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Duality for multiobjective fractional programming problem using convexifactors

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Abstract

In this paper, the concept of ∂^*-quasiconvexity is introduced by using convexifactors. Mond-Weir-type and Schaible-type duals are associated with a multiobjective fractional programming problem, and various duality results are established under the assumptions of ∂^*-pseudoconvexity and ∂^*-quasiconvexity.

Keywords: Multiobjective fractional programming problem, Convexifactors, Duality

Introduction

Duality plays a very important role in optimization problems. Many authors have made contributions in the development of duality theory for multiple objective programming problems. There has been tremendous development in area of multiobjective optimization problems during the past years. For the most recent developments in this area, one can refer to the book by Ansari and Yao [1]. In this book, several aspects of multiobjective optimization starting from the very beginning to the most recent ones have been discussed in the form of various research papers in this field. An important class of such problems, namely, multiple objective fractional programming problems, is of great interest in many areas such as transportation, production, information theory, and numerical analysis. Some of the papers by Schaible [2-4] review the early work done in fractional programming. For some recent work on duality in fractional programming, one can see the study of Lyall et al. [5], Liang [6], etc. Duality in generalized fractional programming has been studied by Barros et al. [7], Liu [8], etc. Weir [9] studied a multiobjective fractional programming problem with the same denominators. Since then, a great deal of research was started in this area under the assumptions of convexity and generalized convexity by many researchers such as Singh [10], Egudo [11], Singh and Hanson [12], Weir [9,13], Suneja and Gupta [14], Suneja and Lalitha [15], etc. Duality for multiobjective fractional programming problem under various assumptions has also been studied by authors like Liu [16], Kim et al. [17], Kim [18], Nobakhtian [19], Mishra and Upadhyay [20], etc.

The concept of convexifactor was first introduced by Demyanov [21] as a generalization of the notion of upper convex and lower concave approximations. In [21], convexifactor was defined as convex and compact set and was termed as convexificator. However, Jeyakumar and Luc [22] in their further study suggested that one can use a closed, nonconvex set instead of a compact and convex one to define a convexificator. Dutta and Chandra [23] called them as convexifactors. They have been further studied by Dutta and Chandra [24], Li and Zhang [25], Gadhi [26], etc. Dutta and Chandra [24] introduced ∂^*-pseudoconvex functions by using this concept of convexifactor. The importance of convexifactors lies in the fact that they are useful even when they are unbounded or nonconvex, and the use of a nonconvex set to define convexifactors has an advantage that in many situations one may just have a convexifactor consisting of finite number of points which is more amenable to various applications. Further, for locally Lipschitz function, one can have convexifactors smaller than the Clarke subdifferential, Michel-Penot subdifferential, etc., so optimality conditions and duality results obtained in terms of convexifactors are sharper. In multiobjective programming problems, generalized convexity plays an important role in deriving duality results. Gadhi [26] has proved necessary and sufficient optimality conditions for a multiobjective fractional programming problem in terms of convexifactors. In this paper, we introduce the notion of ∂^*-quasiconvex functions. We associate Mond-Weir-type and Schaible-type

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duals to the multiobjective fractional programming problem and derive duality results under the assumptions of ∂^*-pseudoconvexity and ∂^*-quasiconvexity. Our results in terms of convexifactors recast the well-known results in the multiobjective fractional programming problem and hence present more general results.

The paper is organized as follows: In the 'Preliminaries' section, we introduce the concept of ∂^*-quasiconvex functions in terms of convexifactors. Kuhn-Tucker-type optimality conditions for multiobjective fractional programming problem have been given in the 'Optimality conditions' section. Finally, in the 'Duality' section, we prove duality results by associating Mond-Weir-type and Schaible-type duals to the problem.

Preliminaries

Throughout the paper, we are concerned with finite dimensional spaces.

Let f: $R^n \to R \cup \{+\infty\}$ be an extended real valued function.

$$(f)_d^-(x,v) = Lim\inf_{t\downarrow 0}\frac{f(x+tv)-f(x)}{t},$$
$$(f)_d^+(x,v) = Lim\sup_{t\downarrow 0}\frac{f(x+tv)-f(x)}{t}$$

denote, respectively, the lower and upper Dini directional derivatives of f at x in direction v.

We begin with the definitions of convexifactors given by Dutta and Chandra [23].

Definition 2.1. The function f: $R^n \to R \cup \{+\infty\}$is said to have an *upper convexifactor* $\partial^u f(x)$ at x if $\partial^u f(x) \subset R^n$ is closed, and for each $v \in R^n$,

$$(f)_d^-(x,v) \le \sup_{x^* \in \partial^u f(x)} \langle x^*, v\rangle.$$

Definition 2.2. The function f: $R^n \to R \cup \{+\infty\}$ is said to have a *lower convexifactor* $\partial_l f(x)$ at x if $\partial_l f(x) \subset R^n$ is closed, and for each $v \in R^n$,

$$(f)_d^+(x,v) \ge \inf_{x^* \in \partial_l f(x)} \langle x^*, v\rangle.$$

Definition 2.3. The function f: $R^n \to R \cup \{+\infty\}$ is said to have a *convexifactor* $\partial^* f(x)$ at x if it is both an upper and lower convexifactor of f at x.

