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The pure injectives and pure projectives in the category of totally disconnected, locally compact abelian groups

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Abstract

Let $\wp$ be the category of totally disconnected, locally compact abelian groups. In this paper, we determine the discrete or compact pure injective groups in $\wp$. Also, we determine the compact pure projective groups in $\wp$.

Keywords: Pure injective; Pure projective; Pure extension; Totally disconnected

Introduction

Throughout, all groups are Hausdorff abelian topological groups and will be written additively. Let $\pounds$ denote the category of locally compact abelian (LCA) groups with continuous homomorphisms as morphisms. A morphism is called proper if it is open onto its image, and a short exact sequence $0 \to A \xrightarrow{\phi} B \xrightarrow{\psi} C \to 0$ in $\pounds$ is said to be proper exact if ϕ and ψ are proper morphisms. In this case, the sequence is called an extension of A by C (in $\pounds$). A subgroup H of a group C is called pure if $nH = H \bigcap nC$ for all positive integers n. An extension $0 \to A \xrightarrow{\phi} B \xrightarrow{\psi} C \to 0$ is called a pure extension if $\phi(A)$ is pure in B. Following Fulp and Griffith [1], we let $\text{Ext}(C, A)$ denote the (discrete) group of extensions of A by C. The elements represented by pure extensions of A by C form a subgroup of $\text{Ext}(C, A)$ which is denoted by $P\text{ext}(C, A)$. Assume that $\Im$ is any subcategory of $\pounds$ such that whenever $0 \to A \xrightarrow{\phi} B \xrightarrow{\psi} C \to 0$ is an extension in $\pounds$, and A and C are in $\Im$, then B is in $\Im$. Following Fulp [2], G in $\Im$ is called a pure projective group if and only if $P\text{ext}(G, X) = 0$ for all X in $\Im$. Similarly, G is a pure injective group in $\Im$ if and only if $P\text{ext}(X, G) = 0$ for all X in $\Im$. Fulp [2] has described the pure injective and pure projective in some categories such as the category of connected, locally compact abelian groups. Let $\wp$ be the category of totally disconnected, locally compact abelian groups. In this paper, we determine the discrete or compact groups which are pure injective in $\wp$. We show that a discrete (torsion or torsion-free) group is pure injective in $\wp$ if and only if it is divisible (Theorems 1 and 2). We show that a compact group G is pure injective in $\wp$ if and only if $G = 0$ (Corollary 1). We also introduce a result on the pure projective of $\wp$. We show that if a compact dual cotorsion group is a pure projective of $\wp$, then it is a torsion group (Corollary 2).

The additive topological group of real numbers is denoted by R. Q is the group of rationales, Z is the group of integers, and $Z(n)$ is the cyclic group of order n. By G_d, we mean the group G with discrete topology. tG is the torsion part of G, and G_0 is the identity component of G. The Pontrjagin dual group of a group G is denoted by $\hat{G}$. The topological isomorphism will be denote by '$\cong$'.

Pure injective in $\wp$

Let $\wp$ be the category of totally disconnected, locally compact abelian groups. In this section, we determine the structure of a discrete or compact pure injective group in $\wp$. Recall that a group $B$is said to be bounded if $nB = 0$ for some integer n.

Lemma 1. *Suppose B is a discrete bounded group. Then, B is pure injective in $\wp$ if and only if $B = 0$.*

Proof. Assume that E is a torsion-free group in $\pounds$. Let E_0 be the identity component of E. Then, the sequence

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$$0 \to E_0 \to E \to E/E_0 \to 0$$

is a proper pure exact. Thus, the sequence

$$0 = Pext(E/E_0, B) \to Pext(E, B) \to Pext(E_0, B)$$

is exact (Proposition 4 in [2]). By Theorem 2.11 in [3], $Pext(E_0, B) = 0$. Thus, $Pext(E, B) = 0$. It follows that B is divisible (Corollary 10 in [4]). Since B is bounded, so $B = 0$. □

Theorem 1. *Let A be a discrete torsion group. Then, A is pure injective in $\wp$ if and only if A is a divisible group.*

