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Eigenvalue problem for *p*-Laplacian with mixed boundary conditions

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Abstract
Eigenvalue problem for *p*-Laplacian with mixed boundary conditions is concerned on a bounded domain. The existence of nonnegative eigenvalues are obtained by using the Lusternik-Schnirelman principle. Boundedness of eigenfunctions is obtained by using the Moser iteration. The simplicity and isolation of the first eigenvalue are proved. The existence of the second eigenvalue is also illustrated.

Keywords: Eigenvalue problem, *p*-Laplacian, Mixed boundary conditions, Ljusternik-Schnirelman principle, Variational methods

MSC: 35P30; 47F05

Introduction

In this paper, we study the following eigenvalue problem

$$\begin{cases} -\triangle_p u(x) = \lambda |u|^{p-2}u, & \text{in } \Omega, \\ u = 0, & \text{on } \sigma, \\ |\nabla u(x)|^{p-2}\frac{\partial u}{\partial n} = \lambda |u|^{p-2}u, & \text{on } \Gamma, \end{cases} \tag{1.1}$$

where $\triangle_p u = div(|\nabla u|^{p-2}|\nabla u|)$ is the p-Laplacian operator, $1 < p < +\infty, \Omega$ is a bounded domain in $\mathbb{R}^N$ with C^1 boundary $\partial\Omega$, $\sigma \cup \Gamma = \partial\Omega$ and $\sigma \cap \Gamma = \emptyset$, Γ is a sufficiently smooth $(N-1)$-dimensional manifold, and n is the outward normal vector on $\partial\Omega$.

Throughout the paper we define $X := \{u \in W^{1,p}(\Omega) : u|_\sigma = 0\}$ is a closed subspace of $W^{1,p}(\Omega)$ with the norm $\|u\|^p = \int_\Omega (|\nabla u|^p + |u|^p)dx$. Eigenvalue problems for the p-Laplacian and $p(x)$-Laplacian have been studied extensively for many years and many interesting results have been obtained. These results are on the structure of the spectrum of Dirichlet, no-flux, Niemann, Robin, and Steklov problems as demonstrated in [1-8]. Problem (1.1) is a mixed boundary value problem, and is different from the classical ones. References [9,10] studied the following problem

$$\begin{cases} -\Delta u = \lambda u, & \text{in } \Omega, \\ u = 0, & \text{on } \sigma, \\ \frac{\partial u}{\partial n} = \lambda \beta u, & \text{on } \Gamma, \end{cases} \tag{1.2}$$

where $\beta = \frac{a^2}{b^2}$. Problem (1.1) is a generalization of (1.2) as $p = 2$ and $\beta = 1$. In this paper, we extend their results and study the complete character of eigenvalue problem (1.1) which is an abstract one and has never been known.

Methods

Since our methods of proofs of the theorem are different from the others, we must consider the boundary σ and Γ. We use the multiplicative inequality in [11,12] to proof the boundedness of eigenfunctions. For example, problem (1.1) includes the following classical problem

$$\begin{cases} X''(x) + \lambda X(x) = 0, & 0 < x < l, \\ X(0) = 0, & \\ X'(l) - \lambda X(l) = 0, & \end{cases} \tag{1.3}$$

as its special case, which leads to the equation for eigenvalue $\lambda > 0$,

$$\tan\sqrt{\lambda} l = \frac{1}{\sqrt{\lambda}}.$$

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Thus, we get the sequence of eigenvalues $\lambda_k = \theta_k^2$, satisfying

$$\frac{(k-1)\pi}{l} < \theta_k < \frac{(2k-1)\pi}{2l}, k = 1, 2, \cdots .$$

Related eigenfunctions are $\{sin(\theta_k x)\}_{k=1,2,\cdots}$.

It is well known that an eigenvalue problem plays a very important role in the studying of all kinds of linear and nonlinear problems. Therefore, the research in present paper would be useful to the understanding of spectrum of nonlinear operator and related problems.

The sketch of the paper is as follows. We first establish the eigenvalue sequence in next section. Next, we consider the boundedness of eigenfunctions in section 'Boundedness of eigenfunctions'. The simplicity and isolation of the first eigenvalue are considered in the section 'Simplicity and isolation of the first eigenvalue'. In the section 'Existence of the second eigenvalue', we consider the existence of the second eigenvalue.

