Jahanshahi *et al. Mathematical Sciences* 2013, 7:42
http://www.iaumath.com/content/7/1/42

Mathematical Sciences
a SpringerOpen Journal

ORIGINAL RESEARCH Open Access

A special successive approximations method for solving boundary value problems including ordinary differential equations

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Abstract

This paper deals with the solutions of boundary value problems based on the converting differential equation with boundary conditions to a mixed Voletrra and Fredholm integral equation; then, we will use the special case of successive approximations method for solving obtained equation. Convergence analysis and error estimate is discussed; also, a numerical example is presented to demonstrate the accuracy of the proposed technique.

Keywords: Special successive approximations method; Boundary value problems; Ordinary differential equations

Introduction

Consider the second-order linear differential equation

$$y''(x) + m^2 y(x) = f(x) + p(x)y(x), \quad x \in I = [a, b], \tag{1}$$

where $f(x)$ and $p(x)$ are continuous in I, and $m > 0$. Together with the DE (1), we consider the boundary conditions of the form

$$\begin{aligned} l_1(y) &= \alpha_{10}y(a) + \alpha_{11}y'(a) + \beta_{10}y(b) + \beta_{11}y'(b) = 0, \\ l_2(y) &= \alpha_{20}y(a) + \alpha_{21}y'(a) + \beta_{20}y(b) + \beta_{21}y'(b) = 0, \end{aligned} \tag{2}$$

where α_{jk}, β_{jk} $(j = 1, 2, \quad k = 0, 1)$ and a, b are given constants. We are interested in finding the solution to this boundary value problem (BVP) by the special case of successive approximations method. With this method, we obtain an integral equation that is equivalent to the BVP, and solution of integral equation is defined as the solution of the BVP. In this study, unlike differential equation (1) with initial condition, we obtain a Fredholm-Volterra integral equation. By applying n integration of the proposed successive approximations method and then ignoring the Volterra integral as an error term, at the end, we will have a second kind Fredholm integral equation, so solving Fredhelm integral equation yields to a solution for BVP (1)-(2).

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The rest of this paper is divided into the following five sections. In the section 'Mathematical statement of problem,' we introduce the application of the proposed method. In the section 'Convergence analysis and estimate of the error,' we will give the convergence analysis and error estimate of our method. Numerical experiments for a problem will be presented in the section 'Numerical experiments.' A conclusion is given in the section 'Conclusions.' At the end of the paper, in the 'Appendix' section, we will discuss about the application of proposed method to the linear fractional differential equations with the boundary conditions.

Mathematical statement of problem

Suppose $y''(x) + m^2 y(x) = 0$ as a homogenous part of Eq. (1), its two linear independent solutions are given by

$$y_1(x) = \cos mx, \qquad y_2(x) = \sin mx. \tag{3}$$

Therefore, general solution of Eq. (1) is

$$y(x) = c_1 \cos mx + c_2 \sin mx + \int_a^x \frac{\sin m(x-\xi)}{m} [f(\xi) + p(\xi)y(\xi)]\, d\xi, \tag{4}$$

where c_1 and c_2 are arbitrary constants. Now, applying boundary conditions (2) yields

$$y(x) = -\frac{y_1(x)}{\Delta}\int_a^b \begin{vmatrix} \beta_{11}\cos m(b-\xi)+\beta_{10}\frac{\sin m(b-\xi)}{m} & l_1y_2 \\ \beta_{21}\cos m(b-\xi)+\beta_{20}\frac{\sin m(b-\xi)}{m} & l_2y_2 \end{vmatrix} \times [f(\xi)+p(\xi)y(\xi)]\,d\xi + \frac{y_2(x)}{\Delta}\int_a^b \begin{vmatrix} \beta_{11}\cos m(b-\xi)+\beta_{10}\frac{\sin m(b-\xi)}{m} & l_1y_1 \\ \beta_{21}\cos m(b-\xi)+\beta_{20}\frac{\sin m(b-\xi)}{m} & l_2y_1 \end{vmatrix} \times [f(\xi)+p(\xi)y(\xi)]\,d\xi + \int_a^x \frac{\sin m(x-\xi)}{m}[f(\xi)+p(\xi)y(\xi)]\,d\xi, \tag{5}$$

where

$$\Delta = \begin{vmatrix} l_1y_1 & l_1y_2 \\ l_2y_1 & l_2y_2 \end{vmatrix} \neq 0. \tag{6}$$

