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A Suzuki type unique common fixed point theorem for hybrid pairs of maps under a new condition in partial metric spaces

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Abstract

In this paper, we introduce a new condition namely, the (W.C.C) condition and give some Suzuki-type, unique, common fixed-point theorems for pairs of hybrid mappings in partial metric spaces using a partial Hausdorff metric. These results generalize and extend the several comparable results in this literature in metric and partial metric spaces.

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Introduction and preliminaries

The study of fixed points for multi-valued maps using a Hausdorff metric was initiated by Nadler [1] who proved the following:

Theorem 1. *Let (X, d) be a complete metric space and $T : X \to CB(X)$ be a mapping satisfying $H(Tx, Ty) \le kd(x, y)$, where $k \in [0, 1)$ then there exists $x \in X$ such that $x \in Tx$.*

Later, an interesting and rich fixed-point theory was developed and extended Theorem 1 using weak and generalized contraction mappings (see [2-7]). The theory of multi-valued maps has many applications in control theory, convex optimization, differential equations, and economics (see [8]). On the other hand, the basic notion of a partial metric space was introduced by Mathews [9] as a part of the study of denotational semantics of data flow networks. He presented a modified version of the Banach contraction principle, which is more suitable in this context (see also [10,11]). In fact, the partial metric spaces constitute a suitable framework to model several distinguished examples of the theory of computation and also to model metric spaces via the domain theory (see [12-31]). In this direction, Aydi et al. [32] introduced the concept of a partial Hausdorff metric and extended Nadler's fixed-point theorem in the setting of partial metric spaces.

Consistent with [9,32,33], the following definitions and results will be needed in the sequel:

Definition 1. ([9]). A partial metric on a nonempty set X is a function $p : X \times X \to \mathbb{R}^+$ such that for all $x, y, z \in X$:

(p_1) $x = y \Leftrightarrow p(x, x) = p(x, y) = p(y, y)$,
(p_2) $p(x, x) \le p(x, y)$,
(p_3) $p(x, y) = p(y, x)$,
(p_4) $p(x, y) \le p(x, z) + p(z, y) - p(z, z)$.

In this case, (X, p) is called a partial metric space.

It is clear that $|p(x, y) - p(y, z)| \le p(x, z)$ $\forall x, y, z \in X$. It is also clear that $p(x, y) = 0$ implies $x = y$ from (p_1) and (p_2). However, if $x = y$, $p(x, y)$ may not be zero. A basic example of a partial metric space is the pair $(\mathbb{R}^+, p)$, where $p(x, y) = \max\{x, y\}$ for all $x, y \in \mathbb{R}^+$. Each partial metric p on X generates a τ_0 topology τ_p on X which has a base, the family of open p - balls $\{B_p(x, \epsilon) \mid x \in X, \epsilon > 0\}$ for all $x \in X$ and $\epsilon > 0$, where $B_p(x, \epsilon) = \{y \in X \mid p(x, y) < p(x, x)+\epsilon\}$ for all $x \in X$ and $\epsilon > 0$. If p is a partial metric on

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X, then the function $p^s : X \times X \to \mathbb{R}^+$ given by $p^s(x, y) = 2p(x, y) - p(x, x) - p(y, y)$ is a metric on X.

Definition 2. ([9]). Let (X, p) be a partial metric space:

(i) A sequence $\{x_n\}$ in (X, p) is said to converge to a point $x \in X$ if and only if $p(x, x) = \lim_{n\to\infty} p(x, x_n)$.
(ii) A sequence $\{x_n\}$ in (X, p) is said to be a Cauchy sequence if $\lim_{n,m\to\infty} p(x_n, x_m)$ exists and is finite.
(iii) (X, p) is said to be complete if every Cauchy sequence $\{x_n\}$ in X converges, with respect to τ_p, to a point $x \in X$ such that $p(x, x) = \lim_{n,m\to\infty} p(x_n, x_m)$.

Lemma 1. *([9]). Let (X, p) be a partial metric space:*

(a) $\{x_n\}$ is a Cauchy sequence in (X, p) if and only if it is a Cauchy sequence in the metric space (X, p^s).
(b) (X, p) is complete if the metric space (X, p^s) is complete. Furthermore, $\lim_{n\to\infty} p^s(x_n, x) = 0$ if and only if

$$p(x, x) = \lim_{n\to\infty} p(x_n, x) = \lim_{n,m\to\infty} p(x_n, x_m).$$

Lemma 2. *([33]). Let (X, p) be a partial metric space and A any nonempty set in X. Then, $a \in \overline{A}$ if and only if $p(a, A) = p(a, a)$, where $\overline{A}$ denotes the closure of A with respect to the topology of the partial metric p.*

Note that A is closed in (X, p) if and only if $A = \overline{A}$.

