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Reductions for Kundu-Eckhaus equation via Lie symmetry analysis

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Abstract

In this paper, using Lie symmetry method, we find classical symmetry operators for Kundu-Eckhaus equation (KE). Also, we obtain one-dimensional optimal system and reduction Lie invariants corresponding to infinitesimal symmetries of the KE equation. Finally, differential invariants of the KE equation are presented.

Keywords: Lie symmetry; Group-invariant solutions; Complex Kundu-Eckhaus equation; Optimal system

Introduction

Kundu [1] and Eckhaus [2,3] independently derived in 1984 to 1985 what can now be called the Kundu-Eckhaus equation as a linearizable form of the nonlinear Schrödinger equation. Levi and Scimiterna in [4] show that the complex Burgers and the Kundu-Eckhaus equations are related by a Miura transformation, and they use this relation to discretize the Kundu-Eckhaus equation. One of the most important discoveries of Sophus Lie in differential equation is to show that it is possible to transform non-linear conditions in a system to linear conditions by infinitesimal invariants, corresponding to the symmetry group generators of the system [5,6]. In this article, our aim is to obtain a set of symmetries of KE equation:

$$i\psi_t + \psi_{xx} + 2|\psi|_x^2\psi + |\psi|^4\psi = 0. \tag{1}$$

Which, in that complex equation, we assume that

$$\psi(t,x) = u(t,x) + iv(t,x). \tag{2}$$

By substituting (2) in the KE equation, we have

$$\begin{aligned}(u_t + v_t i)i + u_{xx} + v_{xx}i + 2(2uu_x + 2vv_x)(u+vi)\\ + (u^2+v^2)^2(u+vi) = 0.\end{aligned} \tag{3}$$

The real and imaginary parts of the equation are

$$u_t + v_{xx} + 2v(2u_x u + 2v_x v) + v(u^2+v^2)^2 = 0$$

$$-v_t + u_{xx} + 2u(2v_x v + 2u_x u) + u(v^2+u^2)^2 = 0.$$

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The classic Lie symmetries are obtained using the Lie symmetry method. This requires the utilization of computer softwares because working with continuous groups involves computations that follow from the algorithmic process. Having the symmetry group of a system of equations has a lot of advantages, one of which is the classification of the solutions of the system. This classification is to consider two solutions in one class if they can be converted to each other, by an element of the symmetry group. If we have an ordinary system, the symmetry group will help us obtain the exact solution. If the equation is order 1, it is possible to get the general solution, but it is not the case for PDE, unless the system is convertible to a linear system. Another application of the symmetry group is the probable reduction of the number of independent variables, and the ideal condition is converting to ODE.

Lie symmetry of KE equation

We used a general method for the determination of the symmetries of a system of PDE based on [7] and [8]. In general case, let us have a non-linear PDE system:

$$\Xi_\nu(x, u^{(n)}) = 0, \qquad \nu = 1, \ldots, l, \tag{4}$$

that has l equations of order n, each of which involving p-independent and q-dependent variables, where $x = (x^1, \ldots, x^p)$, $u = (u^1, \ldots, u^q)$ and $u^{(n)}$ derivation of u with respect to x of order n. Now, let us suppose that we have a one-parametric Lie group of infinitesimal transformations that acts on independent and dependent variables $(t, x, u, v) \in M = J^0_{t,x,u,v} \cong \mathbf{R}^3$ as follows:

$$(\tilde{t}, \tilde{x}, \tilde{u}, \tilde{v}) = (t, x, u, v) + s(\xi^1, \xi^2, \phi^1, \phi^2)(t, x, u, v) + O(s^2), \tag{5}$$

where s is the group parameter and ξ^1, ξ^2, and ϕ^1, ϕ^2 are the infinitesimals parts of transformations. To calculate the Lie symmetry group for KE equation, let us suppose in the general case

$$\mathbf{v} = \xi^1(t,x,u,v)\frac{\partial}{\partial x} + \xi^2(t,x,u,v)\frac{\partial}{\partial t} + \phi^1(t,x,u,v)\frac{\partial}{\partial u} + \phi^2(t,x,u,v)\frac{\partial}{\partial v}, \tag{6}$$

is the infinitesimal transformation group of (3). Now, we prolong the vector field $\mathbf{v}$ to order 2, using the following formula:

