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On elongations of *QTAG*-modules

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Abstract

Mehdi studied $(\omega + k)$-projective *QTAG*-modules with the help of their submodules contained in $H^k(M)$ (the submodule generated by the elements of exponents at most k). These modules contain nice submodules N contained in $H^k(M)$ such that M/N is a direct sum of uniserial modules. Here, we investigate the class $\mathcal{A}$ of *QTAG*-modules, containing nice submodules $N \subseteq H^k(M)$ such that M/N is totally projective. We also study strong ω-elongation of totally projective *QTAG*-modules by $(\omega + k)$-projective *QTAG*-modules.

Keywords: QTAG-module; ω-elongation; Totally projective; $(\omega + k)$-projective etc

Mathematics subject classification (2000): 16 K 20

Introduction

Throughout this paper, all rings will be associative with unity, and modules M are unital *QTAG*-modules. An element $x \in M$ is uniform, if xR is a non-zero uniform (hence uniserial) module and for any R-module M with a unique composition series, $d(M)$ denotes its composition length. For a uniform element $x \in M$, $e(x) = d(xR)$ and $H_M(x) = \sup \left\{ d\left(\frac{yR}{xR}\right) \mid y \in M,\ x \in yR \text{ and } y \text{ uniform} \right\}$ are the exponent and height of x in M, respectively. $H_k(M)$ denotes the submodule of M generated by the elements of height at least k and $H^k(M)$ is the submodule of M generated by the elements of exponents at most k. M is h-divisible if $M = M^1 = \bigcap_{k=0}^{\infty} H_k(M)$ and it is h-reduced if it does not contain any h-divisible submodule. In other words, it is free from the elements of infinite height. A h-reduced *QTAG*-module M is called totally projective if it has a nice system.

A submodule N of M is h-pure in M if $N \cap H_k(M) = H_k(N)$, for every integer $k \geq 0$. For a limit ordinal α, $H_\alpha(M) = \bigcap_{\rho<\alpha} H_\rho(M)$, for all ordinals $\rho < \alpha$ and it is α-pure in M if $H_\sigma(N) = H_\sigma(M) \cap N$ for all ordinals $\sigma < \alpha$.

A submodule $N \subset M$ is nice [1] Definition 2.3 in M, if $H_\sigma(M/N) = (H_\sigma(M) + N)/N$ for all ordinals σ, i.e. every coset of M modulo N may be represented by an element of the same height. A *QTAG*-module M is said to be separable, if $M^1 = 0$. The cardinality of the minimal generating set of M is denoted by $g(M)$. For all ordinals α, $f_M(\alpha)$ is the αth-Ulm invariant of M and it is equal to $g(\mathrm{Soc}(H_\alpha(M))/\mathrm{Soc}(H_{\alpha+1}(M)))$.

For a *QTAG*-module M, there is a chain of submodules $M^0 \supset M^1 \supset M^2 \cdots \supset M^\tau = 0$, for some ordinal τ. $M^{\sigma+1} = (M^\sigma)^1$, where M^σ is the σth-Ulm submodule of M. Singh [2] proved that the results which hold for *TAG*-modules also hold good for *QTAG*-modules. Notations and terminology are followed from [3,4]

Elongations of totally projective *QTAG*-modules by $(\omega + k)$-projective *QTAG*-modules

Recall that a *QTAG*-module M is $(\omega+1)$-projective if there exists submodule $N \subset H^1(M)$ such that M/N is a direct sum of uniserial modules and a *QTAG* module M is $(\omega + k)$-projective if there exists submodule $N \subset H^k(M)$ such that M/N is a direct sum of uniserial modules [5].

A *QTAG*-module is an ω-elongation of a totally projective *QTAG*-module by a $(\omega+k)$-projective *QTAG*-module if and only if $H_\omega(M)$ is totally projective and $M/H_\omega(M)$ is $(\omega + k)$-projective.

Suppose $\mathcal{A}_k$ denotes the family of QTAG-modules M which contain nice submodules $N \subseteq H^k(M)$ free from the elements of infinite height, such that M/N is totally projective. The main goal of this section is to find a condition for the modules of the family $\mathcal{A}_k$ to be isomorphic. To achieve this goal we need some results. We start with the following:

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Lemma 1. *Let M be a QTAG-module and $N \subseteq M$ such that $N \cap H_\omega(M) = 0$, then N is nice in M if and only if $N \oplus H_\omega(M)$ is nice in M.*

Proof. Suppose N is nice in M. Since a submodule K is nice in M if M/K is separable, it is sufficient to show that $M/(N \oplus H_\omega(M))$ is separable. If $H(\bar{x})$ is infinite in $M/(N \oplus H_\omega(M))$, where $\bar{x} = x + N \oplus H_\omega(M)$, then there exist a sequence $\{x_k\}$ in $N \oplus H_\omega(M)$ such that $H(x + x_k) \geq k$, for every $k \in Z^+$.