Results and discussion

Eigenvalue problem for the *p*-Laplacian

Weak solutions

Definition 2.1. A pair $(u, \lambda) \in X \times \mathbb{R}$ is a weak solution of (1.1) provided that

$$\int_\Omega |\nabla u|^{p-2}\nabla u \nabla v dx = \lambda \left(\int_\Omega |u|^{p-2} uv dx + \int_\Gamma |u|^{p-2} uv ds \right), \tag{2.1}$$

for any $v \in X$ as $u = 0$ on σ. Where u is nontrivial, λ is an eigenvalue, and u is called an associated eigenfunction.

It follows from (2.1) that all eigenvalues λ are nonnegative (by choosing $v = u$). It shows that if Γ is of class $C^{1,\gamma}$, then eigenfunction of (2.1) belongs to $C^{1,\alpha}(\overline{\Omega})$. Hence, ∇u exists on Γ, and the boundary conditions of the problem (1.1) make sense. The following lemma assures that if an eigenfunction u is smooth enough, then u solves the corresponding partial differential equation.

Lemma 2.2. *Let (u, λ) be an eigenpair, i.e., a weak solution of (2.1) such that $u \in W^{2,p}(\Omega)$, then (u, λ) solves (1.1).*

Proof. Let $(u, \lambda) \in W^{2,p}(\Omega) \times \mathbb{R}^+$ be an eigenpair of (2.1). We recall the first formula of Green [13], it follows from (2.1) that

$$\int_\Omega \left(-\triangle_p u\right) v dx + \int_\Gamma |\nabla u|^{p-2} \frac{\partial u}{\partial n} v ds$$
$$= \lambda \left(\int_\Omega |u|^{p-2} uv dx + \int_\Gamma |u|^{p-2} uv ds \right)$$

for any $v \in X$. Thus, taking any v in $C_0^\infty(\Omega)$ we have

$$\int_\Omega \left(\triangle_p u + \lambda |u|^{p-2} u\right) v dx = 0,$$

which implies $-\triangle_p u = \lambda |u|^{p-2} u$ in Ω. Furthermore, since the range of the trace mapping $X \hookrightarrow L^p(\Gamma)$ is continuous and compact (see [14]), and $v = 0$ on σ, we have

$$\int_\Gamma |\nabla u|^{p-2} \frac{\partial u}{\partial n} v ds = \lambda \int_\Gamma |u|^{p-2} uv ds, \qquad \forall v \in L^p(\Gamma).$$

□

Therefore, $|\nabla u|^{p-2} \frac{\partial u}{\partial n} = \lambda |u|^{p-2} u$ on Γ.

Existence of L-S sequence for (1.1)

The existence of a sequence of eigenvalues can be proved by the Ljusternik-Schnirelman principle, we call this sequence as L-S sequence $\{\lambda_n\}$.

Let Ω be a bounded domain in $\mathbb{R}^n$ with C^1 boundary. We define the following functionals F and G on X

$$F(u) = \int_\Omega a(x)|u(x)|^p dx + \int_\Gamma b(s)|u(s)|^p ds, \tag{2.2}$$

$$G(u) = \int_\Omega \left(|\nabla u(x)|^p + |u(x)|^p\right) dx + \int_\Gamma \beta(s)|u(s)|^p ds, \tag{2.3}$$

where $a \in L^\infty(\Omega)$ and $b, \beta \in L^\infty(\Gamma)$ such that $a, b, \beta > 0$. Consider the following eigenvalue problem

$$F'(u) = \mu G'(u), u \in S_G, \mu \in \mathbb{R},$$

where S_G is the level $S_G = \{u \in X : G(u) = 1\}$.

For any positive integer n, denoted by $\mathbb{A}_n$ the class of all compact, symmetric subsets K of S_G such that $F(u) > 0$ is on K and $\gamma(K) \geq n$, where $\gamma(K)$ denotes the genus of K, i.e., $\gamma(K) := \inf\{k \in \mathbb{N} : \exists h : K \to \mathbb{R}^k \backslash \{0\}$ such that h is continuous and odd$\}$.