Definition 2.4. The function f: $R^n \to R \cup \{+\infty\}$ is said to have an *upper regular convexifactor* $\partial^u f(x)$ at x if $\partial^u f(x)$ is an upper convexifactor of f at x, and for each $v \in R^n$,

$$(f)_d^+(x,v) = \sup_{x^* \in \partial^u f(x)} \langle x^*, v\rangle.$$

Definition 2.5. The function f: $R^n \to R \cup \{+\infty\}$is said to have a *lower regular convexifactor* $\partial_l f(x)$ at x if $\partial_l f(x)$ is lower convexifactor of f at x, and for each $v \in R^n$,

$$(f)_d^-(x,v) = \inf_{x^* \in \partial_l f(x)} \langle x^*, v\rangle.$$

Convexifactors are not necessarily convex or compact. These relaxations allow applications to a large class of nonsmooth functions.

The important question arises regarding the existence of convexifactors or regular convexifactors at a given point for a general real valued function. In this regard, we present the following theorem from Dutta and Chandra [24].

Theorem 2.6. *Let f: $R^n \to R \cup \{+\infty\}$ and let $x \in R^n$ be a given point where $f(x)$ is finite. Moreover, assume that the lower Dini directional derivative $(f)_d^-(x,v)$ is bounded above. Then, there exists a compact upper convexifactor of f at x. If the upper Dini directional derivative $(f)_d^+(x,v)$ is bounded below, then there exists a compact lower convexifactor of f at x.*

Now, we consider the following multiobjective fractional programming problem:

$$(P)\text{minimize } \phi(x) = \left(\frac{f_1(x)}{g_1(x)}, \ldots, \frac{f_p(x)}{g_p(x)}\right)$$

subject to

$$h_j(x) \le 0, \qquad j = 1, 2, \ldots, m$$

Let $E = \{x \in R^n$: $h_j(x) \le 0$, $j = 1, 2, \ldots, m\}$denote the feasible set for problem (P).

Here, f_i, g_i, $i = 1, 2, \ldots, p$ and h_j, $j = 1, 2, \ldots, m$ are continuous real valued functions defined on R^n such that $f_i(x) \ge 0$ and $g_i(x) > 0$, $i = 1, 2, \ldots, p$ for all $x \in E$, and minimization means finding weak efficient solutions in the following sense:

Definition 2.7. $\bar{x} \in E$ is a *weak efficient* solution of (P) if there does not exist any feasible solution $x \in E$ such that

$$\frac{f_i(x)}{g_i(x)} < \frac{f_i(\bar{x})}{g_i(\bar{x})}, \qquad i = 1, 2, \ldots, p.$$

Definition 2.8. $\bar{x} \in E$ is a *local weak efficient* solution of (P) if there exists a neighborhood U of $\bar{x}$ such that for any feasible solution $x \in U \cap E$, the following does not hold:

$$\frac{f_i(x)}{g_i(x)} < \frac{f_i(\bar{x})}{g_i(\bar{x})}, \qquad i = 1, 2, \ldots, p.$$

We give below the definition of ∂^*-pseudoconvex function given by Dutta and Chandra [24].

Definition 2.9. A function f: $R^n \to R$ is said to be ∂^**-pseudoconvex* at $\bar{x} \in R^n$ if for $x \in R^n$,

$$f(x) < f(\bar{x}) \Rightarrow \langle \xi, x - \bar{x}\rangle < 0, \text{ for all } \xi \in \partial^* f(\bar{x}),$$

where $\partial^* f(\bar{x})$ is a convexifactor of f at $\bar{x}$.

We now introduce ∂^*-quasiconvex function.

Definition 2.10. A function f: $R^n \to R$ is said to be ∂^**-quasiconvex* at $\bar{x} \in R^n$ if for $x \in R^n$,

$$f(x) \leq f(\bar{x}) \Rightarrow \langle \xi, x - \bar{x} \rangle \leq 0, \text{ for all } \xi \in \partial^* f(\bar{x}).$$

Remark 2.11. (i) (Dutta and Chandra [24]) If f is a differentiable function and $\partial^* f(\bar{x})$ is an upper regular convexifactor, then $\partial^* f(\bar{x}) = \{\nabla f(\bar{x})\}$, and the above definition reduces to the definition of quasiconvex function.

(ii) If f is a locally Lipschitz function and $\partial^* f(\bar{x}) = \partial^c f(\bar{x})$ where $\partial^c f(\bar{x})$ is the Clarke generalized gradient, then the above definition reduces to the definition of ∂^c-quasiconvex function defined by Bector et al. [27].