If $x_k = y_k + z_k$ where $y_k \in N$, $z_k \in H_\omega(M)$; then $H(x + y_k) \geq k$ and the coset $x + N$ has infinite height in M/N. Now for some $u \in N$, $H(x+u) \geq \omega$ and $x = -u+(x+u) \in N \oplus H_\omega(M)$, thus $N \oplus H_\omega(M)$ is nice in M.

For the converse suppose $N \oplus H_\omega(M)$ is nice in M. Since $H_\omega(M) \subseteq N \oplus H_\omega(M)$, $M/(N \oplus H_\omega(M))$ must be separable. By the previous argument, an element $x + N$ has height ω in M/N if and only if it can be represented by an element of $H_\omega(M)$ and the result follows. □

Lemma 2. *If N is nice submodule of $H^k(M) \subseteq M$ which is bounded by k such that $N \cap H_\omega(M) = 0$ and M/N is totally projective, then*

(i) *$M/(N \oplus H_\omega(M))$ is a direct sum of uniserial modules and*
(ii) *$M/H_\omega(M)$ is $(\omega + k)$-projective.*

Proof. Since N is a nice submodule we have $H_\omega(M/N) = (H_\omega(M) + N)/N$. Now, $M/(N \oplus H_\omega(M)) \cong (M/N)/H_\omega(M/N)$ and M/N is totally projective; therefore, $(M/N)/H_\omega(M/N)$ is a direct sum of uniserial modules. Thus, $M/(N \oplus H_\omega(M))$ is also a direct sum of uniserial modules.

Again, $(N \oplus H_\omega(M))/H_\omega(M)$ is a submodule of $M/H_\omega(M)$, which is bounded by k. Thus, $M/H_\omega(M)$ is $(\omega + k)$-projective module. □

Lemma 3. *Let M be a QTAG-module and N a submodule of $H^k(M) \subseteq M$ such that $N \cap H_\omega(M) = 0$. If $H_\omega(M)$ is totally projective and $M/(N \oplus H_\omega(M))$ is a direct sum of uniserial modules, then M/N is totally projective.*

Proof. Now, $N \oplus H_\omega(M)$ is nice in M; therefore, by Lemma 1, N is a nice submodule of M. This implies that $H_\omega(M/N) = (N \oplus H_\omega(M))/N \cong H_\omega(M)$ because $N \cap H_\omega(M) = 0$.

Again,

$$(M/N)/H_\omega(M/N) = (M/N)/[(N \oplus H_\omega(M))/N] \cong M/(N \oplus H_\omega(M))$$

is a direct sum of uniserial modules implying that M/N is totally projective. □

Lemma 4. *Let N be a submodule of $H^k(M) \subseteq M$ such that $N \cap H_\omega(M) = 0$. Then the Ulm-invariants of $N \oplus H_\omega(M)$ with respect to M can be determined by $H^k(M)$.*

Proof. The σth Ulm-invariant of $N \oplus H_\omega(M)$ with respect to M is

$$g\big(\operatorname{Soc}(H_\sigma(M))/((H_{\sigma+1}(M)+(N \oplus H_\omega(M))) \cap \operatorname{Soc}(H_\sigma(M)))\big) \quad [5].$$

If σ is an integer, then $H_{\sigma+1}(M) + N \oplus H_\omega(M) = H_{\sigma+1}(M) + N$ and if $x \in H_{\sigma+1}(M)$, $y \in N$ such that $x + y \in \operatorname{Soc}(H_{\sigma+1}(M) + N)$, then there exist x', y' such that $d(x'R/xR) = k - 1 = d(y'R/yR)$.

This implies that $x \in H_{\sigma+1}(H^k(M))$ and $\operatorname{Soc}(H_{\sigma+1}(M) + N + H_\omega(M)) = \operatorname{Soc}[H_{\sigma+1}(H^k(M)) + N]$ and if $\sigma \geq \omega$, then $H_\sigma(M) \subseteq N + H_\omega(M)$ and the σth relative Ulm-invariant is zero. □

Definition 1. A *QTAG*-module M is *h*-distinctive if there is a monomorphism from M into a direct sum of uniserial modules that does not decrease heights.