Theorem 2.3. *Let $F(u)$, $G(u)$ be defined in (2.2) and (2.3) with $a(x) = b(x) = \beta(x) = 1$. Then there exists a nondecreasing sequence of nonnegative $\{\lambda_n\}$ of (2.1) obtained by using the L-S principle such that $\lambda_n = \frac{1}{\mu_n} - 1 \to +\infty$ as $n \to +\infty$, where each μ_n is an eigenvalue of the corresponding equation $F'(u) = \mu G'(u)$ that satisfies $\mu_1 \geq \mu_2 \geq \cdots \geq \mu_k \geq \cdots > 0$, $\lim_{k \to +\infty} \mu_k = 0$.*

Proof. With $a(x) = b(x) = \beta(x) = 1$ in (2.2) and (2.3), $F(u)$ and $G(u)$ become

$$F(u) = \int_\Omega |u(x)|^p dx + \int_\Gamma |u(s)|^p ds, G(u)$$
$$= \int_\Omega \left(|\nabla u(x)|^p + |u(x)|^p\right) dx + \int_\Gamma |u(s)|^p ds.$$

□

Thus, $F'(u) = \mu G'(u)$ is equivalent to

$$\int_\Omega |u|^{p-2}uvdx + \int_\Gamma |u|^{p-2}uvds$$
$$= \mu \left(\int_\Omega \left(|\nabla u|^{p-2}\nabla u \nabla v + |u|^{p-2}uv \right) dx + \int_\Gamma |u|^{p-2}uvds \right),$$

for any $v \in X$, or

$$\int_\Omega |\nabla u|^{p-2}\nabla u\nabla v dx = \left(\frac{1}{\mu} - 1\right) \left(\int_\Omega |u|^{p-2}uvdx + \int_\Gamma |u|^{p-2}uvds \right), \forall v \in X.$$

Combining (2.1) and the existence of the L-S sequence principle, we obtain $\lambda_n = \frac{1}{\mu_n} - 1 \to +\infty$ as $n \to +\infty$.

Boundedness of eigenfunctions

Let Ω be a bounded domain in $\mathbb{R}^n$ with C^1 boundary and $1 < p < +\infty$. We shall show the eigenfunctions are in $L^\infty(\Omega)$, which is the boundedness for solutions of (1.1).

Theorem 3.1. *Let (u, λ) be an eigensolution of the weak form (2.1), then $u \in L^\infty(\Omega)$.*

Proof. In this proof, we use the Moser iteration technique in [15]. We assume first that $u \geq 0$. We define $v_M(x) = \min\{u(x), M\}$ for $M > 0$ and $\varphi = v_M^{kp+1}$ for $k > 0$, then $\nabla\varphi = (kp+1)v_M^{kp}\nabla v_M$. It follows that $\varphi \in X \cap L^\infty(\Omega)$ and $v_M|_\Gamma = \min\{u|_\Gamma, M\}$. Taking φ as a test function we have

$$(kp+1)\int_\Omega |\nabla u|^{p-2}\nabla u v_M^{kp}\nabla v_M dx = \lambda \left(\int_\Omega |u|^{p-2}uv_M^{kp+1}dx + \int_\Gamma |u|^{p-2}uv_M^{kp+1}ds \right),$$

which implies that

$$\frac{kp+1}{(k+1)^p}\int_\Omega |\nabla v_M^{k+1}|^p dx \leq \lambda \left(\int_\Omega |u|^{(k+1)p}dx + \int_\Gamma |u|^{(k+1)p}ds \right).$$

□

Let $M \to \infty$; by Fatou's lemma we obtain

$$\frac{kp+1}{(k+1)^p}\int_\Omega |\nabla u^{k+1}|^p dx \leq \lambda \left(\int_\Omega |u|^{(k+1)p}dx + \int_\Gamma |u|^{(k+1)p}ds \right).$$

That is,

$$\frac{kp+1}{(k+1)^p}\int_\Omega \left(|\nabla u^{k+1}|^p + |u^{k+1}|^p \right) dx \leq \left(\lambda + \frac{kp+1}{(k+1)^p}\right)\int_\Omega |u|^{(k+1)p}dx + \lambda\int_\Gamma |u|^{(k+1)p}ds,$$

$$\frac{kp+1}{(k+1)^p}\|u^{k+1}\|^p \leq \left(\lambda + \frac{kp+1}{(k+1)^p}\right)\|u^{k+1}\|^p_{L^p(\Omega)} + \lambda\|u^{k+1}\|^p_{L^p(\Gamma)}. \tag{3.1}$$