$$\mathrm{Pr}^{(2)}\mathbf{v} = \mathbf{v} + \phi^x\frac{\partial}{\partial u_x} + \phi^t\frac{\partial}{\partial u_t} + \phi^{xx}\frac{\partial}{\partial u_{xx}} + \phi^{xt}\frac{\partial}{\partial u_{xt}} + \phi^{tt}\frac{\partial}{\partial u_{tt}} \tag{7}$$

with coefficients

$$\phi^J = D_J Q + \sum_{i=1}^{2} \xi^i u_{J,i}, \tag{8}$$

in which $Q = \phi - \sum_{i=1}^{2} \xi^i u_i^\alpha$ and $J = (j_1, \cdots, j_k)$, $1 \leq j_k \leq 2$, $1 \leq k \leq 2$, and the sum is all over Js of order $0 < \#J \leq n$ and $u_i^\alpha := \partial u^\alpha / x^i$ and $u_{J,i}^\alpha := \partial u_J^\alpha / x^i$. The invariant conditions, [9], for the KE equation are

$$\begin{cases} \mathrm{Pr}^{(2)}\mathbf{v}(u_t + v_{xx} + 2v(2u_x u + 2v_x v) + v(u^2+v^2)^2) = 0 \\ \mathrm{Pr}^{(2)}\mathbf{v}(-v_t + u_{xx} + 2u(2v_x v + 2u_x u) + u(v^2+u^2)^2) = 0 \\ u_t + v_{xx} + 2v(2u_x u + 2v_x v) + v(u^2+v^2)^2) = 0 \\ -v_t + u_{xx} + 2u(2v_x v + 2u_x u) + u(v^2+u^2)^2) = 0 \end{cases} \tag{9}$$

The solution of which yields the system of PDE as the functions of ξ^1, ξ^2, and ϕ^1, ϕ^2. Here, KE equation is a manifold in the jet space $J^2_{t,x;u,v} \cong \mathbf{R}^9$, and $\mathrm{Pr}^{(2)}\mathbf{v}$ is the prolongation of $\mathbf{v}$ up to the order 2. As a result, we have the PDE system:

$$\begin{array}{llll} \xi^1_x = 0, & \xi^2_u = 0, & \xi^1_v = \xi^1_u = 0, & \phi^2_x = 0, \\ \xi^2_t = 0, & \phi^2_u = 0, & \xi^2_v = 0, & \xi^2_x v = -2\phi^2, \\ \phi^2_t = 0, & \xi^1_t v = -4\phi^2 & & \\ \phi^2_v v = \phi^2, & \phi^1 v = \phi^2 u. & & \end{array} \tag{10}$$

By solving the above system, we will have the following theorem:

Theorem 1. *The Lie group of point symmetries of the KE equation has a Lie algebra generator in the form of the vector field* $\mathbf{v}$*, with the following functional coefficients:*

$$\xi^1(t,x,u,v) = c_1 t + c_2, \qquad \xi^2(t,x,u,v) = \frac{c_1 x}{2} + c_3,$$
$$\phi^1(t,x,u,v) = -\frac{1}{4}c_1 u, \qquad \phi^2(t,x,u,v) = -\frac{1}{4}c_1 v,$$

where c_i, $(i = 1,2,3)$ *are arbitrary constants.*

Theorem 2. *The infinitesimal generators from the Lie one-parameter group of the symmetries of the KE equation are as follows:*

$$\mathbf{v}_1 = t\frac{\partial}{\partial t} + \frac{x}{2}\frac{\partial}{\partial x} - \frac{1}{4}u\frac{\partial}{\partial u} - \frac{1}{4}v\frac{\partial}{\partial v},$$
$$\mathbf{v}_2 = \frac{\partial}{\partial t}, \qquad \mathbf{v}_3 = \frac{\partial}{\partial x}.$$

These vector fields produce a Lie algebra space $\mathcal{G}$ with the following commutator table (Table 1):

Group invariant solutions of KE equation

To obtain the group of transformations which are generated by infinitesimal generators $\mathbf{v}_i$ for $i = 1,2,3$, we should solve the first-order system involving first-order equations in correspondence to each of the generators simultaneously. By solving this system, the one parameter group of $g_k(s) : M \to M$ generated by $\mathbf{v}_i$ for $i = 1,2,3$ involved in Theorem (2) is obtained in the following way:

$$\begin{aligned} g_1 &: (t,x,u,v) \longmapsto (te^s, xe^{\frac{s}{2}}, ue^{-\frac{s}{4}}, ve^{-\frac{s}{4}}), \\ g_2 &: (t,x,u,v) \longmapsto (t+s, x, u, v), \\ g_3 &: (t,x,u,v) \longmapsto (t, x+s, u, v). \end{aligned} \tag{11}$$