Consistent with [32], let (X, p) be a partial metric space. Let $\mathrm{CB}^p(X)$ be the family of all nonempty, closed, and bounded subsets of the partial metric space (X, p), induced by the partial metric p. For $A, B \in \mathrm{CB}^p(X)$ and $x \in X$, define

$$p(A, B) = \inf\left\{p(a, b) : a \in A, b \in B\right\},$$

$$p(x, A) = \inf\left\{p(x, a) : a \in A\right\},$$

and

$$\delta_p(A, B) = \sup\left\{p(a, B) : a \in A\right\}, \quad \delta_p(B, A) = \sup\left\{p(b, A) : b \in B\right\}.$$

Also,

$$H_p(A, B) = \max\left\{\delta_p(A, B), \delta_p(B, A)\right\}.$$

H_p is called the partial Hausdorff metric induced by a partial metric p.

Also, Aydi et al. [32] proved that any Hausdorff metric is a partial Hausdorff metric and the converse is not true (see Example 2.6 in [32]):

Lemma 3. *([32]). Let (X, p) be a partial metric space. For any $A, B, C \in CB^p(X)$, we have*

(i) $\delta_p(A, A) = \sup\{p(a, a) : a \in A\}$,
(ii) $\delta_p(A, A) \leq \delta_p(A, B)$,
(iii) $\delta_p(A, B) = 0$ implies that $A \subseteq B$,
(iv) $\delta_p(A, B) \leq \delta_p(A, C) + \delta_p(C, B) - \inf_{c\in C} p(c, c)$.

Lemma 4. *([32]). Let (X, p) be a partial metric space. For any $A, B, C \in CB^p(X)$, we have*

(i) $H_p(A, A) \leq H_p(A, B)$,
(ii) $H_p(A, B) = H_p(B, A)$,
(iii) $H_p(A, B) \leq H_p(A, C) + H_p(C, B) - \inf_{c\in C} p(c, c)$.

Lemma 5. *([32]). Let (X, p) be a partial metric space. For any $A, B \in CB^p(X)$, the following holds*

$$H_p(A, B) = 0 \quad \textit{implies that} \quad A = B.$$

In [32], they also show that $H_p(A, A)$ need not be zero by an example.

Lemma 6. *([32]). Let (X, p) be a partial metric space, $A, B \in CB^p(X)$, and $h > 1$. For any $a \in A$, there exists $b \in B$ such that $p(a, b) \leq hH_p(A, B)$.*

Theorem 2. *([32]). Let (X, p) be a complete partial metric space and $T : X \to CB^p(X)$ is a multi-valued mapping such that for all $x, y \in X$*

$$H_p(Tx, Ty) \leq k\, p(x, y),$$

where $k \in (0, 1)$, then T has a fixed point.

Very recently, Abbas et al. [34] generalized Theorem 2 by proving the following Suzuki type theorem:

Theorem 3. *Let (X, p) be a complete partial metric space. Take $T : X \to CB^p(X)$ a multi-valued mapping and $\varphi : [0, 1) \to (0, 1]$ a nonincreasing function defined by*

$$\varphi(r) = \begin{cases} 1 & \text{if } 0 \leq r < \frac{1}{2}, \\ 1 - r & \text{if } \frac{1}{2} \leq r < 1. \end{cases} \tag{1}$$

If there exists $r \in [0, 1)$ such that T satisfies the condition

(A) $\varphi(r)p(x, Tx) \leq p(x, y)$ implies
$$H_p(Tx, Ty) \leq r \max\left\{\begin{array}{c} p(x, y), p(x, Tx), p(y, Ty), \\ \frac{1}{2}[p(x, Ty) + p(y, Tx)] \end{array}\right\}$$

for all $x, y \in X$, then T has a fixed point, that is, there exists a point $z \in X$ such that $z \in Tz$.

Now, we give the following commutativity definitions mentioned in [35].

Definition 3. ([35]) Let (X, p) be a partial metric space. Let $f : X \to X$ and $S : X \to CB^p(X)$. The pair (f, S) is called

(i) commuting if $fSx = Sfx, \forall x \in X$,
(ii) weakly compatible if the pair (f, S) commutes at their coincidence points, that is, $fSx = Sfx$ whenever $fx \in Sx$ for $x \in X$,
(iii) IS-commuting at $x \in X$ if $fSx \subseteq Sfx$.

Generally, to prove a coincidence point or a common fixed-point theorem for hybrid mappings, one has to assume a commutativity condition and continuity of mappings. In this paper, we introduce a new condition and prove a unique common fixed-point theorem for hybrid mappings in partial metric spaces without using any standard arguments as commutativity and continuity conditions.

