proof. By applying Theorem 9 under the assumption that T_1 is continuous ϕ- hemicontractive, we obtain Theorem 16 which proves strong convergence of the iteration process defined by (4.1). Consider the following estimates by taking $T_1 = T$ and $v_n = y^1_n$,

$$\|v_n - x_n\| \leq \|v_n - x_{n-1}\| + \|x_{n-1} - x_n\|, \tag{4.5}$$

$$\begin{aligned} \|v_n - x_{n-1}\| &= \left\|a^1_n x_{n-1} + b^1_n T_2 y^2_n + c^1_n v^1_n - x_{n-1}\right\| \\ &= \left\|b^1_n \left(T_2 y^2_n - x_{n-1}\right) + c^1_n \left(v^1_n - x_{n-1}\right)\right\| \\ &\leq b^1_n \left\|T_2 y^2_n - x_{n-1}\right\| + c^1_n \left\|v^1_n - x_{n-1}\right\| \\ &\leq 2M\left(b^1_n + c^1_n\right), \end{aligned} \tag{4.6}$$

$$\begin{aligned} \|x_{n-1} - x_n\| &= \left\|x_{n-1} - a'_n x_{n-1} - b'_n T v_n - c'_n u_n\right\| \\ &= \left\|b'_n(x_{n-1} - T v_n) - c'_n(u_n - x_{n-1})\right\| \\ &\leq b'_n \|x_{n-1} - T v_n\| + c'_n \|u_n - x_{n-1}\| \\ &\leq 2M(b'_n + c'_n). \end{aligned} \tag{4.7}$$

Substituting (4.6 to 4.7) in (4.5), we have

$$\begin{aligned} \|v_n - x_n\| &\leq 2M(b^1_n + c^1_n + b'_n + c'_n) \\ &\to 0, \end{aligned}$$

as $n \to \infty$. □

Corollary 17. *Let K be a nonempty closed convex subset of an arbitrary Banach space X and $T_1, T_2, \ldots, T_p (p \geq 2)$ be self-mappings of K. Let T_1 be a Lipschitz ϕ-hemicontractive mapping, and $R(T_2)$ is bounded. Let $\{a'_n\}$, $\{b'_n\}$, $\{c'_n\}$, $\{a^i_n\}$, $\{b^i_n\}$, and $\{c^i_n\}$ be real sequences in $[0,1]$; $a'_n + b'_n + c'_n = 1 = a^i_n + b^i_n + c^i_n$, $i = 1, 2, \ldots, p-1$ satisfying (i) $\lim_{n\to\infty} b'_n = 0$, (ii) $c'_n = 0(b'_n)$, and (iii) $\sum_{n=1}^{\infty} b'_n = \infty$, $\lim_{n\to\infty} b^1_n = 0 = \lim_{n\to\infty} c^1_n$. For arbitrary $x_0 \in K$, define the sequence $\{x_n\}$ by (4.1). Then, $\{x_n\}$ converges strongly to the common fixed point of $\bigcap_{i=1}^{p} F(T_i) \neq \emptyset$.*

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The author has no competing interests.

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References
1. Chidume, CE: Iterative approximation of fixed point of Lipschitz strictly pseudocontractive mappings. Proc. Amer. Math. Soc. **99**, 283–288 (1987)
2. Ćirić, LB, Rafiq, A, Cakić, N, Ume, JS: Implicit Mann fixed point iterations for pseudo-contractive mappings. Appl. Math. Lett. **22**(4), 581–584 (2009)
3. Mann, WR: Mean value methods in iteraiton. Proc. Amer. Math. Soc. **26**, 506–510 (1953)
4. Zhou, HY, Cho, YJ: Ishikawa and Mann iterative processes with errors for nonlinear ϕ-strongly quasi-accretive mappings in normed linear spaces. J. Korean Math. Soc. **36**, 1061–1073 (1999)
5. Liu, LS: Ishikawa and Mann iterative process with errors for nonlinear strongly accretive mappings in Banach spaces. J. Math. Anal. Appl. **194**, 114–125 (1995)
6. Liu, LW: Approximation of fixed points of a strictly pseudocontractive mapping. Proc. Amer. Math. Soc. **125**, 1363–1366 (1997)
7. Liu, Z, Kim, JK, Kang, SM: Necessary and sufficient conditions for convergence of Ishikawa iterative schemes with errors to ϕ-hemicontractive mappings. Commun. Korean Math. Soc. **18**(2), 251–261 (2003)
8. Liu, Z, Xu, Y, Kang, SM: Almost stable iteration schemes for local strongly pseudocontractive and local strongly accretive operators in real uniformly smooth Banach spaces. Acta. Math. Univ. Comenianae. **LXXVII**(2), 285–298 (2008)
9. Tan, KK, Xu, HK: Iterative solutions to nonlinear equations of strongly accretive operators in Banach spaces. J. Math. Anal. Appl. **178**, 9–21 (1993)
10. Xu, Y: Ishikawa and Mann iterative processes with errors for nonlinear strongly accretive operator equations. J. Math. Anal. Appl. **224**, 91–101 (1998)
11. Xue, Z: Iterative approximation of fixed point for ϕ-hemicontractive mapping without Lipschitz assumption. Int. J. Math. Math. Sci. **17**, 2711–2718 (2005)
12. Xu, HK, Ori, R: An implicit iterative process for nonexpansive mappings. Numer. Funct. Anal. Optim. **22**, 767–773 (2001)
13. Chidume, CE, Abbas, M, Ali, B: Convergence of the Mann iteration algorithm for a class of pseudocontractive mappings. Appl. Math. Comput. **94**(1), 1–6 (2007)
14. Osilike, MO: Implicit iteration process for common fixed points of a finite family of strictly pseudocontractive maps. J. Math. Anal. Appl. **294**(1), 73–81 (2004)
15. Su, Y, Li, S: Composite implicit iteration process for common fixed points of a finite family of strictly pseudocontractive maps. J. Math. Anal. Appl. **320**(2), 882–891 (2006)
16. Ciric, L, Ume, JS: Ishikawa iterative process for strongly pseudocontractive operators in Banach spaces. Math. Commun. **8**, 43–48 (2003)
17. Rafiq, A: On Mann iteration in Hilbert spaces. Nonlinear Anal. TMA. **66**(10), 2230–2236 (2007)
18. Rafiq, A: Implicit fixed point iterations for pseudocontractive mappings. Kodai Math. J. **32**(1), 146–158 (2009)
19. Gu, F: The new composite implicit iterative process with errors for common fixed points of a finite family of strictly pseudocontractive mappings. J. Math. Anal. Appl. **329**(2), 766–776 (2007)
20. Ishikawa, S: Fixed point by a new iteration method. Proc. Amer. Math. Soc. **44**, 147–150 (1974)

21. Kato, T: Nonlinear semigroups and evolution equations. J. Math. Soc. Japan. **19**, 508–520 (1967)
22. Schu, J: On a theorem of C. E. Chidume concerning the iterative approximation of fixed points. Math. Nachr. **153**, 313–319 (1991)
23. Xu, HK: Inequality in Banach spaces with applications. Nonlinear Anal. **16**, 1127–1138 (1991)

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