Remark 2.12. ∂^*-quasiconvex function is not necessarily ∂^*-pseudoconvex as can be seen from the following example:

Example 2.13. Let f: $R^2 \to R$ be a function defined by

$$f(x,y) = \begin{cases} \sqrt{x^2 y}, & x \geq 0, y > 0, \\ x, & x \geq 0, y = 0, \\ \sqrt{-xy}, & x \geq 0, y < 0, \\ -\sqrt{-xy^2}, & x < 0, y \geq 0, \\ \sqrt{-x^2 y}, & x < 0, y < 0, \end{cases}$$

$$\partial^* f(0,0) = \{(x^*, 0) : x^* \geq 0\}.$$

f is ∂^*-quasiconvex at $(\bar{x},\bar{y}) = (0,0)$, but f is not ∂^*-pseudoconvex at $(\bar{x},\bar{y}) = (0,0)$ because for $(x,y) = (-1, 2)$, $\xi = (0, 0)$

$$f(x,y) < f(0,0) \text{ but } \langle \xi, x - y \rangle = 0.$$

Remark 2.14. It may be noted that every ∂^c-pseudoconvex function is ∂^c-quasiconvex when f is locally Lipschitz as can be seen from Remark 3.1 in the study of Rezaie and Zafarani [28] by taking $\eta(x,y) = x - y$. However, the next example shows that the ∂^*-pseudoconvex function is not ∂^*-quasiconvex.

Example 2.15. Let f: $R^2 \to R$ be a function defined by

$$f(x,y) = \begin{cases} -\sqrt{x} - \sqrt{y}, & x \geq 0, y \geq 0 \\ 0, & x > 0, y < 0 \\ \sqrt{-x} + \sqrt{y}, & x < 0, y \geq 0 \\ \sqrt{-x} + \sqrt{-y}, & x \leq 0, y < 0 \end{cases}$$

$$\partial^* f(0,0) = \{(x^*, y^*) : x^* < 0, y^* < 0\}.$$

f is ∂^*-pseudoconvex at $(\bar{x}, \bar{y}) = (0,0)$, but f is not ∂^*-quasiconvex at $(\bar{x}, \bar{y}) = (0,0)$ because for $(x, y) = (1, -1)$, $\xi = (-1, -2)$

$$f(x,y) = f(0,0) \text{ but } \langle \xi, x - y \rangle > 0.$$

The following result is given by Li and Zhang [25].

Lemma 2.16. *Let $\partial^* f(x)$ be a convexifactor of f at x. Then, $\forall \lambda \in R$, $\lambda\, \partial^* f(x)$ is a convexifactor of λf at x.*

We now give the following result given by Jeyakumar and Luc [22].

Remark 2.17. [22] Assume that the functions f, g:$R^n \to R$ admit upper convexifactors $\partial^u f(x)$ and $\partial^u g(x)$ at x, respectively, and that one of the convexifactors is upper regular at x. Then, $\partial^u f(x) + \partial^u g(x)$ is an upper convexifactor of $f + g$ at x.

Similarly, if one of the convexifactors is lower regular at x. Then, $\partial_l f(x) + \partial_l g(x)$ is a lower convexifactor of $f + g$ at x.

Optimality conditions

Gadhi [26] gave the following necessary optimality conditions for (P).

Theorem 3.1. Let $\bar{x} \varepsilon E$ *be a local weak efficient solution of (P). Assume that f_i, g_i, $i = 1, 2, \ldots, p$ and h_j, $j = 1, 2, \ldots, m$ are continuous and admit bounded convexifactors $\partial^* f_i(\bar{x})$, $\partial^* g_i(\bar{x})$, $i = 1, 2, \ldots, p$ and $\partial^* h_j(\bar{x})$, $j = 1, 2, \ldots, m$ at $\bar{x}$, respectively, and that $x \mapsto \partial^* f_i(x)$, $x \mapsto \partial^* g_i(x)$, $i = 1, 2, \ldots, p$ and $x \mapsto \partial^* h_j(x)$, $j = 1, 2, \ldots, m$ are upper semicontinuous at $\bar{x}$. Then, there exist vectors $\alpha^* = (\alpha_1^*, \alpha_2^*, \ldots, \alpha_p^*) \in R_+^p$ and $\mu^* = (\mu_1^*, \mu_2^*, \ldots, \mu_m^*) \in R_+^m$ (not both zero) such that*

$$0 \in \sum_{i=1}^{p} \alpha_i^* (\partial^* f_i(\bar{x}) - \phi_i(\bar{x}) \partial^* g_i(\bar{x})) + \sum_{j=1}^{m} \mu_j^* \partial^* h_j(\bar{x}) \quad (1)$$

$$\mu_j^* h_j(\bar{x}) = 0, \quad j = 1, 2, \ldots, m \quad (2)$$

$$\mu_j^* \geq 0, h_j(\bar{x}) \leq 0, j = 1, 2, \ldots, m \quad (3)$$

where

$$\phi_i(x) = \frac{f_i(x)}{g_i(x)}, i = 1, 2, \ldots, p.$$

We now deduce the Kuhn-Tucker-type necessary optimality conditions for (P) under the assumption of the Slater-type weak constraint qualification which is defined as follows on the lines in the study of Mangasarian [29].

Definition 3.2. The function h is said to satisfy the Slater-type weak constraint qualification at $\bar{x} \in E$ if h_J is ∂^*-pseudoconvex at $\bar{x}$, and there exists an $x^o \in R^n$ such that $h_J(x^o) < 0$ where $J = \{j | h_j(\bar{x}) = 0\}$.