Therefore, we have:

Theorem 3. *If* $u = f(t,x)$*, and* $v = g(t,x)$ *is one of the solutions of KE equation, then the following functions that have been produced through acting* $g_k(s)$ *on* $u = f(t,x)$ *and* $v = g(t,x)$ *will also be the solution of KE equation:*

$$\begin{aligned} g_1(s) \cdot f(t,x) &= f(te^{-s}, xe^{-\frac{s}{2}})e^{-\frac{s}{4}}, \\ g_2(s) \cdot f(t,x) &= f(t-s, x), \\ g_3(s) \cdot f(t,x) &= f(t, x-s), \end{aligned}$$

$$\begin{aligned} g_1(s) \cdot g(t,x) &= g(te^{-s}, xe^{-\frac{s}{2}})e^{-\frac{s}{4}}, \\ g_2(s) \cdot g(t,x) &= g(t-s, x), \\ g_3(s) \cdot g(t,x) &= g(t, x-s). \end{aligned}$$

Optimal system of KE equation

Now, we want to obtain one-dimensional optimal system of the KE equation using its symmetry group. The optimal system is in fact a standard method for the classification of one-dimensional sub-algebras in which each class involves conjugate equivalent members [10]. Also, they involve the

Table 1 Commutator table

[,]	$\mathbf{v}_1$	$\mathbf{v}_2$	$\mathbf{v}_3$
$\mathbf{v}_1$	0	$-\mathbf{v}_2$	$-\frac{\mathbf{v}_3}{2}$
$\mathbf{v}_2$	$\mathbf{v}_2$	0	0
$\mathbf{v}_3$	$\frac{\mathbf{v}_3}{2}$	0	0

group adjoint representation which establishes an equivalent relation among all conjugate sub-algebra elements. In fact, the classification problem for one-dimensional sub-algebra is the same as the problem of the classification of the representation of its adjoint orbits. In this way, the optimal system is constructed. The set of invariant solutions corresponding to a one-dimensional sub-algebra is a list of minimal solutions where all the other invariant solutions can be obtained by transformations. To calculate the adjoint representation, we consider the following Lie series:

$$\mathrm{Ad}(\exp(s\mathbf{v}_i)\mathbf{v}_j) = \mathbf{v}_j - s\,\mathrm{ad}_{\mathbf{v}_j}\mathbf{v}_j + \frac{s^2}{2}\,\mathrm{ad}^2_{\mathbf{v}_j}\mathbf{v}_j - \cdots, \quad (12)$$

for the vector fields $\mathbf{v}_i,\ \mathbf{v}_j$ in which $\mathrm{ad}_{\mathbf{v}_j}\mathbf{v}_j = [\mathbf{v}_i, \mathbf{v}_j]$ is the Lie algebra communicator, s is the group parameter, $i, j = 1, 2, 3$ [7]. Now, we consider an optional member from $\mathcal{G}$ of the form

$$\mathbf{v} = a_1\mathbf{v}_1 + a_2\mathbf{v}_2 + a_3\mathbf{v}_3, \quad (13)$$

and for the simplicity, we write $a = (a_1, a_2, a_3) \in \mathbf{R}^3$; therefore, the adjoint action can be considered as a type of linear transformation group of vectors, so we have the following theorem:

Theorem 4. *The one-dimensional optimal system of Lie algebra $\mathcal{G}$ for the KE equation is*

$$(i)\quad \mathbf{v}_1, \qquad\qquad (ii)\quad a\mathbf{v}_2 + b\mathbf{v}_3. \quad (14)$$

In it, $a, b \in \mathbf{R}$ is arbitrary constant.

Proof. We define $F_i^s : \mathcal{G} \to \mathcal{G}$ by $\mathbf{v} \mapsto \mathrm{Ad}(\exp(s\mathbf{v}_i)\mathbf{v})$ as a linear map, for $i = 1, 2, 3$. So, the matrices M_i^s corresponding to each of the F_i^s, $i = 1, 2, 3$, with respect to the basis $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ will be as follows:

$$\begin{aligned} M_1^s &= \mathrm{E}_{11} + e^{-s}\mathrm{E}_{22} + e^{-\frac{s}{2}}\mathrm{E}_{33}, \\ M_2^s &= \mathrm{I}_3 + s\mathrm{E}_{21}, \\ M_3^s &= \mathrm{I}_3 + \frac{s}{2}\mathrm{E}_{31}, \end{aligned}$$