Main results

We start with the following lemma which is needed to prove our main results:

Lemma 7. *Let $x_n \to x$ as $n \to \infty$ in a partial metric space (X,p) such that $p(x,x) = 0$, then $\lim_{n\to\infty} p(x_n, B) = p(x, B)$ for any $B \in CB^p(X)$.*

Proof. Since $x_n \to x$, we have $\lim_{n\to\infty} p(x_n, x) = p(x, x) = 0$. Applying a triangular inequality for $x_n \in X$ and $y \in B$, we get

$$\begin{aligned} p(x_n, B) \le p(x_n, y) &\le p(x_n, x) + p(x, y) - p(x, x) \\ &\le p(x_n, x) + p(x, y) \end{aligned}$$

which implies that $\lim_{n\to\infty} p(x_n, B) \le p(x, y)$ for all $y \in B$. Therefore,

$$(i) \qquad \lim_{n\to\infty} p(x_n, B) \le p(x, B).$$

Similarly,

$$p(x, y) \le p(x, x_n) + p(x_n, y) - p(x_n, x_n)$$

so $p(x, y) \le p(x, x_n) + p(x_n, y)$. Thus, $p(x, B) \le p(x, x_n) + p(x_n, B)$. Therefore,

$$(ii) \qquad p(x, B) \le \lim_{n\to\infty} p(x_n, B).$$

From (i) and (ii), we have $\lim_{n\to\infty} p(x_n, B) = p(x, B)$. □

Now, we introduce the following new condition, namely the W.C.C. condition, on mappings which are not necessarily continuous and commutative.

Definition 4. Let (X, p) be a partial metric space. Let $f : X \to X$ and $S : X \to CB^p(X)$ be mappings. Then, the pair (f, S) is said to satisfy the W.C.C. condition if $p(fx, fy) \le p(y, Sx), \forall x, y \in X$.

The following example illustrates the W.C.C. condition:

Example 1. Let $X = [0, 1]$ and $p(x, y) = \max\{x, y\}$, $\forall x, y \in X$. Let $f : X \to X$ and $S : X \to CB^p(X)$ be defined by

$$fx = \begin{cases} 0 & \text{if } x \in [0, \frac{1}{2}] \\ \frac{3x}{4} & \text{if } x \in (\frac{1}{2}, 1] \end{cases}$$

and $Sx = [\frac{3}{4}, 1], \ \forall x, y \in X$. We consider the following four cases:

Case 1: $x \in [0, \frac{1}{2}]$ and $y \in [0, \frac{1}{2}]$. Here, $p(fx, fy) = 0 < p(y, Sx)$.
Case 2: $x \in [0, \frac{1}{2}]$ and $y \in (\frac{1}{2}, 1]$. Then, $p(fx, fy) = \frac{3y}{4} \le \frac{3}{4} = p(y, Sx)$.
Case 3: $x \in (\frac{1}{2}, 1]$ and $y \in [0, \frac{1}{2}]$. Here, $p(fx, fy) = \frac{3x}{4} \le \frac{3}{4} = p(y, Sx)$.
Case 4: $x \in (\frac{1}{2}, 1]$ and $y \in (\frac{1}{2}, 1]$. Then, $p(fx, fy) = \max\{\frac{3x}{4}, \frac{3y}{4}\} \le \frac{3}{4} = p(y, Sx)$.

Thus (f, S) satisfies the W.C.C. condition. In this example, the pair (f, S) does not satisfy any type of commutativity mentioned in Definition 3.

The following example shows that the pair (f, S) satisfying the W.C.C condition need not be continuous even when S is a single-valued mapping:

Example 2. Let $X = [0, 1]$ and $p(x, y) = \max\{x, y\}$, $\forall x, y \in X$. Let $f, S : X \to X$ be defined by

$$fx = \begin{cases} \frac{x}{6} & \text{if } x \ne 1 \\ \frac{1}{4} & \text{if } x = 1 \end{cases}$$

and

$$Sx = \begin{cases} x & \text{if } x \ne 1 \\ \frac{1}{2} & \text{if } x = 1. \end{cases}$$

We distinguish the following cases:

Case (i): $x \ne 1$ and $y \ne 1$. We have $p(fx, fy) = \max\{\frac{x}{6}, \frac{y}{6}\} = \frac{1}{6}\max\{x, y\} = \frac{1}{6}p(y, Sx)$.
Case (ii): $x \ne 1$ and $y = 1$. Then, $p(fx, fy) = \max\{\frac{x}{6}, \frac{1}{4}\} = \frac{1}{4} < 1 = p(y, Sx)$.
Case (iii): $x = 1$ and $y \ne 1$. We have $p(fx, fy) = \max\{\frac{1}{4}, \frac{y}{6}\} = \frac{1}{4} < \frac{1}{2} \le p(y, Sx)$.
Case (iv): $x = 1$ and $y = 1$. Here, $p(fx, fy) = \frac{1}{4} < 1 = p(y, Sx)$.

Thus (f, S) satisfies the W.C.C. condition.

In this example, note that f and S are discontinuous.

Now, we state and prove our main results.