When $u = 0$ in σ, by the multiplicative inequality stated (see Chapter 1, Section 1.4.7, Corollary 2 in [11]) and the Moser iteration done in [12] of the form

$$\|u\|^p_{L^p(\partial\Omega)} \leq \varepsilon\|u\|^p + C(\varepsilon)\|u\|^p_{L^p(\Omega)}, \varepsilon > 0,$$

we obtain

$$\|u^{k+1}\|^p_{L^p(\Gamma)} \leq \varepsilon\|u^{k+1}\|^p + C(\varepsilon)\|u^{k+1}\|^p_{L^p(\Omega)}, \varepsilon > 0. \tag{3.2}$$

Combining (3.1) and (3.2), it has

$$\left(\frac{kp+1}{(k+1)^p} - \lambda\varepsilon\right)\|u^{k+1}\|^p \leq \left(\lambda(1 + C(\varepsilon)) + \frac{kp+1}{(k+1)^p}\right) \times \|u^{k+1}\|^p_{L^p(\Omega)}.$$

Since $\varepsilon \to 0$, we may assume that $\frac{kp+1}{(k+1)^p} - \lambda\varepsilon > 0$, then

$$\|u\| \leq \left[\left(\lambda\,(1 + C(\varepsilon)) + \frac{kp+1}{(k+1)^p}\right) \frac{1}{\frac{kp+1}{(k+1)^p} - \lambda\varepsilon} \right]^{\frac{1}{(k+1)p}} \times \|u\|_{L^{(k+1)p}(\Omega)}. \tag{3.3}$$

By Sobolev's embedding function $X \hookrightarrow L^{p^*}(\Omega)$, where $p^* = \frac{Np}{N-p}$, if $p < N$ and $p^* = 2p$, if $p = N$. Then there exists a constant $c_1 > 0$ such that

$$\|u^{k+1}\|_{L^{p^*}(\Omega)} \leq c_1\|u^{k+1}\|,$$

which is

$$\|u\|_{L^{(k+1)p^*}(\Omega)} \leq c_1^{\frac{1}{k+1}}\|u\|. \tag{3.4}$$

By (3.3) and (3.4), for any $k > 0$, we can find a constant $c_2 > 0$ such that

$$\left[\left(\lambda\,(1 + C(\varepsilon)) + \frac{kp+1}{(k+1)^p}\right) \frac{1}{\frac{kp+1}{(k+1)^p} - \lambda\varepsilon} \right]^{\frac{1}{(k+1)p}} < c_2^{\frac{1}{\sqrt{k+1}}},$$

which is

$$\|u\|_{L^{(k+1)p^*}(\Omega)} \le c_1^{\frac{1}{k+1}} c_2^{\frac{1}{\sqrt{k+1}}} \|u\|_{L^{(k+1)p}(\Omega)}. \tag{3.5}$$

Choosing k_1 such that $(k_1+1)p = p^*$, taking $k = k_1$ in (3.5), it has

$$\|u\|_{L^{(k_1+1)p^*}(\Omega)} \le c_1^{\frac{1}{k_1+1}} c_2^{\frac{1}{\sqrt{k_1+1}}} \|u\|_{L^{p^*}(\Omega)}.$$

Next, we choose k_2 such that $(k_2+1)p = (k_1+1)p^*$, then taking $k = k_2$ in (3.5), we have

$$\|u\|_{L^{(k_2+1)p^*}(\Omega)} \le c_1^{\frac{1}{k_2+1}} c_2^{\frac{1}{\sqrt{k_2+1}}} \|u\|_{L^{(k_1+1)p^*}(\Omega)}.$$

Therefore,

$$\|u\|_{L^{(k_n+1)p^*}(\Omega)} \le c_1^{\frac{1}{k_n+1}} c_2^{\frac{1}{\sqrt{k_n+1}}} \|u\|_{L^{(k_{n-1}+1)p^*}(\Omega)},$$

where the sequence $\{k_n\}$ is chosen such that $(k_n+1)p = (k_{n-1}+1)p^*, k_0 = 0$.