Remark 1. Let M be a *QTAG*-module and N a submodule of M such that M/N is a direct sum of uniserial modules. If N is *h*-distinctive, then M is also a direct sum of uniserial modules.

Now, we consider the family $\mathcal{A}_k$ of *QTAG*-modules M which contains nice submodules $N \subseteq H^k(M)$ free from the elements of infinite height, such that M/N is totally projective.

In fact, any module in $\mathcal{A}_k$ is an extension of a totally projective module $H_\omega(M)$ by a separable $(\omega + k)$-projective module $M/H_\omega(M)$ or M is a ω-elongation of a totally projective module by a separable $(\omega + k)$-module.

Theorem 1. *A direct summand of a module in $\mathcal{A}_k$ is again in $\mathcal{A}_k$.*

Proof. Let $M \in \mathcal{A}_k$, such that $M = T \oplus K$ and $N \subseteq H^k(M)$ a nice submodule of M, $N \cap H_\omega(M) = 0$ and M/N totally projective. We define

$$M_1 = T \cap (N \oplus H_\omega(M)) \text{ and } M_2 = K \cap (N \oplus H_\omega(M)).$$

Now, by Lemma 2, $M/(N \oplus H_\omega(M))$ is a direct sum of uniserial modules; therefore

$$T/M_1 \cong (T + (N \oplus H_\omega(M)))/(N \oplus H_\omega(M)) \subseteq M/(N \oplus H_\omega(M))$$

is also a direct sum of uniserial modules.

Again, $H_\omega(M) \subseteq M_1 \oplus M_2 \subseteq N \oplus H_\omega(M)$, therefore

$$M_1 \oplus M_2 = H_\omega(M) \oplus (N \cap (M_1 \oplus M_2)).$$

Since $H_\omega(M) = H_\omega(T) \oplus H_\omega(K)$,

$$M_1 = H_\omega(T) \oplus [M_1 \cap (H_\omega(K) \oplus (N \cap (M_1 \oplus M_2)))] .$$

Now, the submodule $M_1 \cap (H_\omega(K) \oplus (N \cap (M_1 \oplus M_2)))$ is contained in $H^k(M)$ and free from the elements of infinite height. Since $H_\omega(T)$ is a summand of the totally projective module $H_\omega(M)$, by applying Lemma 3, on T and $M_1 \cap ((N \cap (M_1 \oplus M_2)) \oplus H_\omega(K))$, $T \in \mathcal{A}_k$, which completes the proof. □

Theorem 2. *Let $M, M' \in \mathcal{A}_k$. Then M is isomorphic to M' if and only if there is a height-preserving isomorphism $f : H^k(M) \to H^k(M')$.*

Proof. Consider the height-preserving isomorphism f: $H^k(M) \to H^k(M')$. Since $M, M' \in \mathcal{A}_k$, there are nice submodules $N \subseteq H^k(M) \subseteq M$ and $N' \subseteq H^k(M') \subseteq M'$ such that $N \cap H_\omega(M) = 0$, $N' \cap H_\omega(M') = 0$ and $M/N, M'/N'$ are totally projective. By Lemma 2, $M/(N \oplus H_\omega(M))$ and $M'/(N' \oplus H_\omega(M'))$ are direct sums of uniserial modules. We put

$$K = (N \oplus H_\omega(M)) \cap (f^{-1}(N') \oplus H_\omega(M))$$

and consider the exact sequence

$$0 \to (N \oplus H_\omega(M))/K \to M/K \to M/(N \oplus H_\omega(M)) \to 0.$$

Let $x \in N, y \in H_\omega(M)$ such that $H(x+y+K) \geq m$. Since $y \in K, x+y+K = x+K$ and $H(x+K) \geq m$, and there exists some $z \in M$ such that $d[(z+K)R/(x+K)R] = m$. Now there is some $z' \in (x+K)R$ such that $z' - x \in K$. Therefore, $(z'-x) \in (f^{-1}(N') \oplus H_\omega(M))$ and for some $u' \in N'$, $z'' = x + f^{-1}(u')$ where $H_{M'}(f(x)+u') = H_M(z'') \geq m$. This implies that the height of the coset $f(x) + u' + (N' \oplus H_\omega(M'))$ is greater than equal to m in $M'/(N' \oplus H_\omega(M'))$. The map $\bar{f} : ((N \oplus H_\omega(M))/K) \to M'/(N' \oplus H_\omega(M))$ is a monomorphism which does not decrease heights; thus, $(N \oplus H_\omega(M))/K$ is h-distinctive, and by Remark 1, M/K is a direct sum of uniserial modules. Similarly, M'/K' is a direct sum of uniserial modules, where

$$K' = (f(N) \oplus H_\omega(M')) \cap (N' \oplus H_\omega(M')).$$

Since f is height-preserving isomorphism, it maps $H^k(K)$ onto $H^k(K')$, where

$$H^k(K) = \left(N \oplus H_\omega(H^k(M))\right) \cap \left(f^{-1}(N') \oplus H_\omega(H^k(M))\right).$$