Theorem 4. *Let (X, p) be a complete partial metric space. Let $S, T : X \to CB^p(X)$ and $f : X \to X$. Assume that there exists $r \in [0, 1)$ such that for every $x, y \in X$*

$(A1)$ *$\varphi(r)$ min $\{p(fx, Sx), p(fy, Ty)\} \le p(fx, fy)$ implies*

$$H_p(Sx, Ty) \le r \max \left\{ \begin{array}{c} p(fx, fy), p(fx, Sx), p(fy, Ty), \\ \frac{1}{2}[p(fx, Ty) + p(fy, Sx)] \end{array} \right\}$$

where φ is defined by (1),

$(A2)$ $\bigcup_{x\in X} Sx \subseteq f(X)$ *and* $\bigcup_{x\in X} Tx \subseteq f(X)$,

$(A3)$ *The pair (f, S) or the pair (f, T) satisfies the W.C.C condition.*

Then f, S and T have a unique common fixed point in X.

Proof. Let $x_0 \in X$ and suppose that $h = \frac{1}{\sqrt{r}} > 0$, $y_0 = fx_0$. Now from $(A2)$, we have $Sx_0 \subseteq f(X)$, so there exists $x_1 \in X$ such that $y_1 = fx_1 \in Sx_0$.

By Lemma 6 with $h = \frac{1}{\sqrt{r}}$, there exists $y_2 \in Tx_1$ such that

$$p(fx_1, y_2) \le \frac{1}{\sqrt{r}} H_p(Sx_0, Tx_1).$$

Since $Tx_1 \subseteq f(X)$, we may find a point $x_2 \in X$ such that $y_2 = fx_2 \in Tx_1$. Therefore,

$$p(fx_1, fx_2) \le \frac{1}{\sqrt{r}} H_p(Sx_0, Tx_1).$$

Since $\varphi(r)p(fx_0, Sx_0) \le p(fx_0, Sx_0) \le p(fx_0, fx_1)$, we have

$$\varphi(r) \min \{p(fx_0, Sx_0), p(fx_1, Tx_1)\} \le p(fx_0, fx_1).$$

By $(A1)$, we have

$$\begin{aligned} p(fx_1, fx_2) &\le hH_p(Sx_0, Tx_1) = \tfrac{1}{\sqrt{r}} H_p(Sx_0, Tx_1), \\ &\le \sqrt{r} \max \left\{ \begin{array}{c} p(fx_0, fx_1), p(fx_0, Sx_0), p(fx_1, Tx_1), \\ \frac{1}{2}[p(fx_0, Tx_1) + p(fx_1, Sx_0)] \end{array} \right\} \\ &\le \sqrt{r} \max \left\{ \begin{array}{c} p(y_0, y_1), p(y_0, y_1), p(y_1, y_2), \\ \frac{1}{2}[p(y_0, y_2) + p(y_1, y_1)] \end{array} \right\} \\ p(y_1, y_2) &\le \sqrt{r} \max \{p(y_0, y_1), p(y_1, y_2), \tfrac{1}{2}[p(y_0, y_1) \\ &\quad + p(y_1, y_2)]\}, \text{ from } (p_4) \\ &\le \sqrt{r} \max \{p(y_0, y_1), p(y_1, y_2)\}. \end{aligned}$$

If $p(y_0, y_1) < p(y_1, y_2)$ then $p(y_1, y_2) \le \sqrt{r} p(y_1, y_2)$ which is a contradiction. Hence, $p(y_0, y_1) \ge p(y_1, y_2$. Thus, we have

$$p(y_1, y_2) \le \beta p(y_0, y_1), \tag{2}$$

where $\beta = \sqrt{r} < 1$.

As $fx_2 \in Tx_1$, from Lemma 6, we choose $y_3 \in Sx_2$ such that

$$p(fx_2, y_3) \le \frac{1}{\sqrt{r}} H_p(Sx_2, Tx_1).$$

Since $Sx_2 \subseteq g(X)$, we find a point $x_3 \in X$ such that $y_3 = fx_3 \in Sx_2$. Therefore,

$$p(fx_2, fx_3) \le \frac{1}{\sqrt{r}} H_p(Sx_2, Tx_1).$$

Since $\varphi(r)p(fx_1, Tx_1) \le p(fx_1, Tx_1) \le p(fx_2, fx_1)$, we have

$$\varphi(r) \min \{p(fx_2, Sx_2), p(fx_1, Tx_1)\} \le p(fx_2, fx_1).$$

Hence, by $(A1)$, we have

$$\begin{aligned} p(fx_2, fx_3) &\le \tfrac{1}{\sqrt{r}} H_p(Sx_2, Tx_1), \\ &\le \sqrt{r} \max \left\{ \begin{array}{c} p(fx_2, fx_1), p(fx_2, Sx_2), p(fx_1, Tx_1), \\ \frac{1}{2}[p(fx_2, Tx_1) + p(fx_1, Sx_2)] \end{array} \right\} \\ &\le \sqrt{r} \max \left\{ \begin{array}{c} p(y_2, y_1), p(y_2, y_3), p(y_1, y_2), \\ \frac{1}{2}[p(y_2, y_2) + p(y_1, y_3)] \end{array} \right\} \\ p(y_2, y_3) &\le \sqrt{r} \max \{p(y_1, y_2), p(y_2, y_3)]\}, \quad \text{from} \quad (p_4). \end{aligned}$$