and E_{ij}s are 3×3 elementary matrices for $i, j = 1, 2, 3$, where $(i; j)$ entry of E_{ij} is 1, and those of others are zero. Suppose $\mathbf{v} = a_1\mathbf{v}_1 + a_2\mathbf{v}_2 + a_3\mathbf{v}_3$, we have the combination:

$$\begin{aligned} F_3^s \circ F_2^s \circ F_1^s : \mathbf{v} \mapsto & [a_1]\,\mathbf{v}_1 + [e^{-s}sa_1 + e^{-s}a_2]\,\mathbf{v}_2 \\ & + [\frac{1}{2}e^{-\frac{s}{2}}sa_1 + e^{-\frac{s}{2}}a_3]\,\mathbf{v}_3. \end{aligned}$$

If $a_1 \neq 0$, then by substituting $s = -\frac{a_2}{a_1}$ and $s = -\frac{2a_3}{a_1}$ using F_2^s and F_3^s, we can vanish the coefficient of $\mathbf{v}_2$ and $\mathbf{v}_3$, and by scaling of $\mathbf{v}$, we can suppose $a_1 = 1$. In this case, $\mathbf{v}$ is reduced to form (i), and if $a_1 = 0$, then $\mathbf{v}$ is reduced to form (ii). □

Similarity reduction of KE equation

The KE equation has been stated in the $(t, x; u, v)$ coordinate, but we are looking for a new coordinate that the equation will reduce to ODE. For example, the first element of the optimal system is $\mathbf{v}_1$. It has the determining equation in the form:

$$\frac{2dx}{x} = \frac{dt}{t} = -\frac{4du}{u} = -\frac{4dv}{v}.$$

Solving this equation will result in two invariants $y = \frac{x}{\sqrt{t}}$, $f = ut^{1/4}$, $g = vt^{1/4}$. Now, if we consider $u(x, t) = f(y)t^{-1/4}$, and $v(x, t) = g(y)t^{-1/4}$ as a function of $y = \frac{x}{\sqrt{t}}$, we can state the derivatives of u and v with respect to x and t in the form of f, g, and y, and the derivatives of f, g with respect to y. Substituting it in the KE equation, we get an ODE as follows:

$$\begin{aligned} -1/2yf' - 1/4f + g'' + 2g(2gg' + 2ff') + g(f^2 + g^2)^2 &= 0, \\ 1/2yg' + 1/4g + f'' + 2f(2gg' + 2ff') + f(f^2 + g^2)^2 &= 0. \end{aligned}$$

If we assume the $\phi(y) = f(y) + g(y)i$ in complex manner, we have

$$i(1/2y\phi_y + 1/4\phi) + \phi_{yy} + 2|\phi|_y^2\phi + |\phi|^4\phi = 0. \quad (15)$$

For the rest of the optimal system elements and symmetry group, the reduced equations will be as the following Table 2:

Characterization of differential invariants

Suppose that G is a transformation group. It is well known that a smooth real differential function $I : J^n \longrightarrow \mathbf{R}$, where J^n is the corresponding n-th jet space, is a differential invariant for G if and only if for all $\mathbf{v} \in \mathcal{G}$, its nth prolongation annihilates I, i.e., $\mathbf{v}^{(n)}(I) = 0$. To obtain the differential invariant of the KE equation, up to order 2, we solve the following system:

$$\frac{\partial I}{\partial t} = 0, \qquad \frac{\partial I}{\partial x} = 0, \qquad t\frac{\partial I}{\partial t} + \frac{x}{2}\frac{\partial I}{\partial x} - \frac{u}{4}\frac{\partial I}{\partial u} - \frac{v}{4}\frac{\partial I}{\partial v} = 0, \quad (16)$$

Table 2 Reduced equations

C_i	y_i	f_i	g_i	Similarity reduced equation
v_1	$\frac{x}{\sqrt{t}}$	$ut^{1/4}$	$vt^{1/4}$	$i(1/2y\phi_y + 1/4\phi) + \phi_{yy} + 2\lvert\phi\rvert_y^2\phi + \lvert\phi\rvert^4\phi = 0$
v_2	x	u	v	$\phi_{yy} + 2\lvert\phi\rvert_y^2\phi + \lvert\phi\rvert^4\phi = 0$
v_3	t	u	v	$i\phi_y + \lvert\phi\rvert^4\phi = 0$
$v_2 + v_3$	$x - t$	u	v	$i\phi_y + \phi_{yy} + 2\lvert\phi\rvert_y^2\phi + \lvert\phi\rvert^4\phi = 0$