Proof. If A is a divisible group, it is clear that $Pext(X, A) = 0$ for all totally disconnected groups X (Theorem 3.4 in [1]). So, A is pure injective in $\wp$. Conversely, suppose that A is a discrete torsion, pure injective group in $\wp$. Then, $\text{Ext}(Q, A) = 0$. Hence, A is a cotorsion group. By Corollary 54.4 in [5], $A = B \oplus D$ where B and D are bounded and divisible groups, respectively. Clearly B is pure injective in $\wp$. So by Lemma 1, $B = 0$. Hence, $A = D$ is a divisible group. □

Theorem 2. *Let A be a discrete torsion-free group. Then, A is pure injective in $\wp$ if and only if A is a divisible group.*

Proof. Suppose that A is a discrete torsion-free, pure injective group in $\wp$. Then, $\text{Ext}(\widehat{Q/Z}, A) = Pext(\widehat{Q/Z}, A) = 0$. By corollary 2.10 in [6], we have the exact sequence

$$(*) \quad Hom(\widehat{Q/Z}, A) \to \text{Ext}(\hat{Z}, A) \to \text{Ext}(\hat{Q}, A) \to \text{Ext}(\widehat{Q/Z}, A) = 0.$$

Since $\hat{A}$ is connected, so $Hom(\widehat{Q/Z}, A) \cong Hom(\hat{A}, Q/Z) = 0$. It follows from $(*)$ that $\text{Ext}(\hat{Z}, A) \cong \text{Ext}(\hat{Q}, A)$. By Proposition 2.17 in [1], $A \cong \text{Ext}(\hat{Z}, A)$. So, $A \cong \text{Ext}(\hat{Q}, A)$. Since $\text{Ext}(\hat{Q}, A)$ is divisible (p. 223(I) in [3]), so A is a divisible group. □

Definition 1. A locally compact abelian group G will be called an *£-cotorsion* if and only if $\text{Ext}(X, G) = 0$ for each torsion-free group X in £ [4].

Theorem 3. *Let G be a compact group. Then, $Pext(X, G) = 0$ for any totally disconnected group X in £ if and only if $G \cong (R/Z)^\sigma$ where σ is a cardinal number.*

Proof. Suppose $Pext(X, G) = 0$ for any totally disconnected group X. First, we show that G is an *£-cotorsion* group. Let X be torsion-free in £ and X_0 the component of identity. Since X_0 is pure in X, so X/X_0 is torsion-free. Hence, $0 = Pext(X/X_0, G) = \text{Ext}(X/X_0, G)$. Consider the exact sequence

$$(*) \quad 0 = \text{Ext}(X/X_0, G) \to \text{Ext}(X, G) \to \text{Ext}(X_0, G) \to 0.$$

Recall that since X_0 is a compact torsion-free group, $\hat{X}_0$ is a discrete divisible group. Consequently, $\text{Ext}(X_0, G) \cong \text{Ext}(\hat{G}, \hat{X}_0) = 0$. By (*), $\text{Ext}(X, G) = 0$. So G is an *£-cotorsion*. By Corollary 9 in [4], G is connected. It follows that $\text{Ext}(X, G) = 0$ for any totally disconnected group in £. By Theorem 5.1 in [1], $G \cong (R/Z)^\sigma$. The converse is clear. □

Corollary 1. *A compact group G is pure injective in $\wp$ if and only if $G = 0$.*

Proof. Let G be a compact, pure injective group in $\wp$. Then, G is totally disconnected, and $Pext(X, G) = 0$ for any totally disconnected group X. So, by Theorem 3, $G = 0$. □

Pure projective in $\wp$

In this section, we show that if a compact group is pure projective in $\wp$, then it is a torsion group.

Lemma 2. *A discrete group A is pure projective in $\wp$ if and only if A is a direct sum of cyclic groups.*

Proof. Let A be a discrete pure projective group in $\wp$. So, $Pext(A, X) = 0$ for any discrete group X. By Theorem 30.2 in [5], A is a direct sum of cyclic groups. The converse is clear. □

Recall that a discrete group A is said to be a cotorsion if for any discrete torsion-free group B, $\text{Ext}(B, A) = 0$ [5]. A compact group G is called dual cotorsion if and only if the dual group of G is a cotorsion.