Thus, we have

$$p(y_2, y_3) \le \beta p(y_1, y_2) \le \beta^2 p(y_0, y_1). \tag{3}$$

Continuing in this way, we obtain a sequence $\{y_n\}$ in X such that for any $n \in \mathbb{N}$,

$$y_{2n+1} = fx_{2n+1} \in Sx_{2n}, \quad y_{2n+2} = fx_{2n+2} \in Tx_{2n+1}$$

and

$$p(y_n, y_{n+1}) \le \beta^n p(y_0, y_1). \tag{4}$$

Clearly,

$$p(y_{n+1}, y_n) \to 0 \quad \text{as} \quad n \to \infty. \tag{5}$$

For $m > n$, we have

$$\begin{aligned} p(y_n, y_m) &\le p(y_n, y_{n+1}) + p(y_{n+1}, y_{n+2}) + \ldots + p(y_{m-1}, y_m), \\ &\le (\beta^n + \beta^{n+1} + \ldots + \beta^{m-1}) p(y_1, y_0), \text{ from (4)} \\ &\le \frac{\beta^n}{1-\beta} p(y_1, y_0) \to 0 \quad \text{as} \quad n \to \infty. \end{aligned} \tag{6}$$

Thus, $\{y_n\}$ is a Cauchy sequence in X. Hence from Lemma 1, we have $\{y_n\}$ is a Cauchy sequence in the metric space (X, p^s).

Since (X, p) is complete and again from Lemma 1, it follows that (X, p^s) is complete. So, $\{y_n\}$ converges to some z in (X, p^s). That is

$$\lim_{n\to\infty} p^s(y_n, z) = 0.$$

Now, from Lemma 1 and (6), we have

$$p(z, z) = \lim_{n\to\infty} p(y_n, z) = \lim_{n\to\infty} p(y_n, y_m) = 0. \tag{7}$$

Suppose the pair (f, S) satisfies the W.C.C condition. Then,

$$p(fx,fy) \leq p(y, Sx) \quad \text{for all} \quad x, y \in X. \tag{8}$$

From (8), we have

$$p(fx_{2n},fz) \leq p(z, Sx_{2n}) \leq p(z,fx_{2n+1}).$$

Letting $n \to \infty$ and using Lemma 7 and (7), we can obtain

$$p(z,fz) \leq 0 \quad \text{so that} \quad fz = z. \tag{9}$$

Claim : $p(fz, Sx) \leq r \max\{p(fx,fz), p(fx, Sx)\}$ for any $fx \in X - \{fz\}$. (10)

Let $fx \in X - fz$. Since $y_{2n+1} \to z = fz$, $y_{2n+2} \to z = fz$ and $p(z,z) = \lim_{n\to\infty} p(y_n, z) = 0$, there exists a positive integer n_0 such that for all $n \geq n_0$, we have

$$p(fz,fx_{2n+1}) \leq \frac{1}{3}p(fz,fx)$$

and

$$p(fz,fx_{2n+2}) \leq \frac{1}{3}p(fz,fx).$$

So, for any $n \geq n_0$, we have

$$\begin{aligned}
\varphi(r)p(fx_{2n+1}, Tx_{2n+1}) &\leq p(fx_{2n+1}, Tx_{2n+1}),\\
&\leq p(fx_{2n+2},fx_{2n+1}),\\
&\leq p(fx_{2n+2},fz) + p(fz,fx_{2n+1}),\\
&\leq \tfrac{2}{3}p(fz,fx),\\
&= p(fx,fz) - \tfrac{1}{3}p(fx,fz),\\
&\leq p(fx,fz) - p(fz,fx_{2n+1}),\\
&\leq p(fx,fx_{2n+1}).
\end{aligned}$$

Hence, we have

$$\varphi(r) \min\left\{p(fx, Sx), p(fx_{2n+1}, Tx_{2n+1})\right\} \leq p(fx,fx_{2n+1})$$

which implies that

$$\begin{aligned}
&p(fx_{2n+2}, Sx)\\
&\leq H_p(Sx, Tx_{2n+1}),\\
&\leq r \max\left\{\begin{array}{c} p(fx,fx_{2n+1}), p(fx, Sx), p(fx_{2n+1}, Tx_{2n+1}),\\ \frac{1}{2}\left[p(fx, Tx_{2n+1}) + p(fx_{2n+1}, Sx)\right]\end{array}\right\}.
\end{aligned}$$

Letting $n \to \infty$, we get

$$\begin{aligned}
p(fz, Sx) &\leq r \max\left\{\begin{array}{c} p(fz,fx), p(fx, Sx), p(fz,fz),\\ \frac{1}{2}[p(fx,fz) + p(fz, Sx)]\end{array}\right\}\\
&\leq r \max\left\{p(fx,fz), p(fx, Sx), \tfrac{1}{2}[p(fx,fz)\right.\\
&\qquad \left. +p(fz, Sx)]\right\}, \quad \text{from} \quad (p_2).
\end{aligned}$$