It is easy to see that $k_n + 1 = \left(\frac{p^*}{p}\right)^n$, hence

$$\|u\|_{L^{(k_n+1)p^*}(\Omega)} \le c_1^{\sum_{i=1}^n \frac{1}{k_i+1}} c_2^{\sum_{i=1}^n \frac{1}{\sqrt{k_i+1}}} \|u\|_{L^{p^*}(\Omega)}.$$

There exists $C > 0$ such that

$$\|u\|_{L^{(k_n+1)p^*}(\Omega)} \le C\|u\|_{L^{p^*}(\Omega)},$$

for any $n = 1, 2, \cdots$, with $r_n = (k_n+1)p^* \to +\infty$ as $n \to +\infty$.

Next, we will prove $u \in L^\infty(\Omega)$. Suppose $u \notin L^\infty(\Omega)$, then there exists $\varepsilon_1 > 0$ and a set A of positive measure in Ω such that $|u(x)| > C\|u\|_{L^{p^*}(\Omega)} + \varepsilon_1 = K$, for all $x \in A$.

Hence,

$$\liminf_{n\to+\infty} \|u\|_{L^{r_n}(\Omega)} \ge \liminf_{n\to+\infty} \left(\int_A K^{r_n}\right)^{\frac{1}{r_n}} = \liminf_{n\to+\infty} K|A|^{\frac{1}{r_n}}$$
$$= K > C\|u\|_{L^q(\Omega)},$$

which contradicts what has been established above.

If u (as an eigenfunction of (2.1)) changes sign, we consider u^+, and it is easy to know $u^+ \in X$. We define for each $M > 0, v_M(x) = \min(u^+(x), M)$. Taking again $\varphi = v_M^{kp+1}$ as a test function in $W^{1,p}(\Omega)$, we obtain

$$(kp+1)\int_\Omega |\nabla u^+|^{p-2}\nabla u^+ v_M^{kp} \nabla v_M dx$$
$$= \lambda\left(\int_\Omega |u^+|^{p-2}u^+ v_M^{kp+1} dx + \int_\Gamma |u^+|^{p-2}u^+ v_M^{kp+1} ds\right).$$

Proceeding the same way as above, we conclude that $u^+ \in L^\infty(\Omega)$. Similarly we have $u^- \in L^\infty(\Omega)$. Therefore $u = u^+ + u^-$ is in $L^\infty(\Omega)$.

Simplicity and isolation of the first eigenvalue

In this section, we will study the characterization of the first eigenvalue of (1.1). In the succeeding text, we assume that Ω is a bounded domain in $\mathbb{R}^n$ with $C^{1,\gamma}$ boundary, $\gamma > 0$, and $1 < p < +\infty$. By (2.1) we have $\lambda_1 = \inf_{u \in X\setminus\{0\}} \frac{\int_\Omega |\nabla u|^p dx}{\int_\Omega |u|^p dx + \int_\Gamma |u|^p ds}$.

Simplicity of the first eigenvalue

Proposition 4.1. *If (u, λ) is an eigenpair of (2.1) with $\lambda > \lambda_1$, then u has to change sign in Ω.*

Proof. If (u, λ) satisfies (2.1) for any $v \in X$, by choosing $v \equiv 1$, we obtain

$$\int_\Omega |u|^{p-2}u dx + \int_\Gamma |u|^{p-2}u ds = 0.$$

□

Therefore, u has to change sign.

Theorem 4.2. *The principal eigenvalue λ_1 is simple; i.e., if u, v are two eigenfunctions associated with λ_1, then there exists a constant k such that $u = kv$.*

Proof. By proposition 4.1, we can assume that u, v are positive in Ω. We assume u, v are strictly positive in $\overline{\Omega}$, we take

$$\eta_1 = \frac{u^p - v^p}{u^{p-1}}, \eta_2 = \frac{v^p - u^p}{v^{p-1}}$$

as test functions in the weak form of (2.1) satisfied by u, v, respectively. We have

$$\int_\Omega |\nabla u|^{p-2}\nabla u \nabla\left(\frac{u^p - v^p}{u^{p-1}}\right)dx$$
$$= \lambda\left(\int_\Omega |u|^{p-2}u\left(\frac{u^p - v^p}{u^{p-1}}\right)dx + \int_\Gamma |u|^{p-2}u \right. \tag{4.1}$$
$$\left. \times \left(\frac{u^p - v^p}{u^{p-1}}\right)ds\right),$$

$$\int_\Omega |\nabla v|^{p-2}\nabla v \nabla\left(\frac{v^p - u^p}{v^{p-1}}\right)dx$$
$$= \lambda\left(\int_\Omega |v|^{p-2}v\left(\frac{v^p - u^p}{v^{p-1}}\right)dx + \int_\Gamma |v|^{p-2}v \right. \tag{4.2}$$
$$\left. \times \left(\frac{v^p - u^p}{v^{p-1}}\right)ds\right).$$