Remark 3.3. If h is a differentiable function at $\bar{x}$ and admits an upper regular convexifactor $\partial^* h(\bar{x})$ at $\bar{x}$, then the above Slater-type weak constraint qualification reduces to Slater's weak constraint qualification given by Mangasarian [29].

Theorem 3.4. Let $\bar{x} \in E$ *be a weak efficient solution of (P). Suppose that the hypotheses of Theorem 3.1 hold. Then, there exist vectors $\alpha^* \in R_+^p$ and $\mu^* \in R_+^m$* (not both zero) such that (1), (2) and (3) hold. If the Slater-type weak constraint qualification holds at $\bar{x}$, *then* $\alpha^* \neq 0$.

Proof. Suppose on contrary $\alpha^* = 0$, then $\mu^* \neq 0$.

Now using (1), we get that there exists $\zeta_j \in \partial^* h_j(\bar{x})$, $j = 1, 2, \ldots, m$ such that

$$\sum_{j=1}^{m} \mu_j^* \zeta_j = 0. \tag{4}$$

Since h satisfies the Slater-type weak constraint qualification at $\bar{x}$, therefore h_J is ∂^*-pseudoconvex at $\bar{x}$, and there exists an $x^o \in R^n$ such that

$$h_j(x^o) < 0, \quad j \in J,$$

where

$$J = \{j | h_j(\bar{x}) = 0\}.$$

$$\Rightarrow h_j(x^o) < h_j(\bar{x}), \quad j \in J.$$

Using ∂^*- pseudoconvexity of h_j, $j \in J$, we get

$$\langle \zeta_j, x^o - \bar{x} \rangle < 0, j \in J$$

$$\Rightarrow \langle \sum_{j \in J} \mu_j^* \zeta_j, x^o - \bar{x} \rangle < 0 \text{ as } \mu^* \neq 0.$$

Now, (2) gives $\mu_j^* = 0$, for all $j \notin J$ and thus we have

$$\langle \sum_{j=1}^{m} \mu_j^* \zeta_j, x^o - \bar{x} \rangle < 0,$$

which is contradiction to (4).

Hence, $\alpha^* \neq 0$.

Duality

Duality plays a crucial role in mathematical programming as sometimes solving a dual is easier than solving a primal. Wolfe [30] associated a dual problem with a primal nonlinear programming problem and proved various duality theorems under the assumptions of convexity. Since certain duality theorems may fail to hold for the Wolfe model if the objective and/or the constraint functions are generalized convex, Mond and Weir [31] presented a new model for studying duality which allowed the weakening of the convexity requirements for the objective and the constraint functions. In this section, we have introduced two types of duals: Mond-Weir-type and Schaible-type duals in terms of convexifactors which are more general than the duals existing in the literature.

We associate the following Mond-Weir-type dual with problem (P).

$$(\mathrm{D}_1) \text{ Maximize } \phi(u) = \left\{ \frac{f_1(u)}{g_1(u)}, \frac{f_2(u)}{g_2(u)}, \ldots, \frac{f_p(u)}{g_p(u)} \right\}$$

subject to

$$\begin{aligned} &0 \in \sum_{i=1}^{p} \lambda_i (\partial^* f_i(u) - v_i \partial^* g_i(u)) + \sum_{j=1}^{m} \gamma_j \partial^* h_j(u) \\ &\gamma_j h_j(u) \geq 0, \quad j = 1, 2, \ldots, m \end{aligned} \tag{5}$$

where $0 \neq \lambda \in R_+^p, \gamma \in R_+^m, v_i = \phi_i(u) = \frac{f_i(u)}{g_i(u)}, i = 1, 2, \ldots, p.$

Here, maximizing means finding weak efficient solutions in the following sense:

A feasible solution $(u^*, \lambda^*, \gamma^*, v^*)$ of the dual (D_1) is said to be a weak efficient solution of (D_1) if there does not exist any feasible solution (u, λ, γ, v) of (D_1) such that

$$\frac{f_i(u)}{g_i(u)} > \frac{f_i(u^*)}{g_i(u^*)}, i = 1, 2, \ldots, p.$$

We shall now prove the weak duality theorem.

Theorem 4.1. *(Weak Duality). Let x be feasible for (P) and (u, λ, γ, v) be feasible for (D_1). Suppose that $\partial^* f_i(u)$, $i = 1, 2, \ldots, p$ is an upper regular convexifactor of $f_i(.)$, $i = 1, 2, \ldots, p$ at u and $\partial^* g_i(u)$, $i = 1, 2, \ldots, p$ is a lower regular convexifactor of $g_i(.)$, $i = 1, 2, \ldots, p$ at u. If $f_i(.) - v_i g_i(.)$, $i = 1, 2, \ldots, p$ is ∂^*-pseudoconvex at u, $\gamma_j h_j(.)$, $j = 1, 2, \ldots, m$ is ∂^*-quasiconvex at u, then*

$$\phi(x) \nless \phi(u).$$

Proof. Since $\partial^* f_i(u)$, $i = 1, 2, \ldots, p$ is an upper regular convexifactor of $f_i(.)$, $i = 1, 2, \ldots, p$ at u and $\partial^* g_i(u)$, $i = 1, 2, \ldots, p$ is a lower regular convexifactor of $g_i(.)$, $i = 1, 2, \ldots, p$ at u, using Remark 2.17 and Lemma 2.16, we have that $\partial^* f_i(u) - v_i \partial^* g_i(u)$, $i = 1, 2, \ldots, p$ is a convexifactor of $f_i(.) - v_i g_i(.)$, $i = 1, 2, \ldots, p$ at u.