Thus, by this technique, problem (1)-(2) is reduced to integral equation (5). We can rewrite this equation as follows:

$$y(x) = \int_a^b K(x,\xi)y(\xi)d\xi + \int_a^x R(x,\xi)y(\xi)d\xi + F(x), \tag{7}$$

where

$$K(x,\xi) = y_1(x)g_1(\xi) + y_2(x)g_2(\xi) \tag{8}$$

is the degenerate kernel of Fredholm part and will remain degenerate during iteration processes. Also, $g_1(\xi)$, $g_2(\xi)$, $R(x,\xi)$, and $F(x)$ are as follows:

$$g_1(\xi) = -\frac{1}{\Delta}\begin{vmatrix} \beta_{11}\cos m(b-\xi)+\beta_{10}\frac{\sin m(b-\xi)}{m} & l_1y_2 \\ \beta_{21}\cos m(b-\xi)+\beta_{20}\frac{\sin m(b-\xi)}{m} & l_2y_2 \end{vmatrix} p(\xi)$$
$$g_2(\xi) = \frac{1}{\Delta}\begin{vmatrix} \beta_{11}\cos m(b-\xi)+\beta_{10}\frac{\sin m(b-\xi)}{m} & l_1y_1 \\ \beta_{21}\cos m(b-\xi)+\beta_{20}\frac{\sin m(b-\xi)}{m} & l_2y_1 \end{vmatrix} p(\xi) \tag{9}$$

$$R(x,\xi) = \frac{\sin m(x-\xi)}{m}p(\xi) \tag{10}$$

$$F(x) = \int_a^x \frac{\sin m(x-\xi)}{m}p(\xi)f(\xi)d\xi - \frac{y_1(x)}{\Delta}\int_a^b \begin{vmatrix} \beta_{11}\cos m(b-\xi)+\beta_{10}\frac{\sin m(b-\xi)}{m} & l_1y_2 \\ \beta_{21}\cos m(b-\xi)+\beta_{20}\frac{\sin m(b-\xi)}{m} & l_2y_2 \end{vmatrix} f(\xi)d\xi + \frac{y_2(x)}{\Delta}\int_a^b \begin{vmatrix} \beta_{11}\cos m(b-\xi)+\beta_{10}\frac{\sin m(b-\xi)}{m} & l_1y_1 \\ \beta_{21}\cos m(b-\xi)+\beta_{20}\frac{\sin m(b-\xi)}{m} & l_2y_1 \end{vmatrix} f(\xi)d\xi \tag{11}$$

Now, we apply the special case of successive approximations method to integral equation (7), that is, as a zero-order approximation we take

$$y(x) = y_0(x) = \int_a^b K(x,\xi)y(\xi)d\xi + \int_a^x R(x,\xi)y(\xi)d\xi + F(x), \tag{12}$$

by substitution $y_0(x)$ just into the Volterra integral, first approximation, and by repeating this process, $(n+1)$th approximation, i.e.,

$$y(x) = y_{n+1}(x) = \int_a^b K_{n+1}(x,\xi)y(\xi)d\xi + \int_a^x R_{n+1}(x,\xi)y_0(\xi)d\xi + F_{n+1}(x), n = 0, 1, 2, ... \tag{13}$$

will be obtained. The first approximation is

$$\begin{aligned} y_1(x) &= \int_a^b K(x,\xi)y(\xi)d\xi + \int_a^x R(x,\xi)y_0(\xi)d\xi + F(x) \\ &= \int_a^b K(x,\xi)y(\xi)d\xi + \int_a^x R(x,\xi)\left[\int_a^b K(\xi,\eta)y(\eta)d\eta + \int_a^\xi R(\xi,\eta)y(\eta)d\eta + F(\xi)\right]d\xi + F(x) \\ &= \int_a^b K_1(x,\eta)y(\eta)d\eta + \int_a^x R_1(x,\eta)y(\eta)d\eta + F_1(x), \end{aligned} \tag{14}$$

where

$$K_1(x,\eta) = y_1(x)g_1(\eta) + y_2(x)g_2(\eta) + g_1(\eta)\int_a^x R(x,\xi)y_1(\xi)d\xi + g_2(\eta)\int_a^x R(x,\xi)y_2(\xi)d\xi \tag{15}$$

which is a degenerated kernel and

$$R_1(x,\eta) = \int_\eta^x R(x,\xi)R(\xi,\eta)d\xi, F_1(x) = F(x) + \int_a^x R(x,\xi)F(\xi)d\xi. \tag{16}$$

