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Non-normal edge-transitive directed Cayley graphs of abelian groups

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Abstract

For a group G, and a subset S of G such that $1_G \notin S$, let $X = Cay(G, S)$ be the corresponding Cayley graph. Then X is said to be normal edge transitive if $N_{Aut(X)}(G)$ is transitive on edges. In this paper, we determine all connected directed Cayley graphs of finite abelian groups with valency at most 3 which are normal edge transitive but not normal.

Keywords: Cayley graph, Vertex transitive, Edge transitive, Normal edge transitive, Normal graph, Automorphism group

AMS: 05C10; 05C25

Introduction

Throughout this paper, graphs are finite, simple and directed. For a graph X, let $V(X)$, $E(X)$ and $Aut(X)$ denote its vertex set, edge set and automorphism group, respectively. Let G be a finite group and S a subset of G not containing the identity 1_G. The Cayley graph $X = Cay(G, S)$ of G with respect to S is a graph defined by $V(X) = G$, $E(X) = \{(g, sg) | g \in G, s \in S\}$. In particular, if $S^{-1} = S$, such a graph can be viewed as an undirected graph by coalescing each pair, (g, sg) and (sg, g), of directed edges into a single undirected edge $\{g, sg\}$. A Cayley graph $X = Cay(G, S)$ is called normal for G if the right regular representation of G is a normal subgroup of the automorphism group of X (see [1]). A graph X is called arc transitive or symmetric if $Aut(X)$ acts transitively on the arc set of X.

Let X and Y be two graphs. The *direct product* $X \times Y$ is defined as the graph with vertex set $V(X \times Y) = V(X) \times V(Y)$ such that, for any two vertices, $u = (x_1, y_1)$ and $v = (u_2, v_2)$ in $V(X \times Y)$, (u, v) is an edge in $X \times Y$ whenever $x_1 = x_2$ and $(y_1, y_2) \in E(Y)$ or $y_1 = y_2$ and $(x_1, x_2) \in E(X)$. The graphs are called relatively prime if they have no nontrivial common direct factor. The lexicographic product $X[Y]$ is defined as the graph with vertex set $V(X \times Y) = V(X) \times V(Y)$ such that, for any two vertices, $u = (x_1, x_2)$ and $v = (v_1, v_2)$ in $V(X[Y])$, (u, v) in an edge in $X[Y]$ whenever $(x_1, x_2) \in E(X)$ or $x_1 = x_2$ and $(y_1, y_2) \in E(Y)$.

The concept of normality of the Cayley graph is known to be of fundamental importance for the study of arc transitive graphs. So, for a given finite group G, a natural problem is to determine all the normal or non-normal Cayley graph of G. Some meaningful results in this direction, especially for the undirected Cayley graphs, have been obtained. Baik et al. [2] determined all non-normal Cayley graphs of abelian groups with valency at most 4 and later [3] dealt with valency 5. For directed Cayley graphs, Xu et al., [4] determined all non-normal Cayley graphs of abelian groups with valency at most 3.

An approach to analysing the family of Cayley graphs for a finite group G is given, which identifies normal edge transitive Cayley graphs as a subfamily of central importance. These are the Cayley graphs for G for which a subgroup of automorphisms exists, which both normalises G and acts transitively on edges. It is shown that, for a nontrivial group G, each normal edge transitive Cayley graph for G has at least one homomorphic image which is a normal edge transitive Cayley graph for a characteristically simple quotient group of G. For example, Alaeiyan et al. [5] determined all normal edge transitive undirected connected Cayley graphs of abelian groups with valancy at most 5 which are not normal. Our main result is as follows, the proof of which will be given in section 'The proof of 1.1.'

Theorem 1.1. Let $X = Cay(G, S)$ be a connected directed Cayley graph of an abelian group G with respect

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to S, and the size of S is at most 3. Then X is normal edge transitive if one of the following cases happens:

(1) $G = Z_{2n} = \langle a \rangle (n > 2, n = 2k), s = \{a, a^{n+1}\}$, $X = C_n[2K_1]$
(2) $G = Z_n \times Z_2 = \langle a \rangle \times \langle u \rangle (n > 2, n = 2k)$, $S = \{a, au\}, X = C_n[2K_1]$

The following corollaries are immediate consequences of the theorem.

Corollary 1.2. All non-normal connected Cayley graphs with valency 3 of a finite abelian group are not normal edge transitive.

Corollary 1.3. A non-normal connected Cayley graph with valency 2 of a finite abelian group is normal edge transitive if and only if $o(G) = 4k$.