On the contrary, suppose that $\phi(x) < \phi(u)$.

Then,

$$f_i(x) - v_i g_i(x) < 0, \quad i = 1, 2, \ldots, p, \tag{6}$$

$$\text{where } v_i = \phi_i(u) = \frac{f_i(u)}{g_i(u)}, i = 1, 2, \ldots, p.$$

Since (u, λ, γ, v) is feasible for (D_1), therefore, there exist $\xi_i \in \partial^* f_i(u) - v_i \partial^* g_i(u)$, $i = 1, 2, \ldots, p$, $\zeta_j \in \partial^* h_j(u)$, $j = 1, 2, \ldots, m$ such that

$$\sum_{i=1}^{p} \lambda_i \xi_i + \sum_{j=1}^{m} \gamma_j \zeta_j = 0. \tag{7}$$

Using (6), the feasibility of x for (P), and the feasibility of (u, λ, γ, v) for (D_1), we get

$$f_i(x) - v_i g_i(x) < 0 = f_i(u) - v_i g_i(u), \quad i = 1, 2, \ldots, p \tag{8}$$

and

$$\gamma_j h_j(x) \leq 0 \leq \gamma_j h_j(u), \quad j = 1, 2, \ldots, m. \tag{9}$$

Since $f_i(.) - \nu_i g_i(.)$, $i = 1, 2,\ldots, p$ is ∂^*-pseudoconvex at u, we have from (8),

$\langle \xi_i, x - u\rangle < 0$ for all $\xi_i \in \partial^* f_i(u) - \nu_i\, \partial^* g_i(u)$, $i = 1, 2,\ldots, p$.

As $0 \neq \lambda \in R^p_+$, we get

$$\langle \sum_{i=1}^{p} \lambda_i \xi_i, x - u\rangle < 0 \text{ for all } \xi_i \in \partial^* f_i(u) - \nu_i \partial^* g_i(u). \tag{10}$$

Now, the ∂^*-quasiconvexity of $\gamma_j h_j$, $j = 1, 2,\ldots, m$ and (9) gives us

$\langle \zeta_j, x - u\rangle \leq 0$ for all $\zeta_j \in \partial^*(\gamma_j h_j)(u)$, $j = 1, 2,\ldots, m$

which on using Lemma 2.16 implies that

$\langle \gamma_j \zeta_j, x - u\rangle \leq 0$ for all $\zeta_j \in \partial^* h_j(u)$, $j = 1, 2,\ldots, m$.

As $\gamma_j \geq 0$, $j = 1, 2,\ldots, m$, we have

$$\langle \sum_{j=1}^{m} \gamma_j \zeta_j, x - u\rangle \leq 0, \quad \text{for all } \zeta_j \in \partial^* h_j(u). \tag{11}$$

Adding (10) and (11), we get

$$\langle \sum_{i=1}^{p} \lambda_i \xi_i + \sum_{j=1}^{m} \gamma_j \zeta_j, x - u\rangle < 0$$

which is a contradiction to (7).

In the next theorem, we shall prove the strong duality result.

Theorem 4.2. *(Strong Duality). Let x^* be a weak efficient solution of (P). Assume that the hypotheses of Theorem 3.4 hold. Then, there exists $(\lambda^*, \gamma^*, \nu^*) \in R^p \times R^m \times R^p$ such that $(x^*, \lambda^*, \gamma^*, \nu^*)$ is feasible for dual (D_1). If for each feasible x for (P) and $(u, \lambda, \gamma, \nu)$ for (D_1) hypotheses of Theorem 4.1 hold, then $(x^*, \lambda^*, \gamma^*, \nu^*)$ is a weak efficient solution of (D_1).*

Proof. Since x^* is a weak efficient solution of (P) and all the assumptions of Theorem 3.4 are satisfied, therefore, there exist vectors $0 \neq \lambda^* \in R^p_+$ and $\gamma^* \in R^m_+$ such that (1), (2), and (3) hold.

That is,

$$0 \in \sum_{i=1}^{p} \lambda_i^* (\partial^* f_i(x^*) - \phi_i(x^*) \partial^* g_i(x^*)) + \sum_{j=1}^{m} \gamma_j^* \partial^* h_j(x^*)$$

$$\gamma_j^* h_j(x^*) = 0, \;\; j = 1, 2, \ldots, m$$

$$\gamma_j^* \geq 0, h_j(x^*) \leq 0, j = 1, 2, \ldots, m, \phi_i(x^*) = \frac{f_i(x^*)}{g_i(x^*)}, i = 1, 2, \ldots, p$$

which implies that $(x^*, \lambda^*, \gamma^*, \nu^*)$ is feasible for dual where $\nu_i^* = \frac{f_i(u^*)}{g_i(u^*)}, i = 1, 2, \ldots, p$. Let, if possible, $(x^*, \lambda^*, \gamma^*, \nu^*)$ be not a weak efficient solution of (D_1). Then, there exists $(u, \lambda, \gamma, \nu)$ feasible for dual such that $\phi(u) > \phi(x^*)$.