□

Combining (4.1) and (4.2) yields

$$0=\int_\Omega |\nabla u|^{p-2}\nabla u\nabla\left(\frac{u^p-v^p}{u^{p-1}}\right)dx + \int_\Omega |\nabla v|^{p-2}\nabla v\nabla\left(\frac{v^p-u^p}{v^{p-1}}\right)dx. \tag{4.3}$$

Using $\nabla\left(\frac{u^p-v^p}{u^{p-1}}\right) = \nabla u - p\frac{v^{p-1}}{u^{p-1}}\nabla v + (p-1)\frac{v^p}{u^p}\nabla u$, we obtain

$$\int_\Omega |\nabla u|^{p-2}\nabla u\nabla\left(\frac{u^p-v^p}{u^{p-1}}\right)dx = \int_\Omega |\nabla lnu|^p u^p - p\int_\Omega v^p|\nabla lnu|^{p-2}\langle\nabla lnu,\nabla lnv\rangle + (p-1)\int_\Omega |\nabla lnu|^p v^p \tag{4.4}$$

and

$$\int_\Omega |\nabla v|^{p-2}\nabla v\nabla\left(\frac{v^p-u^p}{v^{p-1}}\right)dx = \int_\Omega |\nabla lnv|^p v^p - p\int_\Omega u^p|\nabla lnv|^{p-2}\langle\nabla lnv,\nabla lnu\rangle + (p-1)\int_\Omega |\nabla lnv|^p u^p. \tag{4.5}$$

By (4.3), (4.4), and (4.5), we obtain

$$0=\int_\Omega u^p\left(|\nabla lnu|^p-|\nabla lnv|^p-p|\nabla lnv|^{p-2}\langle\nabla lnv,\nabla lnu-\nabla lnv\rangle\right) + \int_\Omega v^p\left(|\nabla lnv|^p-|\nabla lnu|^p-p|\nabla lnu|^{p-2} \times \langle\nabla lnu,\nabla lnv-\nabla lnu\rangle\right). \tag{4.6}$$

When $p\geq 2$ by reference [1], we have

$$|\nabla lnu|^p-|\nabla lnv|^p-p|\nabla lnv|^{p-2}\langle\nabla lnv,\nabla lnu-\nabla lnv\rangle \geq C(p)|\nabla lnv-\nabla lnu|^p,$$

$$|\nabla lnv|^p-|\nabla lnu|^p-p|\nabla lnu|^{p-2}\langle\nabla lnu,\nabla lnv-\nabla lnu\rangle \geq C(p)|\nabla lnu-\nabla lnv|^p.$$

Therefore, (4.6) implies that

$$0\geq\int_\Omega C(p)|\nabla lnu-\nabla lnv|^p(u^p+v^p).$$

Hence,

$$0=|\nabla lnu-\nabla lnv|.$$

This also implies that $u=kv$, as we wanted to prove.

When $p<2$, we have

$$|\nabla lnu|^p-|\nabla lnv|^p-p|\nabla lnv|^{p-2}\langle\nabla lnv,\nabla lnu-\nabla lnv\rangle \geq C(p)\frac{|\nabla lnv-\nabla lnu|^p}{(|\nabla lnu|+|\nabla lnv|)^{2-p}},$$

$$|\nabla lnv|^p-|\nabla lnu|^p-p|\nabla lnu|^{p-2}\langle\nabla lnu,\nabla lnv-\nabla lnu\rangle \geq C(p)\frac{|\nabla lnu-\nabla lnv|^p}{(|\nabla lnu|+|\nabla lnv|)^{2-p}}.$$

Arguing as above, we also conclude $u=kv$.