Preliminary and result

For a graph X, we denote the automorphism group of X by $Aut(X)$. The following propositions are basic.

Proposition 2.1. [6] Let $X = Cay(G, S)$ be a Cayley graph of group G relative on S.

(1) $Aut(X)$ contains the right regular permutation of G, so X is vertex transitive.
(2) X is connected if and only if $G =< S >$.
(3) X is undirected if and only if $S^{-1} = S$.

Proposition 2.2. [7] Let $\Gamma = Cay(G, S)$ be a Cayley graph for a finite group G with $S \neq \phi$. Then Γ is normal edge transitive if and only if $Aut(G, S)$ is transitive on S, and if Γ is undirected, then Γ is normal edge transitive as an undirected graph if and only if $Aut(G, S)$ is either transitive on S or has two orbits in S which are inverses of each other.

Proposition 2.3. [4] Let $X = Cay(G, S)$ be a connected directed Cayley graph of an abelian group G with respect to S, and the valency of S at most 3. Then X is normal except when one of the following cases happens:

(1) $G = Z_{2n} = \langle a \rangle (n > 2), S = \{a, a^{n+1}\}, X = C_n[2K_1]$
(2) $G = Z_n \times Z_2 = \langle a \rangle \times \langle u \rangle (n > 2), S = \{a, au\}$, $X = C_n[2K_1]$
(3) $G = Z_4 = \langle a \rangle, S = G \setminus \{1\}, X = K_4$
(4) $G = Z_6 = \langle a \rangle, S = \{a, a^3, a^5\}, X = K_{3,3}$
(5) $G = Z_4 \times Z_2 = \langle a \rangle \times \langle b \rangle, S = \{a, a^{-1}, b\}, X = Q_3$
(6) $G = Z_{2n} \times Z_m = \langle a \rangle \times \langle c \rangle (n > 2, m > 1)$, $S = \{a, a^{n+1}, c\}, X = C_n[2K_1] \times C_m$
(7) $G = Z_n \times Z_2 \times Z_m = \langle a \rangle \times \langle u \rangle \times \langle c \rangle (n > 2$, $m > 1), S = \{a, au, c\}, X = C_n[2K_1 \times C_m]$
(8) $G = Z_{2n} = \langle a \rangle, (n > 2), S = \{a, a^{n+1}, a^n\}$
(9) $G = Z_n \times Z_2 = \langle a \rangle \times \langle u \rangle (n > 2), S = \{a, au, u\}$
(10) $G = Z_{2k} \times Z_2 = \langle a \rangle \times \langle u \rangle (k > 2), S = \{a, au, a^k\}$
(11) $G = Z_{2k} \times Z_2 = \langle a \rangle \times \langle u \rangle (k > 2), S = \{a, au, a^k u\}$
(12) $G = Z_{4n} = \langle a \rangle (n = 4k + 1, k > 0)$, $S = \{a, a^{2n+1}, a^{n+1}\}$
(13) $G = Z_{4n} \times Z_2 = \langle x \rangle \times \langle y \rangle (n = 2k + 1, k > 0)$, $S = \{x, x^{2n+1}, x^{n+1}y\}$
(14) $G = Z_n \times Z_4 = \langle a \rangle \times \langle u \rangle (n = 4k, k > 0)$, $S = \{a, av^2, av\}$
(15) $G = Z_k \times Z_t = \langle x \rangle \times \langle y \rangle$, $S = \{x^{k/nh}y, x^{k/nh}yu, x^{k/mh}y^{-1}\}, u = (x^{k/nh}y)^{nh/2}$
(16) $G = Z_k \times Z_t \times Z_2 = \langle x \rangle \times \langle y \rangle \times \langle u \rangle$, $S = \{x^{k/nh}y, x^{k/nh}yu, x^{k/mh}y^{-1}\}$

In both (15) and (16), $k = mnh/(m, n)$ and $t = (m, n)$. In (15), m is a positive integer; $h > 1$, 2 is not a divided h; and $2|n$, $n > 2$ when $n/2$ is odd, and $n > 4$ otherwise. In (16), m is a positive integer, $h > 1$, and $n > 2$.

The proof of 1.1

Let G be a finite abelian group, $X = Cay(G, S)$ a connected directed Cayley graph of G with respect to S with valency at most 3. In this section, 'The proof of 1.1' will be completed by a series of lemmas. We will apply Proposition 2.2.

Lemma 3.1. The graphs X in cases (1) (for n odd), (2) (for n odd), (3), (4), (5), (6) ($2n \neq m$), (7) ($n \neq m$ and $n = m = 2k + 1$), (8), (9), (11), (10), (12) and (13) in Proposition 2.3 are not normal edge transitive.