However, this is a contradiction to Theorem 4.1 as x^* is feasible for (P) and $(u, \lambda, \gamma, \nu)$ is feasible for (D_1).

Hence, $(x^*, \lambda^*, \gamma^*, \nu^*)$ is a weak efficient solution of (D_1).

We now provide an example illustrating Theorem 4.1.

Example 4.3 Consider the problem

(P) Minimize $\left[\frac{f_1(x)}{g_1(x)}, \frac{f_2(x)}{g_2(x)}\right]$

subject to $h_1(x) \leq 0$, $h_2(x) \leq 0$,

where f_i, g_i, h_j: $R \to R$, $i = 1, 2$, $j = 1, 2$ are defined by

$$f_1(x) = \begin{cases} 1 + \frac{x}{10}, x \geq 0 \\ 1 + x^2, \; x < 0 \end{cases}, \qquad f_2(x) = \begin{cases} 2 + x^2 + 2x, & x \geq 0 \\ 2 + x, & x < 0 \end{cases}$$

$$g_1(x) = \begin{cases} -1 + 2x + x^2, & x \geq 0 \\ -1 + 3x, & x < 0 \end{cases}, \quad g_2(x) = \begin{cases} 2 + x^2, & x \geq 0 \\ 2 + \frac{x}{2}, & x < 0 \end{cases}$$

$$h_1(x) = \begin{cases} -x, & x \geq 0 \\ x^2, & x < 0 \end{cases}, \qquad h_2(x) = \begin{cases} 2 - x, & x \geq 0 \\ 2 + x^2, & x < 0 \end{cases}$$

The set of feasible solutions of (P) is $E = [2, \infty[$, and its dual is given by

(D_1) Maximize $\left[\frac{f_1(u)}{g_1(u)}, \frac{f_2(u)}{g_2(u)}\right]$

subject to

$$0 \in \sum_{i=1}^{2} \lambda_i (\partial^* f_i(u) - \nu_i \partial^* g_i(u)) + \sum_{j=1}^{2} \gamma_j \partial^* h_j(u)$$

$$\gamma_j h_j(u) \geq 0, \quad j = 1, 2,$$

where $0 \neq \lambda \in R^2_+, \gamma \in R^2_+$, and $\nu_i = \frac{f_i(u)}{g_i(u)}, i = 1, 2$. $(u, \lambda_1, \lambda_2, \gamma_1, \gamma_2) = \left(0, \frac{1}{4}, 1, 2, \frac{1}{4}\right)$ is feasible for dual (D_1).

$$\partial^* f_1(0) = \left\{0, \frac{1}{10}\right\}, \partial^* f_2(0) = \{1, 2\},$$

$$\partial^* g_1(0) = \{2, 3\}, \partial^* g_2(0) = \left\{0, \frac{1}{2}\right\},$$

$$\partial^* h_1(0) = \{-1, 0\}, \partial^* h_2(0) = \{-2, 0\}$$

$$\partial^* f_1(0) - \nu_1 \partial^* g_1(0) = \left\{2, 3, \frac{21}{10}, \frac{31}{10}\right\},$$

$$\partial^* f_2(0) - \nu_2 \partial^* g_2(0) = \left\{1, \frac{1}{2}, 2, \frac{3}{2}\right\},$$

where

$$\nu_1 = \frac{f_1(u)}{g_1(u)} = -1, \quad \nu_2 = \frac{f_2(u)}{g_2(u)} = 1$$

$f_1(.) - \nu_1 g_1(.)$ and $f_2(.) - \nu_2 g_2(.)$ are ∂^*-pseudoconvex at $u = 0$.

$\gamma_1 h_1(.)$ and $\gamma_2 h_2(.)$ are ∂^*-quasiconvex at $u = 0$.

We can see that for feasible point $x = 2$ for (P) and $(u, \lambda_1, \lambda_2, \gamma_1, \gamma_2) = \left(0, \frac{1}{4}, 1, 2, \frac{1}{4}\right)for(D_1)$,

$$\left(\frac{f_1(x)}{g_1(x)}, \frac{f_2(x)}{g_2(x)}\right) = \left(\frac{6}{35}, \frac{5}{3}\right) \nleq \left(\frac{f_1(u)}{g_1(u)}, \frac{f_2(u)}{g_2(u)}\right) = (-1, 1).$$

Hence, Theorem 4.1 is illustrated.

Remark 4.4. There do exist functions which are both ∂^*-pseudoconvex and ∂^*-quasiconvex as can be seen from the following example.