Again, if we put

$$T = N \cap \left(f^{-1}(N') \oplus H_\omega(H^k(M))\right),$$

$$T' = N' \cap \left(f(N) \oplus H_\omega(H^k(M'))\right),$$

then $K = T \oplus H_\omega(M)$, $K' = T' \oplus H_\omega(M')$. From Lemma 3, M/T and M'/T' are totally projective. Now $f(T) \oplus H_\omega(M') = T' \oplus H_\omega(M')$; therefore, f induces a height-preserving isomorphism $g_1 : T \to T'$.

The Ulm-invariants of $H_\omega(M)$ and $H_\omega(M')$ are determined by the cardinality of the minimal generating sets of their socles and f is height preserving therefore these are equal for $H_\omega(M)$ and $H_\omega(M')$.

As these modules are totally projective, there is an isomorphism $g_2 : H_\omega(M) \to H_\omega(M')$, which is again height preserving. Now, the isomorphisms g_1, g_2 help us to define an isomorphism $\phi : K \to K'$, where K and K' are nice in M and M', respectively. Since the submodules T and T' have elements of finite heights only and the modules $H_\omega(M)$ and $H_\omega(M')$ have elements of height $\geq \omega$ only, ϕ must be height preserving.

Therefore, by Lemma 4, the Ulm-invariants of K with respect to M can be determined with the help of $H^k(M)$. As

$$f(H^k(K)) = H^k(K'), \quad f_\alpha(K, M) = f_\alpha(K', M')$$

for all α and $M \cong M'$ [6,7]. □

Remark 2. Thus, the isomorphic modules M in $\mathcal{A}_k$ can be identified by $H^k(M)$.

Strong ω-elongations of totally projective *QTAG*-modules by $(\omega + k)$-projective *QTAG*-modules

In the last section, we studied ω-elongations of a totally projective module by $(\omega + k)$-projective module where $H_\omega(M)$ is totally projective and $M/H_\omega(M)$ is $(\omega + k)$-projective.

Here, we study strong ω-elongations and separate ω-elongations. We start with the following:

Definition 2. A *QTAG*-module M is a strong ω-elongation of a totally projective module by a $(\omega + k)$-projective module when $H_\omega(M)$ is totally projective and there is a submodule $N \subseteq H^k(M)$ such that $M/(N + H_\omega(M))$ is a direct sum of uniserial modules.

Definition 3. A *QTAG*-module M is a separate strong ω-elongation of a totally projective module by a separable $(\omega + k)$-projective module if there is a submodule $N \subseteq H^k(M)$, with $N \cap H_\omega(M) = 0$, $H_\omega(M)$ is totally projective and $M/(N \oplus H_\omega(M))$ is a direct sum of uniserial modules.

Remark 3. For the separable modules, $M/(N + H_\omega(M)) \cong (M/N)/(N + H_\omega(M))/N$ is a direct sum of uniserial modules, we have $H_\omega(M/N) = (H_\omega(M) + N)/N$ and these are separate strong ω-elongations.

Now, we prove some basic results:

Proposition 1. *A direct summand of a strong ω-elongation of a totally projective module by a $(\omega + k)$-projective module is again a strong ω-elongation of a totally projective module by a $(\omega + k)$-projective module.*

Proof. Let $M = T \oplus K$ and $N \subseteq M$ such that $N \subseteq H^k(M)$ and $M/(N + H_\omega(M))$ is a direct sum of uniserial modules. We put $M_1 = T \cap (N + H_\omega(M))$ to get

$$\begin{aligned} T/M_1 &\cong (T + (N + H_\omega(M))/(N + H_\omega(M)) \\ &\subseteq M/(N + H_\omega(M)), \end{aligned}$$

which is a direct sum of uniserial modules. Since $H_\omega(M)$ is totally projective and $H_\omega(M) = H_\omega(T) \oplus H_\omega(K)$, $H_\omega(T)$ is also totally projective. Again,

$$\begin{aligned} M_1 &= T \cap (N + H_\omega(T) + H_\omega(K)) \\ &= H_\omega(T) + (T \cap (N + H_\omega(K))); \end{aligned}$$

thus,

$$H_k(T \cap (N + H_\omega(K))) \subseteq H_k(T) \cap H_\omega(K) = 0$$

as $H_k(N) = 0$. Consequently, the result follows. □

Remark 4. Direct sums of strong ω-elongations of a totally projective module by a $(\omega + k)$-projective module is a strong ω-elongations of a totally projective module by $(\omega + k)$-projective module.