Proof. In case (1) (for n odd), $(n + 1, 2n) = 2r$ and a is a generator for G, thus there is no automorphism which takes a to a^{n+1} which means that $Aut(G)$ cannot work transitively on S. In case (12), $(n + 1, n) \neq 1$, similarly it is not normal edge transitive.

In case (2) (for n odd), $O(a) \neq O(au)$, so there is no automorphism which takes a to au. In case (3), $O(a^2) = 2$ and $O(a) = 4$. In case (4), $O(a^5) = 6$ and $O(a^3) = 2$. Thus, there is no automorphism which takes a to a^2. In case (5), $O(a) = 4$ and $O(b) = 2$. In case (6) (for $n \neq 2m$), $O(a) \neq O(c)$. In case (7) (for $m \neq n$), $O(a) \neq O(c)$, and (for $m = n = 2k + 1$), $O(a) \neq O(au)$. In case (8), $O(a) \neq O(a^n)$. In case (9), $O(a) \neq O(u)$. In case (10), $O(a) \neq O(a^k)$. In case (11), $O(a) \neq O(a^k u)$. In case (13), $O(x^{n+1}y) \neq O(x)$. □

Lemma 3.2. The graphs X in cases (1) (for n even) and (2) (for n even) in Proposition 2.3 are normal edge transitive.

Proof. In case (1) (for n even), $(n, n+1) = 1$, and since a is a generator of G, there is an automorphism which takes a to a^{n+1}. It means that $Aut(G, S)$ acts transitively on S.

In case (2) (for n even and $r \in Z$), define ϕ by $\phi(a^{2r}) = a^{2r}$, $\phi(a^{2r+1}) = a^{2r+1}u$, $\phi(a^{2r}u) = a^{2r}u$ and $\phi(a^{2r+1}u) = a^{2r+1}$. Obviously, $\phi \in Aut(G,S)$, and so $Aut(G,S)$ acts transitive on S and G is normal edge transitive. □

Lemma 3.3. The graphs X in cases (6) ($2n = m$), (7) ($m = n = 2k$) and (14) in Proposition 2.3 are not normal edge transitive.

Proof. For case (6) ($2n = m$), let $\phi \in Aut(G,S)$ and $\phi(a) = c$, then $\phi(a^{n+1}) = c^{n+1}$ but it has to take a^{n+1} to a or a^{n+1}. □

Similarly, for cases (7) ($m = n = 2k$) and (14), there are no automorphism to take a to c and av^2 to av, respectively.

Lemma 3.4. The graphs X in cases (15) and (16) in Proposition 2.3 are normal edge transitive.

Proof. In case (15), if $(m,n) = 1$ then $S = \{x^m y, x^m y(x^m y)^{(nh)/2}, x^n y^{-1}\}$, clearly $O(x^m y) \neq O(x^m y^{-1})$, and so it is not normal edge transitive. □

Now let $(m,n) = l$. We have $n = lr$ and $m = ls$, so $x^m y = x^s y$ and $x^n y^{-1} = x^r y^{-1}$. Clearly, $O(x^s y) \neq O(x^r y^{-1})$.

Case (16) is similar to (15).

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
MA and JM carried out the algebraic graph theory studies together and both of them participated in each part of this paper. Both authors read and approved the final manuscript.

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References

1. Xu, MY: Automorphism groups and isomorphism of Cayley graphs. Discerete Math. **182**, 309-319 (1998)
2. Baik, YG, Feng, Y, Sim, HS, Xu, M: On the normality of Cayley graph of abelian groups. Algebra Colloq. **5**, 297–304 (1998)
3. Baik, YG, Feng, Y, Sim, HS: The normality of Cayley graphs of finite abelian groups with valency 5. System Sci. Math. Sci. **13**, 425-431 (2000)
4. Xu, MY, Zhang, Q, Zhou, JX: On the normality of directed Cayley graphs of abelian groups. System Sci. Math. Sci. **25**(6), 700-710 (2005)
5. Alaeiyan, M, Tavallaee, H, Talebi, AA: Cayley graphs of abelian groups which are not normal edge-transitive. Vietnam J. Mathematics. **33**(3), 309–318 (2005)
6. Godsil, C, Royle, G: Algebraic Graph Theory. Springer, New York (2001)
7. Praeger, CE: Finite Normal edge-transitive Cayley graphs. Bull. Austral. Math. Soc. **60**, 207-220 (1999)

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