For the kernel of Volterra integral, we have

$$\begin{aligned} |R_1(x,\eta)| &= \frac{1}{m^2}\left|\int_\eta^x \sin m(x-\xi)\sin m(\xi-\eta)p(\xi)p(\eta)d\xi\right| \\ &\leq \frac{1}{m^2}|p(\eta)|\int_\eta^x |p(\xi)|d\xi \\ &= \frac{1}{m^2}|p(\eta)||p(\alpha_1)|(x-\eta), \quad \alpha_1 \in [\eta, x], \end{aligned} \tag{17}$$

which the last equality concluded from mean value theorem for integration. Again, we can apply this successive approximations on Eq. (14) and so on. Finally from above discussion, we can conclude the following theorem:

Theorem 1. *The converted BVP (1)-(2) to the integral equation (7) provided with the condition (6) can be reduced to the integral equation*

$$y(x) = \int_a^b K_n(x,\eta)y(\eta)d\eta + \int_a^x R_n(x,\eta)y(\eta)d\eta + F_n(x), \tag{18}$$

where n is the number of repetition of special successive approximations method. The kernel of Feredholm term will be the degenerate kernel of the form

$$K_n(x,\eta) = k_{n1}(x)g_1(\eta) + k_{n2}(x)g_2(\eta).$$

The kernel of Volterra term can be obtained by recurrence relation

$$R_n(x,\eta) = \int_\eta^x R_{n-1}(x,\xi)R(\xi,\eta)d\xi, \quad n = 2,3,\ldots,$$

which is bounded by the following inequality

$$|R_n(x,\eta)| \le \frac{1}{m^{n+1}}|p(\eta)||p(\alpha_1)||p(\alpha_2)|\ldots|p(\alpha_n)|\frac{(x-\eta)^n}{n!},$$
$$\alpha_i \in [a,b],\ i = 1,2,\ldots,n.$$

At the end, following [1], by ignoring the Volterra integral as an error term in nth approximation, we can solve the rest of Eq. (18), which is Fredholm integral equation with the degenerate kernel.

Convergence analysis and estimate of the error

In each of iteration, the bounded of neglecting term (Volterra term) can be obtained in the following form

$$|E_n| = \left|\int_a^x R_n(x,t)y(t)dt\right| \le \int_a^x |R_n(x,t)|\,|y(t)|dt$$
$$\le \frac{|p(\alpha_1)||p(\alpha_2)|\ldots|p(\alpha_n)|}{m^{n+1}(n)!}\int_a^x |p(t)||y(t)|(x-t)^n dt.$$

By assuming $y(x) \in L^1[a,b]$ and using mean value theorem for integration [2], we have estimate of error as follows:

$$|E_n| \le \frac{|p(\alpha_1)||p(\alpha_2)|\ldots|p(\alpha_n)|}{m^{n+1}(n)!}\underbrace{|(x-\alpha)^n p(\alpha)|}_{\le M_1}\underbrace{\int_0^x |y(t)|\,dt}_{\le M_2}$$
$$\le \frac{1}{m^{n+1}(n)!}M^n\underbrace{M_1M_2}_{M'} = \frac{M^nM'}{m^{n+1}(n)!}. \tag{19}$$

Here, M is maximum value of $p(x)$ in $[a,\ b]$. Hence, this confirms the unconditional convergence of the scheme.

Remark 1. *One may ask, why we did not consider $(n+1)$th approximation as follows?*

$$y(x) = y_{n+1}(x) = \int_a^b K_{n+1}(x,\xi)y(\xi)d\xi + \int_a^x R_{n+1}(x,\xi)y_n(\xi)d\xi$$
$$+ F_{n+1}(x), n = 0,1,2,\ldots$$

To answer this question, in this form, we will lose unconditional convergence of the scheme, and convergence will be obtained by imposing condition on the function $p(x)$, we leave it to the reader to verify that.

Remark 2. *Existence and uniqueness of solution of BVP (1)-(2) is deducible from condition (6), (see [3]).*

Table 1 Numerical results of the Example 1

x	Exact $y(x)$	Abs. Err. $y(x)$ for n=1	Abs. Err. $y(x)$ for n = 2
0	1.000000000000000	5.386012494114079E-5	3.574668085273292E-05
0.1	1.105170918075648	5.879544653235591E-5	3.878670949075680E-05
0.2	1.221402758160170	6.699635764699785E-5	4.132350799124218E-05
0.3	1.349858807576003	8.949568904654814E-5	4.649878941680452E-05
0.4	1.491824697641270	1.412963409449297E-4	6.190356011468445E-05
0.5	1.648721270700128	2.385317584480573E-4	1.005754869259246E-04
0.6	1.822118800390509	3.959505937054741E-4	1.810815599516458E-04
0.7	2.013752707470477	6.248370391718439E-4	3.267218782851522E-04
0.8	2.225540928492468	9.314423503587577E-4	5.639554728629648E-04
0.9	2.459603111156950	1.315967746945468E-3	9.202040193541983E-04
1	2.718281828459046	1.772105615607567E-3	1.421231461255523E-03