Example 4.5. Let f: $R^2 \to R$ be a function defined by

$$f(x,y) = \begin{cases} \sqrt{x}+y, & x>0, y>0, \\ \sqrt{x}-y, & x>0, y\leq 0, \\ \sqrt{-x}+y, & x\leq 0, y>0, \\ -\sqrt{-x}+y, & x\leq 0, y\leq 0, \end{cases}$$

$$\partial^* f(0,0) = \{(x^*, y^*) : x^* \geq 1, y^* \geq 1\}.$$

f is ∂^*-pseudoconvex and ∂^*-quasiconvex at $(\bar{x}, \bar{y})$ = (0,0).

We now associate the Schaible-type dual with (P) which is given as follows:

(D_2) Maximize $v = (v_1, v_2, \ldots, v_p)$

subject to

$$\begin{aligned} &0 \in \sum_{i=1}^{p} \lambda_i(\partial^* f_i(u) - v_i \partial^* g_i(u)) + \sum_{j=1}^{m} \gamma_j \partial^* h_j(u) \\ &\sum_{i=1}^{p} \lambda_i(f_i(u) - v_i g_i(u)) \geq 0, \\ &\sum_{j=1}^{m} \gamma_j h_j(u) \geq 0, \\ &0 \neq \lambda \in R_+^p, \gamma \in R_+^m, v_i \geq 0, i = 1, 2, \ldots, p. \end{aligned} \tag{12}$$

Remark 4.6. If we assume that f_i, g_i, i = 1, 2,. . ., p, h_j, j = 1, 2,. . ., m are differentiable and admit upper regular convexifactors $\partial^* f_i(u)$, $\partial^* g_i(u)$, i = 1, 2,. . ., p and $\partial^* h_j(u)$, j = 1, 2,. . ., m at u, respectively, then the Schaible-type dual reduces to the following:

Maximize $v = (v_1, v_2, \ldots, v_p)$

subject to

$$\begin{aligned} &\sum_{i=1}^{p} \lambda_i(\nabla f_i(u) - v_i \nabla g_i(u)) + \sum_{j=1}^{m} \gamma_j \nabla h_j(u) = 0 \\ &\sum_{i=1}^{p} \lambda_i(f_i(u) - v_i g_i(u)) \geq 0, \\ &\sum_{j=1}^{m} \gamma_j h_j(u) \geq 0, \\ &0 \neq \lambda \in R_+^p, \gamma \in R_+^m, v_i \geq 0, i = 1, 2, \ldots, p, \end{aligned}$$

which is similar to the dual given by Suneja and Gupta [14].

We shall now prove the weak duality and strong duality results.

Theorem 4.7. (Weak Duality). Let x be feasible for (P) and (u, λ, γ, v) be feasible for (D_2). Suppose that $\partial^* f_i(u)$, i = 1, 2, . . ., p is an upper regular convexifactor of f_i (.), i = 1, 2,. . ., p at u and $\partial^* g_i(u)$, i = 1, 2,. . ., p is a lower regular convexifactor of g_i (.), i = 1, 2,. . ., p at u. Also, assume that for some i, and some j, $\partial^* f_i(u) - v_i \partial^* g_i(u)$, $\partial^* h_j(u)$ are respectively upper regular convexifactors of $f_i(.) - v_i g_i(.)$, i = 1, 2,. . ., p and h_j (.), j = 1, 2, ..., m at u, and for some $i_0 \neq i$, $j_0 \neq j$, $\partial^* f_{i_0}(u) - v_{i_0} \partial^* g_{i_0}(u), \partial^* h_{j_0}(u)$ are respectively lower regular convexifactors of $f_{i_0}(.) - v_{i_0} g_{i_0}(.)$, and $h_{j_0}(.)$, at u. *If* $\sum_{i=1}^{p} \lambda_i(f_i(.) - v_i g_i(.))$ is ∂^*-pseudoconvex at u and, $\sum_{j=1}^{m} \gamma_j h_j(.)$ is ∂^*-quasiconvex at u, then

$\phi(x) \nless v$.

Proof. Since $\partial^* f_i(u)$, i = 1, 2,. . ., p is an upper regular convexifactor of $f_i(.)$, i = 1, 2,. . ., p at u and $\partial^* g_i(u)$, i = 1, 2,. . ., p is a lower regular convexifactor of $g_i(.)$, i = 1, 2,. . ., p at u, using Remark 2.17 and Lemma 2.16, we have that $\partial^* f_i(u) - v_i \partial^* g_i(u)$, i = 1, 2,. . ., p is a convexifactor of $f_i(.) - v_i g_i(.)$, i = 1, 2,. . ., p at u. Also, since for some i, and some j, $\partial^* f_i(u) - v_i \partial^* g_i(u)$, $\partial^* h_j(u)$ are respectively upper regular convexifactors of $f_i(.) - v_i g_i(.)$, i = 1, 2,. . ., p and $h_j(.)$, j = 1, 2,.., m at u, and for some $i_0 \neq i$, $j_0 \neq j$, $\partial^* f_{i_0}(u) - v_{i_0} \partial^* g_{i_0}(u), \partial^* h_{j_0}(u)$ are respectively lower regular convexifactors of $f_{i_0}(.) - v_{i_0} g_{i_0}(.)$, and $h_{j_0}(.)$, at u using Remark 2.17 and Lemma 2.16, we have that $\sum_{i=1}^{p} \lambda_i(\partial^* f_i(u) - v_i \partial^* g_i(u)), \sum_{j=1}^{m} \gamma_j \partial^* h_j(u)$ are convexifactors of $\sum_{i=1}^{p} \lambda_i(f_i(.) - v_i g_i(.))$ and $\sum_{j=1}^{m} \gamma_j h_j(.)$at u, respectively.