If $\max\left\{p(fx,fz), p(fx, Sx), \frac{1}{2}[p(fx,fz) + p(fz, Sx)]\right\} = \max\{p(fx,fz), p(fx, Sx)\}$, then

$$p(fz, Sx) \leq r \max\left\{p(fx,fz), p(fx, Sx)\right\}.$$

If $\max\left\{p(fx,fz), p(fx, Sx), \frac{1}{2}[p(fx,fz) + p(fz, Sx)]\right\} = \frac{1}{2}[p(fx,fz) + p(fz, Sx)]$, then

$$p(fz, Sx) \leq \frac{1}{2}[p(fx,fz) + p(fz, Sx)]$$

which implies that

$$(1 - \frac{r}{2})p(fz, Sx) \leq \frac{r}{2}p(fx,fz).$$

So,

$$\begin{aligned}
p(fz, Sx) &\leq \frac{r}{2-r}p(fx,fz) \leq rp(fx,fz)\\
&\leq r \max\left\{p(fx,fz), p(fx, Sx)\right\}.
\end{aligned}$$

Hence, (10) is proved.

Now, we will show that $fz \in Tz$. First, consider the case $0 \leq r < \frac{1}{2}$. On the contrary, suppose that $fz \notin Tz = \overline{Tz}$ as Tz is closed. Hence, by Lemma 2, together with (7) and (9), we have

$$p(fz, Tz) \neq p(fz,fz) = p(z, z) = 0. \tag{11}$$

Then, from (A2) and (11), we can choose $fa \in Tz$ such that

$$2rp(fa,fz) < p(fz, Tz). \tag{12}$$

Having $fa \in Tz$ and $fz \notin Tz$ imply $fa \neq fz$, then by (10)

$$p(fz, Sa) \leq r \max\{p(fz,fa), p(fa, Sa)\}. \tag{13}$$

Since $\varphi(r)p(fz, Tz) \leq p(fz, Tz) \leq p(fa,fz)$, so it follows that

$$\varphi(r) \min\left\{p(fa, Sa), p(fz, Tz)\right\} \leq p(fa,fz).$$

Now by $(A1)$, we have

$$\begin{aligned}
H_p(Sa, Tz) &\leq r \max\left\{\begin{array}{c} p(fa,fz), p(fa, Sa), p(fz, Tz),\\ \frac{1}{2}[p(fa, Tz) + p(fz, Sa)]\end{array}\right\}\\
&\leq r \max\left\{\begin{array}{c} p(fa,fz), p(fa, Sa), p(fz,fa)],\\ \frac{1}{2}[p(fa,fa) + p(fz,fa) + p(fa, Sa) - p(fa,fa)]\end{array}\right\}\\
&\leq r \max\left\{p(fa,fz), p(fa, Sa), \tfrac{1}{2}[p(fa, Sa) + p(fz,fa)]\right\}\\
&\leq r \max\left\{p(fa,fz), p(fa, Sa)\right\}.
\end{aligned}$$

Since $fa \in Tz$, then $p(fa, Sa) \leq H_p(Sa, Tz)$. Therefore, we obtain

$$H_p(Sa, Tz) \leq r \max\left\{p(fa,fz), H_p(Sa, Tz)\right\}.$$

Since $r < 1$, it follows that

$$H_p(Sa, Tz) \leq rp(fa, fz) \tag{14}$$

Thus, we have

$$p(fa, Sa) \leq H_p(Sa, Tz) \leq rp(fa, fz) < p(fa, fz). \tag{15}$$

By (13)

$$p(fz, Sa) \leq rp(fa, fz). \tag{16}$$

Also,

$$\begin{aligned} p(Sa, Tz) &= \inf\{p(x, y) : x \in Sa, y \in Tz\} \\ &\leq \inf\{p(x, fa) : x \in Sa\} \quad \text{since} \quad fa \in Tz \\ &= p(fa, Sa) \\ &\leq H_p(Sa, Tz) \end{aligned}$$

and by (14) and (16), we have

$$\begin{aligned} p(fz, Tz) &\leq p(fz, Sa) + p(Sa, Tz) \\ &\leq p(fz, Sa) + H_p(Sa, Tz) \\ &\leq rp(fa, fz) + rp(fa, fz) = 2rp(fa, fz) \\ &< p(fz, Tz), \quad \text{from} \quad (12). \end{aligned}$$