Theorem 4.3. *Let u be an eigenfunction corresponding to $\lambda\neq\lambda_1$, then u changes sign on Γ, that is, the sets $\{x\in\Gamma : u(x)>0\}$ and $\{x\in\Gamma : u(x)<0\}$ have positive measure.*

Proof. Assume that u does not change sign in Ω, then we can assume that $u>0$ in Ω due to the Harnack inequality. Let u_1 be an eigenfunction with λ_1; making similar calculation as the ones performed in the proof of lemma 4.2, we arrive at

$$(\lambda_1-\lambda)\left(\int_\Omega\left(u_1^p-u^p\right)dx+\int_\Gamma\left(u_1^p-u^p\right)ds\right) = (\lambda_1-\lambda)\int_{\Omega\cup\Gamma}\left(u_1^p-u^p\right)dx \geq C\int_\Omega|\nabla lnu_1-\nabla lnu|^p\left(u_1^p+u^p\right)dx.$$

□

Hence, taking ku instead of u, for any $k>0$, we have

$$\int_{\Omega\cup\Gamma}\left(u_1^p-k^pu^p\right)dx\leq 0,$$

which is a contradiction if $k^p<\int_{\Omega\cup\Gamma}u_1^pdx/\int_{\Omega\cup\Gamma}u^pdx$. Therefore, u changes sign in Ω.

Suppose that u does changes sign on Γ, then we can assume $u\leq 0$ on Γ. Using u^+ as a test function in (2.1), we conclude that

$$\int_\Omega|\nabla u|^{p-2}\nabla u\nabla u^+dx=0.$$

Since u changes sign in Ω, the left hand side is strictly positive. This is a contradiction. Hence, u changes sign on Γ.

Isolation of the first eigenvalue

Given λ, an eigenvalue of (1.1) and u, an eigenfunction associated with λ, we define

$Z(u)=\{x\in\overline{\Omega}:u(x)=0\}$,

$N(u)=$ the number of components of $\overline{\Omega}\backslash Z(u)$,

$N(\lambda)=sup\{N(u):u$ is an eigenfunction associated with $\lambda\}$.

We shall show $N(\lambda)$ is finite.

Theorem 4.4. *Let (u, λ) be a (weak) eigenpair of (1.1), $\lambda \neq \lambda_1$, there exists a constant C such that $|\Gamma^+| \geq C\lambda^{-\beta}$ and $|\Gamma^-| \geq C\lambda^{-\beta}$, where $\Gamma^+ = \Gamma \cap \{u > 0\}, \Gamma^- = \Gamma \cap \{u < 0\}, \beta = (N-1)/(p-1)$ if $1 < p < N$ and $\beta = 2$ if $p \geq N$.*

Proof. If we let $u^- \in W^{1,p}(\Omega)$ be a test function in (2.1), we obtain

$$\int_\Omega |\nabla u^-|^p dx = \lambda \left(\int_\Omega |u^-|^p dx + \int_{\Gamma \cap \{u<0\}} |u^-|^p ds \right), \tag{4.7}$$

that is,

$$\|u^-\|^p = (\lambda + 1)\|u^-\|^p_{L^p(\Omega)} + \lambda \int_{\Gamma \cap \{u<0\}} |u^-|^p ds.$$

□

When $1 < p < N$, we choose $\alpha = (N-1)/(N-p)$ and $\beta = (N-1)/(p-1)$, by the Hölder inequality and the Sobolev embedding functions $X \hookrightarrow L^{\alpha p}(\Gamma)$ and $X \hookrightarrow L^p(\Omega)$, there exists constants $C_1, C_2 > 0$, such that

$$\|u^-\|^p \leq (\lambda + 1)C_1\|u^-\|^p + \lambda C_2\|u^-\|^p|\Gamma^-|^{1/\beta},$$

that is, $|\Gamma^-| \geq C\lambda^{-\beta}$, where $C = \left[\frac{1-(\lambda+1)C_1}{C_2}\right]^\beta$.

When $p \geq N$, we choose $\alpha = \beta = 2$ and by the embedding functions $X \hookrightarrow L^{2p}(\Gamma)$, a similar argument works for u^+ as above.