On the contrary, suppose $\phi(x) < v$.

Then,

$$f_i(x) - v_i g_i(x) < 0, \quad i = 1, 2, \ldots, p. \tag{13}$$

Since (u, λ, γ, v) is feasible for (D_2), therefore, there exist $\xi_i \in (\partial^* f_i(u) - v_i \partial^* g_i(u))$, i = 1, 2,. . ., p, $\zeta_j \in \partial^* h_j(u)$, j = 1, 2,. . ., m such that

$$\sum_{i=1}^{p} \lambda_i \xi_i + \sum_{j=1}^{m} \gamma_j \zeta_j = 0. \tag{14}$$

Using (13), $0 \neq \lambda \in R_+^p, \gamma \in R_+^m$, and the fact that x is feasible for (P), we get that

$$\begin{aligned} &\sum_{i=1}^{p} \lambda_i(f_i(x) - v_i g_i(x)) < 0 \\ &\sum_{j=1}^{m} \gamma_j h_j(x) \leq 0. \end{aligned} \tag{15}$$

Since (u, λ, γ, v) is feasible for (D_2), using (15) we have

$$\begin{aligned} &\sum_{i=1}^{p} \lambda_i(f_i(x) - v_i g_i(x)) < 0 \leq \sum_{i=1}^{p} \lambda_i(f_i(u) - v_i g_i(u)) \\ &\sum_{j=1}^{m} \gamma_j h_j(x) \leq 0 \leq \sum_{j=1}^{m} \gamma_j h_j(u). \end{aligned}$$

Using ∂^*-pseudoconvexity of $\sum_{i=1}^{p} \lambda_i(f_i(.) - v_i g_i(.))$ and ∂^*-quasiconvexity of $\sum_{j=1}^{m} \gamma_j h_j(.)$, we have

$$\langle \sum_{i=1}^{p} \lambda_i \xi_i, x-u \rangle < 0 \text{ for } \xi_i \in \partial^* f_i(u) - \nu_i \partial^* g_i(u), i = 1, 2, \ldots, p \tag{16}$$

and

$$\langle \sum_{j=1}^{m} \gamma_j \zeta_j, x-u \rangle \leq 0 \text{ for } \zeta_j \in \partial^* h_j(u), j = 1, 2, \ldots, m. \tag{17}$$

Adding (16) and (17), we get

$$\langle \sum_{i=1}^{p} \lambda_i \xi_i + \sum_{j=1}^{m} \gamma_j \zeta_j, x-u \rangle < 0$$

which is a contradiction to (14).

Theorem 4.8. *(Strong Duality). Let x^* be a weak efficient solution of (P). Assume that the hypotheses of Theorem 3.4 hold. Then, there exists $(\lambda^*, \gamma^*, \nu^*) \in R^p \times R^m \times R^p$ such that $(x^*, \lambda^*, \gamma^*, \nu^*)$ is feasible for dual. If for each feasible x for (P) and $(u, \lambda, \gamma, \nu)$ for (D_2) hypotheses of Theorem 4.7 hold, then $(x^*, \lambda^*, \gamma^*, \nu^*)$ is a weak efficient solution of (D_2).*

Proof. Since x^* is a weak efficient solution of (P) and all the assumptions of Theorem 3.4 are satisfied, therefore, there exist vectors $0 \neq \lambda^* \in R^p_+$, $\gamma^* \in R^m_+$, and $\nu_i^* \geq 0$, $i = 1, 2, \ldots, p$ such that (1), (2), and (3) hold.

That is,

$$0 \in \sum_{i=1}^{p} \lambda_i^* \left(\partial^* f_i(x^*) - \nu_i^* \partial^* g_i(x^*) \right) + \sum_{j=1}^{m} \gamma_j^* \partial^* h_j(x^*)$$
$$\gamma_j^* h_j(x^*) = 0, \quad j = 1, 2, \ldots, m$$
$$\gamma_j^* \geq 0, h_j(x^*) \leq 0, j = 1, 2, \ldots, m$$

where

$$\nu_i^* = \phi_i(x^*), i = 1, 2, \ldots, p \tag{18}$$

which implies that $(x^*, \lambda^*, \gamma^*, \nu^*)$ is feasible for dual (D_2). Let, if possible, $(x^*, \lambda^*, \gamma^*, \nu^*)$ be not a weak efficient solution of (D_2). Then, there exists $(u, \lambda, \gamma, \nu)$ feasible for dual such that $\nu^* < \nu$.

On using (18), we have

$$\phi(x^*) < \nu.$$

However, this is a contradiction to Theorem 4.7 as x^* is feasible for (P) and $(u, \lambda, \gamma, \nu)$ is feasible for (D_2).

Hence, x^* is a weak efficient solution of (P).

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
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