It is a contradiction, so $fz \in Tz$. Thus, from (9)

$$z = fz \in Tz. \tag{17}$$

Now, from (8), we have

$$p(fz, z) = p(fz, fz) \leq p(z, Sz). \tag{18}$$

Since $fz \in Tz$, so we have

$$\varphi(r)p(fz, Tz) \leq p(fz, Tz) \leq p(fz, fz),$$

which implies that

$$\varphi(r) \min\{p(fz, Sz), p(fz, Tz)\} \leq p(fz, fz).$$

Now, by $(A1)$

$$\begin{aligned} p(Sz, z) \leq H_p(Sz, Tz) &\leq r \max\left\{\begin{array}{c} p(fz, fz), p(fz, Sz), p(fz, Tz), \\ \frac{1}{2}[p(fz, Tz) + p(fz, Sz)] \end{array}\right\} \\ &= r \max\left\{\begin{array}{c} p(z, z), p(z, Sz), p(z, Tz)], \\ \frac{1}{2}[p(z, Tz) + p(z, Sz)] \end{array}\right\} \\ &\leq rp(z, Sz) \quad \text{from (7)} \end{aligned}$$

which in turn yields that $p(z, Sz) = 0$. By Lemma 2 and (7), we have $z \in Sz$. Hence,

$$fz = z = Sz \tag{19}$$

From (17) and (19), z is a common fixed point of f, S, and T. Now, we consider the case $\frac{1}{2} \leq r < 1$. First, we prove that

$$H_p(Sx, Tz) \leq r \max\left\{\begin{array}{c} p(fx, fz), p(fx, Sx), p(fz, Tz), \\ \frac{1}{2}[p(fx, Tz) + p(fz, Sx)] \end{array}\right\} \tag{20}$$

for all $x \in X$ such that $fx \neq fz$.

Assume that $fx \neq fz$. Then, for every $n \in \mathbb{N}$, there exists $z_n \in Sx$ such that

$$p(fz, z_n) \leq p(fz, Sx) + \frac{1}{n}p(fx, fz).$$

Therefore,

$$\begin{aligned} p(fx, Sx) &\leq p(fx, z_n) \\ &\leq p(fx, fz) + p(fz, z_n) \\ &\leq p(fx, fz) + p(fz, Sx) + \tfrac{1}{n}p(fx, fz) \\ &\leq p(fx, fz) + r\max\{p(fz, fx), p(fx, Sx)\} \\ &\quad + \tfrac{1}{n}p(fx, fz), \quad \text{from} \quad (10). \end{aligned}$$

Hence, we have either $p(fx, Sx) \leq (1 + r + \frac{1}{n})p(fx, fz)$ or $(1 - r)p(fx, Sx) \leq (1 + \frac{1}{n})p(fx, fz)$.

Letting $n \to \infty$, we get

$$p(fx, Sx) \leq (1+r)p(fx, fz) \quad \text{or} \quad (1-r)p(fx, Sx) \leq p(fx, fz).$$

Thus,

$$\begin{aligned} \varphi(r)p(fx, Sx) = (1 - r)p(fx, Sx) &\leq \frac{1}{1+r}p(fx, Sx) \\ &\leq p(fx, fz), \end{aligned}$$

or

$$\varphi(r)p(fx, Sx) = (1 - r)p(fx, Sx) \leq p(fx, fz).$$

Hence, we have

$$\varphi(r) \min\{p(fx, Sx), p(fz, Tz)\} \leq p(fx, fz).$$

Now, by $(A1)$, with $y = z$ we get (20).

Since $y_n \to z$, we may assume that $y_n \neq z$ for any n. Taking $x = x_{2n}$ in (20), we get

$$\begin{aligned} p(fx_{2n+1}, Tz) &\leq H_p(Sx_{2n}, Tz) \\ &\leq r\max\{p(fx_{2n}, fz), p(fx_{2n}, Sx_{2n}), p(fz, Tz), \\ &\quad \tfrac{1}{2}\left[p(fx_{2n}, Tz) + p(fz, Sx_{2n})\right]\}. \end{aligned}$$

Letting $n \to \infty$, using Lemma 7, (5), (7), and (9), we get

$$\begin{aligned} p(z, Tz) &\leq r\max\{0, 0, p(z, Tz), \tfrac{1}{2}[p(z, Tz) + 0]\} \\ &\leq rp(z, Tz) \end{aligned}$$

which in turn yields that $p(z, Tz) = 0$ so that $z \in Tz$. Thus $fz = z \in Tz$.

Now, following as in the case $0 \leq r < \frac{1}{2}$, and from (15) to (17), we have $z = fz \in Sz$. Thus, z is a common fixed point of f, S and T. Thus, from the two cases above, we have z is a common fixed point of f, S, and T.

Suppose z' is another common fixed point of f, S, and T. By (8), we have

$$p(z,z') = p(fz,fz') \le p(z',Sz) \le H_p(Sz,Tz'). \tag{21}$$

Using (p_2)

$$\varphi(r)\min\left\{p(fz,Sz),p(fz',Tz')\right\} \le p(fz,fz').$$

Hence, by $(A1)$

$$\begin{aligned} H_p(Sz,Tz') &\le r\max\left\{\begin{array}{c} p(fz,fz'),p(fz,Sz),p(fz',Tz'), \\ \frac{1}{2}\left[p(fz,Tz')+p(fz',Sz)\right] \end{array}\right\} \\ &\le r\max\left\{\begin{array}{c} H_p(Sz,Tz'),H_p(Sz,Sz),H_p(Tz',Tz'), \\ \frac{1}{2}\left[H_p(Sz,Tz')+H_p(Tz',Sz)\right] \end{array}\right\} \\ &\qquad \text{from} \quad (19) \\ &\le rH_p(Sz,Tz') \quad \text{from} \quad \text{Lemma 4 (i).} \end{aligned}$$

Thus, $H_p(Sz,Tz') = 0$, so that from (21), we have $z = z'$. Hence, z is the unique common fixed point of f, S, and T.