Numerical experiments

Example 1. *Consider the boundary value problem*

$$y'' + 4y = (5e^x - 1) + e^{-x}y, y(0) - y'(0) = 0, y(1) - y'(1) = 0$$

with the exact solution $y(x) = e^x$. *By the method of variation of parameters [4], we obtain*

$$y(x) = c_1 \cos 2x + c_2 \sin 2x + \frac{1}{2}\int_0^x \sin 2(x-\xi)\left[(5e^{\xi} - 1) + e^{-\xi}y(\xi)\right] d\xi$$

substituting $y(x)$ *in boundary values yields*

$$y(x) = (\sin 2x + 2\cos 2x)\left(\frac{2}{5} + \frac{1}{20}\tan 1 + \frac{1}{5\sin 2}\int_0^1 [\cos 2(1-x) - \frac{1}{2}\sin 2(1-x)]\, e^{-x}y(x)dx\right) + \frac{1}{2}\int_0^x \sin 2(x-\xi)[\,(5e^{\xi} - 1) + e^{-\xi}y(\xi)]\, d\xi.$$

Table 1 shows the exact values of $y(x)$ *along with the absolute errors (Abs. Err.* $y(x)$ *for* $x = 0 : .1 : 1$*) obtained in the first and second iterations.*

Conclusions

In this paper, the special case of successive approximations method have been applied for solving boundary value problems, and convergence of method have been discussed. At the end, numerical results of Example 1 showed that the method is accurate and reliable.

Appendix

Application of proposed method to the linear fractional differential equaions with the boundary conditions

Definition 1. *The fractional integral of the function* $y \in L^1([a,b], R_+)$ *of order* $q \in R_+$ *is defined by*

$$I^q y(t) = \frac{1}{\Gamma(q)}\int_0^t (t-s)^{q-1}y(s)ds,$$

where Γ *is Gamma function.*

Definition 2. *The Caputo derivative of fractional order q for a function* $y(t)$ *is defined by*

$$({}^C D^q f)(t) = \frac{1}{\Gamma(n-q)}\int_0^t \frac{f^{(n)}(s)}{(t-s)^{q-n+1}}ds,$$

for $n-1 < q < n$ *and* $n = [q] + 1$, *where* $[q]$ *denotes the integral part of the real number q.*

Lemma 1. *Let* $q > 0$ *and* $n = [q] + 1$. *Then,*

$$I^q({}^C D^q y(t)) = y(t) - \sum_{k=0}^{n-1}\frac{y^{(k)}(a)}{k!}(t-a)^k.$$

Remark 3. *Consider the fractional differential equation*

$$ {}^C D^{\alpha} y(x) = p(x)y(x) + f(x), \quad x \in I = [a,b], \quad 1 < \alpha \le 2 \tag{20}$$

where $f(x)$ *and* $p(x)$ *are continuous in I. Together with the FDE (20), we consider the boundary conditions of the form*

$$l_1(y) = \alpha_{10}y(a) + \alpha_{11}y'(a) + \beta_{10}y(b) + \beta_{11}y'(b) = 0,$$
$$l_2(y) = \alpha_{20}y(a) + \alpha_{21}y'(a) + \beta_{20}y(b) + \beta_{21}y'(b) = 0. \tag{21}$$

By applying fractional integral of order α *on both side of Eq. (20), we get*

$$y(x) = y(a) + y'(a)(x-a) + \frac{1}{\Gamma(\alpha)}\int_a^x (x-s)^{\alpha-1}(p(s)y(s) + f(s))ds. \tag{22}$$

To obtain $y(a)$ *and* $y'(a)$ *and in the sequel* $y(x)$, *we substitute Eq. (22) into the boundary condition (21), then by solving obtained system of equations,* $y(a)$ *and* $y'(a)$ *and in the sequel* $y(x)$ *will be obtained. At the end, by substituting* $y(a)$ *and* $y'(a)$ *into Eq. (22), we reach to a Fredholm-Volterra integral equation, then, applying proposed method to the Fredholm-Volterra integral equation yields to a solution for BVP (20)-(21). The details are left to the reader.*

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
All authors read and approved the final manuscript.

Acknowledgements
The authors would like to express their sincere thanks to the reviewers of this paper for their suggestions which led to an improved version of original manuscript.

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Received: 29 December 2012 Accepted: 2 July 2013
Published: 19 Aug 2013

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10.1186/2251-7456-7-42
Cite this article as: Jahanshahi *et al.*: A special successive approximations method for solving boundary value problems including ordinary differential equations. *Mathematical Sciences* 2013, 7:42