Theorem 4. *Let G be a compact dual cotorsion group. If $Pext(G, X) = 0$ for any $X \in \wp$, then $G \cong \Pi_{i \in I} Z(b_i)$.*

Proof. Let G be a compact group such that $\hat{G}$ is a cotorsion group and $Pext(G, X) = 0$ for any totally disconnected group X. By Theorem 2.11 in [3], it is enough to show that $Pext(G, F) = 0$ for any compact group F in £. We have $Pext(G, F) = Pext(\hat{F}, \hat{G})$. Consider the exact sequence

$$(*) \quad \ldots \to Pext(\hat{F}/t\hat{F}, \hat{G}) \to Pext(\hat{F}, \hat{G}) \to Pext(t\hat{F}, \hat{G}) \to 0.$$

Since $\hat{G}$ is a cotorsion group, $Pext(\hat{F}/t\hat{F}, \hat{G}) = 0$. Since $\widehat{(t\hat{F})}$ is totally disconnected, then $Pext(t\hat{F}, \hat{G}) = Pext(G, \widehat{(t\hat{F})}) = 0$. It follows from $(*)$ that $Pext(\hat{F}, \hat{G}) = 0$. Hence, $Pext(G, F) = 0$. □

Corollary 2. *If a compact dual cotorsion group is pure projective in $\wp$, then it is a torsion group.*

Proof. It is clear by Theorem 4. □

Theorem 5. *Let C be a connected group. Then, $\mathrm{Pext}(C, X) = 0$ for all totally disconnected groups X in £ if and only if C is a vector group.*

Proof. Since a vector group is a projective object of £, it is clear that $Pext(C, X)$ for all totally disconnected groups $X \in$ £. Conversely, assume that C is a connected group. Then, $C \cong R^n \oplus K$ where K is a compact connected group (Theorem 9.14 in [7]). Since the sequence

$$0 = Hom(Q\hat{/}Z, \hat{K}) \to \mathrm{Ext}(\hat{Z}, \hat{K}) \to \mathrm{Ext}(\hat{Q}, \hat{K}) = \mathrm{Pext}(\hat{Q}, \hat{K}) = 0$$

is exact, so by Proposition 2.17 in [1], $\hat{K}$ is isomorphic to $\mathrm{Ext}(\hat{Z}, \hat{K}) = 0$ and therefore $K = 0$. It follows that C is a vector group. □

Theorem 6. *Let C be a locally connected group in £. Then, $\mathrm{Pext}(C, X) = 0$ for all totally disconnected groups X in £ if and only if $C \cong R^n \oplus E$ where E is a discrete direct sum of cyclic groups.*

Proof. Let C be a locally connected group in £ and $\mathrm{Pext}(C, X) = 0$ for all totally disconnected groups X in £. By p. 19 in [8] and p. 38 in [9], $C \cong R^n \oplus E \oplus \hat{D}$ where R^n is a vector group with $n \geq 0$, E a discrete group, and D is a discrete torsion-free abelian group in which every subgroup of finite rank is free. Then, $Pext(E, X) = 0$ for all discrete groups X. Hence, E is a direct sum of cyclic groups. We show that $\hat{D} = 0$. Recall that $\mathrm{Ext}(\hat{D}, Q) = Pext(\hat{D}, Q) = 0$. Consider the exact sequence

$$0 = Hom(\hat{D}, Q/Z) \to \mathrm{Ext}(\hat{D}, Z) \to \mathrm{Ext}(\hat{D}, Q) = 0.$$

So $D \cong \mathrm{Ext}(\hat{Z}, D) = \mathrm{Ext}(\hat{D}, Z) = 0$, i.e., $D = 0$. □

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
HS and AA carried out the mathematical studies and participated in data analysis. Both authors read and approved the final manuscript.

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