Table 3 Characterization of differential invariants

Vector field	Ordinary invariant	First order	Second order
v_1	$\frac{x}{\sqrt{t}}, ut^{1/4}, vt^{1/4}$	$*, u_t t^{5/4} u_x t^{3/4}$, $v_t t^{5/4} v_x t^{3/4}$	$*, **, u_{tt} t^{9/4}, u_{tx} t^{7/4}, u_{xx} t^{5/4}$, $v_{tt} t^{9/4}, v_{tx} t^{7/4}, v_{xx} t^{5/4}$
v_2	x, u, v	$*, u_x, u_t, v_x, v_t$	$*, **, u_{xx}, u_{xt}, u_{tt}, v_{xx}, v_{xt}, v_{tt}$
v_3	t, u, v	$*, u_x, u_t, v_x, v_t$	$*, **, u_{xx}, u_{xt}, u_{tt}, v_{xx}, v_{xt}, v_{tt}$

$*$ and $**$ refer back to ordinary and first-order invariants, respectively.

where, I is a smooth function of (x, t, u, v), and

$$\frac{\partial I_1}{\partial t} = 0, \quad \frac{\partial I_1}{\partial x} = 0, \quad t\frac{\partial I_1}{\partial t} + \frac{x}{2}\frac{\partial I_1}{\partial x} - \frac{u}{4}\frac{\partial I_1}{\partial u} - \frac{v}{4}\frac{\partial I_1}{\partial v} - \frac{5u_t}{4}\frac{\partial I_1}{\partial u_t} - \frac{3u_x}{4}\frac{\partial I_1}{\partial u_x} - \frac{5v_t}{4}\frac{\partial I_1}{\partial v_t} - \frac{3v_x}{4}\frac{\partial I_1}{\partial v_x} = 0, \tag{17}$$

where I_1 is a smooth function of $(x, t, u, v, u_x, u_t, v_x, v_t)$,

$$\frac{\partial I_2}{\partial t} = 0, \quad \frac{\partial I_2}{\partial x} = 0, \quad t\frac{\partial I_2}{\partial t} + \frac{x}{2}\frac{\partial I_2}{\partial x} - \frac{u}{4}\frac{\partial I_2}{\partial u} - \frac{v}{4}\frac{\partial I_2}{\partial v} - \frac{5u_t}{4}\frac{\partial I_2}{\partial u_t} - \frac{3u_x}{4}\frac{\partial I_2}{\partial u_x} - \frac{5v_t}{4}\frac{\partial I_2}{\partial v_t} - \frac{3v_x}{4}\frac{\partial I_2}{\partial v_x} - \frac{9u_{tt}}{4}\frac{\partial I_2}{\partial u_{tt}} - \frac{7u_{tx}}{4}\frac{\partial I_2}{\partial u_{tx}} - \frac{5u_{xx}}{4}\frac{\partial I_2}{\partial u_{xx}} - \frac{9v_{tt}}{4}\frac{\partial I_2}{\partial v_{tt}} - \frac{7v_{tx}}{4}\frac{\partial I_2}{\partial v_{tx}} - \frac{5v_{xx}}{4}\frac{\partial I_2}{\partial v_{xx}} = 0, \tag{18}$$

where I_2 is a smooth function of $(x, t, u, v, \cdots, u_{xx}, u_{xt}, u_{tt}, v_{xx}, v_{xt}, v_{tt})$. The solution of Equations (16) up to (18) are listed in Table 3:

Conclusion

In this paper, by applying the criterion of invariance of the equation under the infinitesimal prolonged infinitesimal generators, we find the most general Lie point symmetries group of the Kundu-Eckhaus equation. Also, we have constructed the optimal system of one-dimensional subalgebras of Kundu-Eckhaus equation. The latter, creates the preliminary classification of group invariant solutions. The Lie invariants and similarity reduced equations corresponding to infinitesimal symmetries are obtained.

Competing interests

The authors declare that they have no competing interests.

Authors' contributions

This work was carried out in collaboration between all authors. MT designed the study, performed the statistical analysis, wrote the protocol, and wrote the first draft of the manuscript. NA managed the analyses of the study and managed the literature searches. Both authors read and approved the final manuscript.

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