Theorem 4.5. *The principal eigenvalue λ_1 of (1.1) is isolated. That is, there exists $a > \lambda_1$ such that λ_1 is the unique eigenvalue in $[0, a]$.*

Proof. We can prove this theorem as Theorem 5.16 of [3] by assuming

$$\Gamma_n^- = \{x \in \Gamma : u_n(x) < 0\}, \Gamma_n^+ = \{x \in \Gamma : u_n(x) > 0\}.$$

□

Existence of the second eigenvalue

Proposition 5.1. *For any eigenvalue λ of (2.1), we have*

$$\lambda_{N(\lambda)} \leq \lambda,$$

where $N(\lambda)$ is the maximal number of nodal domains associated with λ (see Theorem 4.4), and $\lambda_{N(\lambda)}$ is the $N(\lambda)$th eigenvalue taken from the L-S sequence of Theorem 2.3.

Proof. Let $r = N(\lambda)$, then there is an eigenfunction $u \neq 0$ associated with λ such that $r = N(u)$. Let $\omega_1, \omega_2, \cdots, \omega_r$ be the r-components of $\overline{\Omega} \backslash Z(u)$. We define

$$v_i(x) = \begin{cases} \frac{u(x)}{[\int_{\Omega\cap\omega_i} |u|^p dx + \int_{\Gamma\cap\omega_i} |u|^p ds]^{1/p}} & \text{if} \quad x \in \overline{\omega_i}, \\ 0 & \text{if} \quad x \in \overline{\Omega} \backslash \overline{\omega_i}. \end{cases}$$

□

Then by the Theorem C.3 in [3], we have $v_i \in X$ for $i = 1, 2, \cdots, r$.

Let X_r denote the subspace of X which is spanned by $\{v_1, v_2, \cdots, v_r\}$. For each $v \in X_r$, $v = \sum_{i=1}^r \alpha_i v_i$, we obtain

$$F(v) = \int_\Omega |v|^p dx + \int_\Gamma |v|^p ds = \sum_{i=1}^r |\alpha_i|^p F(v_i) = \sum_{i=1}^r |\alpha_i|^p.$$

Thus, the map $v \mapsto F(v)^{1/p}$ is a norm on X_r. Hence, the compact set S_r is defined by

$$S_r = \left\{ v \in X_r : F(v) = \frac{1}{\lambda + 1} \right\},$$

which can be identified with the unit sphere of $\mathbb{R}^n$, and which is r. By choosing $v = v_i$ as a test function, we obtain

$$\int_\Omega |\nabla u|^{p-2} \nabla u \nabla v_i dx = \lambda \left(\int_\Omega |u|^{p-2} u v_i dx + \int_\Gamma |u|^{p-2} u v_i ds \right).$$

Hence,

$$\int_{\Omega\cap\omega_i} |\nabla v_i|^p dx = \lambda \left(\int_{\Omega\cap\omega_i} |v_i|^p dx + \int_{\Gamma\cap\omega_i} |v_i|^p ds \right),$$

or

$$G(v_i) = (\lambda + 1)F(v_i), i = 1, 2, \cdots, r.$$

Thus, for $v \in S_r$, we have

$$G(v) = (\lambda + 1) \sum_{i=1}^r |\alpha_i|^p F(v_i) = (\lambda + 1) \sum_{i=1}^r |\alpha_i|^p$$

$$= (\lambda + 1)F(v) = 1.$$

It implies $S_r \subset S_G$. Hence

$$\frac{1}{1 + \lambda_r} = \mu_r = \sup_{H \in \mathbb{A}_r} \inf_{v \in H} F(v) \geq \inf F(v) = \frac{1}{1 + \lambda}.$$

Therefore $\lambda_r \leq \lambda$. This completes the proof.

Proposition 5.2. *For any of the problems, $\lambda_2 = \inf\{\lambda :$ λ is an eigenvalue and $\lambda > \lambda_1\}$.*

Proof. The proof is similar to Theorem 5.19 in [3], we omit it here. □

Conclusions

There are four important conclusions that can really be drawn from this study: (1) there exists a nondecreasing sequence of nonnegative $\{\lambda_n\}$ of (2.1); (2) there is boundedness of eigenfunctions; (3) the first eigenvalue is simple and isolated; and (4) there is an existence of a second eigenvalue.

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
All authors (GL, HL, and BC) contributed equally and significantly in writing this paper. All authors read and approved the final manuscript.

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