After this, we recall some results from previous work, which are helpful in proving the next theorem:

Result 1. A *QTAG*-module M is a Σ-module if and only if $\mathrm{Soc}(M) = \bigcup_{k<\omega} M_k$, $M_k \subseteq M_{k+1}$ and for every $k \in Z^+$, $M_k \cap H_k(M) = \mathrm{Soc}(H_\omega(M))$.

Result 2. Let N be a submodule of a *QTAG*-module M such that M/N is a direct sum of uniserial modules. Then M is a direct sum of uniserial modules if and only if $N = \bigcup_{k<\omega} N_k$, $N_k \subseteq N_{k+1}$ and $N_k \cap H_k(M) = 0$. Equivalently if $\mathrm{Soc}(N) = \bigcup_{k<\omega} (S_k)$, $S_k \subseteq S_{k+1}$ and $S_k \cap H_k(M) = 0$ for every $k \in Z^+$.

It is well known that each totally projective module is a $\sum$-module. The next statement answers under what conditions the converse holds. These additional conditions include the new elongations of totally projective modules by $(\omega + 1)$-projective modules.

Now we are in the state to prove the following:

Theorem 3. *A QTAG-module M which is a strong ω-elongation of a totally projective module by a $(\omega + 1)$-projective module, is a Σ-module if and only if M is a totally projective module.*

Proof. Suppose M is a Σ- module. Since $H_\omega(M)$ is totally projective, in order to prove that M is totally projective, we have to show that $M/H_\omega(M)$ is a direct sum of uniserial modules. By the structure of M, there exists a submodule $N \subseteq \mathrm{Soc}(M)$, such that $M/(N + H_\omega(M))$ is a direct sum of uniserial modules. Also

$$(M/H_\omega(M))/(N + H_\omega(M)/H_\omega(M)) \cong M/(N + H_\omega(M)).$$

Since M is a Σ-module, by Result 1, $\mathrm{Soc}(M) = \bigcup_{k<\omega} M_k$, $M_k \subseteq M_{k+1}$ and $M_k \cap H_k(M) \subseteq H_\omega(M)$ for every $k \in Z^+$. As $N \subseteq \mathrm{Soc}(M)$, $N = \bigcup_{k<\omega} N_k$, $N_k = N \cap M_k$, $N_k \subseteq N_{k+1}$ and $N_k \cap H_k(M) \subseteq H_\omega(M)$. Therefore,

$(N + H_\omega(M))/H_\omega(M) = \bigcup_{k<\omega} [(N_k + H_\omega(M))/H_\omega(M)]$ and

$$\begin{aligned} &[N_k + H_\omega(M)/H_\omega(M)] \cap H_k(M/H_\omega(M)) \\ &= [(N_k + H_\omega(M)) \cap H_k(M)]\,/H_\omega(M) \\ &= [H_\omega(M) + (N_k \cap H_k(M))]\,/H_\omega(M) \\ &= 0. \end{aligned}$$

Now, by Result 2, $M/H_\omega(M)$ is a direct sum of uniserial modules, and the result follows.

The converse is trivial. □

Corollary 1. *A module M is summable and a strong ω-elongation of a totally projective module by a $(\omega + 1)$-projective module if and only if M is a totally projective module of length $\leq \omega + 1$. In other words M is a direct sum of countably generated modules.*

Proof. Every summable module M is a Σ-module and every totally projective module of length $\omega + 1$ is a direct sum of countably generated modules. Therefore M is summable. □

We end this paper with the following remark:

Remark 5. Now we may say that a *QTAG*-module M is a $(\omega + 1)$-projective Σ-module, if and only if it is a direct sum of countably generated modules with lengths at most $\omega + 1$.

Competing interests
The author declares that they have no competing interests.

Authors' contributions

Each author contributed equally in writing this manuscript and all of them have seen the final version of it.

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