Similarly, we can prove the theorem when (f, T) satisfies the W.C.C. condition. □

Next, take $f = I_X$ (the identity map on X) in Theorem 4, we have the following corollary for two multi-valued maps.

Corollary 1. *Let (X,p) be a complete partial metric space and let $S,T : X \to CB^p(X)$. Assume that there exists $r \in [0,1)$ such that for every $x,y \in X$,*

$(B1)$ $\varphi(r)$ min $\left\{p(x,Sx),p(y,Ty)\right\} \le p(x,y)$ implies

$$H_p(Sx,Ty) \le r\max\left\{\begin{array}{c} p(x,y),p(x,Sx),p(y,Ty)], \\ \frac{1}{2}[p(x,Ty)+p(y,Sx)] \end{array}\right\}$$

where φ is a function defined by (1).

$(B2)$ The pair (I_X,S) or the pair (I_X,T) satisfies the W.C.C. condition.

Then, S and T have a common fixed point in X, that is, there exists an element $z \in X$ such that $z \in Sz \cap Tz$.

Taking $S = T$ in the above corollary, we get the following:

Corollary 2. *Let (X,p) be a complete partial metric space and let $T : X \to CB^p(X)$. Assume that there exists $r \in [0,1)$ such that for every $x,y \in X$,*

$(C1)$ $\varphi(r)$ min $\left\{p(x,Tx),p(y,Ty)\right\} \le p(x,y)$ implies

$$H_p(Tx,Ty) \le r\max\left\{\begin{array}{c} p(x,y),p(x,Tx),p(y,Ty)], \\ \frac{1}{2}[p(x,Ty)+p(y,Tx)] \end{array}\right\}$$

where φ is a function defined by (1).

$(C2)$ The pair (I_X,T) satisfies the (W.C.C) condition.

Then, T has a unique fixed point in X, that is, there exists an element $z \in X$ such that $z \in Tz$.

In case of single-valued maps, Theorem 4 reduces to the following corollary:

Corollary 3. *Let (X,p) be a complete partial metric space and $f,S,T : X \to X$. Assume that there exists $r \in [0,1)$ such that for every $x,y \in X$ every $x,y \in X$*

$(D1)$ $\varphi(r)$ min $\left\{p(fx,Sx),p(fy,Ty)\right\} \le p(fx,fy)$ implies

$$H_p(Sx,Ty) \le r\max\left\{\begin{array}{c} p(fx,fy),p(fx,Sx),p(fy,Ty), \\ \frac{1}{2}[p(fx,Ty)+p(fy,Sx)] \end{array}\right\}$$

where φ is defined by (1).

$(D2)$ $\bigcup_{x\in X} Sx \subseteq f(X)$ and $\bigcup_{x\in X} Tx \subseteq f(X)$.

$(D3)$ The pair (f,S) or the pair (f,T) satisfies the W.C.C. condition.

Then f, S, and T have a unique common fixed point in X.

We drop the W.C.C. condition in Corollary 2 to get a fixed-point result (without uniqueness):

Corollary 4. *Let (X,p) be a complete partial metric space and let $T : X \to CB^p(X)$. Assume that there exists $r \in [0,1)$ such that for every $x,y \in X$,*

$(E1)$ $\varphi(r)$ min $\left\{p(x,Tx),p(y,Ty)\right\} \le p(x,y)$ implies

$$H_p(Tx,Ty) \le r\max\left\{\begin{array}{c} p(x,y),p(x,Tx),p(y,Ty)], \\ \frac{1}{2}[p(x,Ty)+p(y,Tx)] \end{array}\right\}$$

where φ is a function defined by (1).

Then, T has a fixed point in X, that is, there exists an element $z \in X$ such that $z \in Tz$.

Similarly, for single-valued maps we have

Corollary 5. *Let (X,p) be a complete partial metric space and $T : X \to X$. Assume that there exists $r \in [0,1)$ such that for every $x,y \in X$ every $x,y \in X$*

$(F1)$ $\varphi(r)$ min $\left\{p(x,Tx),p(y,Ty)\right\} \le p(x,y)$ implies

$$H_p(Tx,Ty) \le r\max\left\{\begin{array}{c} p(x,y),p(x,Tx),p(y,Ty), \\ \frac{1}{2}[p(x,Ty)+p(y,Tx)] \end{array}\right\}$$

where φ is defined by (1).

Then, T has a fixed point in X.

Remark 1. Corollary 4 is a generalization of Theorem 3. Also, Corollary 4 improves and extends the main result of Doricć and Lazović [5] to partial metric spaces.

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
All authors read and approved the final manuscript.

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