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# Szasz-Schurer operators on a domain in complex plane

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**Abstract**
Our purpose is to show the order of approximation and simultaneous approximation, Voronovskaja-type results with quantitative estimate, the exact degree of approximation for complex Szasz-Schurer operators and complex Kantorovich type generalization of Szasz-Schurer operator attached to analytic functions on compact disks.

**Keywords:** Complex Szasz-Schurer operators; Voronovksaja type result; Exact order of approximation; Simultaneous approximation

**MSC:** 30E10, 41A28

## 1 Introduction

The main problem of approximation theory consists in finding for a complicated function a close-by simple function. Weierstrass's approximation theorem stating that every continuous function on a bounded interval can be approximated to arbitrary accuracy by polynomials is such an important example for this process and has been played the significant role in the development of analysis. For complex analytic functions, this theorem has a significant generalization known as Mergelyan's theorem. The mentioned theorem is about: If a function $f$ is defined on compact set $G$ whose complement is connected in the complex plane and is continuous on $G$ and analytic in the interior, $f$ can be approximated on $G$ by polynomials.

By using probability theory Bernstein [1] proved the Weierstrass's theorem and defined approximate polynomials known as Bernstein polynomials in the literature. In the case of the function $f(z)$ defined and analytic in a certain region involving the interval $[0,1]$, the problem was investigated by Wright [2], Kantorovich [3] and then Bernstein [4].

The degree of approximation for previous mentioned work at first was obtained by Gal [5] on compact disks. Also, exact quantitative estimates and quantitative Voronovskaja-type results for these polynomials, together with similar results for complex version of Bernstein-Stancu polynomials, Kantorovich-Stancu polynomials, Szasz operators, Baskakov operators were obtained by Gal in the book [5] which has collected several recent papers of him. Moreover; complex Schurer type generalization of Bernstein and Kantorovich polynomials were studied by Anastassiou-Gal [6], complex genuine Durrmeyer type polynomials were investigate by Gal [7] and other important generalization of known operators were studied by Gal-Gupta ([8,9]), Agarwal-Gupta [10], Mahmudov ([11-13]), Mahmudov-Gupta [14].

In 1950, Szasz defined and studied the approximation properties of the following operators

$$S_n(f;x) = e^{-nx} \sum_{j=0}^{\infty} \frac{(nx)^j}{j!} f\left(\frac{j}{n}\right),$$

whenever $f$ satisfies exponential-type growth condition [15]. Then Gergen, Dressel and Purcell [16] proved that for certain class of analytic function $f(z)$ the complex Szasz operators $S_n(f;z)$ approximate this function in parabolic domain. Gal obtained quantitative estimates of the convergence and Voronovskaja type theorem in compact disk for the complex Szasz operators attached to $f(z)$ which is analytic function and satisfies some suitable exponential-type growth condition in [17] and does not satisfy such type condition in [18].

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The purpose of this paper is to study complex Szasz-Schurer operator defined by

$$S_{n,p}\left(f;z\right)=e^{-(n+p)z}\sum_{j=0}^{\infty}\frac{((n+p)\,z)^j}{j!}f\left(\frac{j}{n}\right), \tag{1}$$

and complex Kantorovich type generalization of Szasz-Schurer operator defined as

$$\mathcal{K}_{n,p}\left(f;z\right)=(n+p+1)\,e^{-(n+p+1)z}\sum_{j=0}^{\infty}\frac{((n+p+1)\,z)^j}{j!}\int_{\frac{j}{n+1}}^{\frac{j+1}{n+1}}f\left(t\right)dt, \tag{2}$$

where $p\in\mathbb{N}_0$, $n\in\mathbb{N}$ and the function $f:\bar{\mathbb{D}}_R\cup[R,\infty)\rightarrow\mathbb{C}$ is analytic in $\mathbb{D}_R$ and bounded on $[0,\infty)$.

The paper is organized as follows: In Section 2, we give some lemmas of background of the main problems. In Section 3, we obtain upper estimate in approximation by $S_{n,p}\left(f;z\right)$, simultaneous approximation by the operators (1), the Voronovskaja-type formula with a quantitative upper estimate and the exact degree of approximation for $S_{n,p}\left(f;z\right)$. In a final Section 4, the same results for complex Kantorovich type generalization of Szasz-Schurer operator are derived from the obtained inequalities in Section 3.

## 2 Some auxiliary results

Before proceeding to the study of order of approximation by the complex Szasz-Schurer operators, it is necessary to analyze the some properties of the mentioned operators. Here the following lemmas are useful.

**Lemma 1.** *Suppose that f is a polynomial having degree m. Then $S_{n,p}\left(f;z\right)$ is a polynomial having the same degree.*

*Proof.* For $m\in\mathbb{N}_0$, let be $f\left(z\right)=e_m\left(z\right)=z^m$. Taking into account of the following fact

$$e^{-(n+p)z}\sum_{j=0}^{\infty}\frac{((n+p)\,z)^j}{j!}j^m=\sum_{j=0}^{m}c_j^{(m)}\,((n+p)\,z)^j\,,$$

where $c_j^{(m)}$ are constants and $c_m^{(m)}=1$, we get

$$\begin{aligned}S_{n,p}\left(e_m;z\right)&=e^{-(n+p)z}\sum_{j=0}^{\infty}\frac{((n+p)\,z)^j}{j!}\left(\frac{j}{n}\right)^m\\&=\frac{1}{n^m}\sum_{j=0}^{m}c_j^{(m)}\,((n+p)\,z)^j\,.\end{aligned}$$

From the above fact, as $n\rightarrow\infty$ one obtains $S_{n,p}\left(e_m;z\right)\rightarrow e_m\left(z\right)$. This mentioned convergence is uniform on every compact subset of complex plane. Hence, by using the linearity property of $S_{n,p}$ operators we deduce the same result for arbitrary polynomials. □

The aim of the next lemma is to represent the operators (1) with the help of divided difference.

**Lemma 2.** *Let $z\in\mathbb{C}$, $n\in\mathbb{N}$ and $p\in\mathbb{N}_0$ be arbitrary. Then for $S_{n,p}$ holds that*

$$S_{n,p}\left(f;z\right)=\sum_{i=0}^{\infty}\left(\frac{n+p}{n}\right)^i\left[0,\frac{1}{n},\frac{2}{n},...,\frac{i}{n};f\right]z^i.$$

*Proof.* By using the definition of finite difference and divided difference of function $f$, we immediately deduce that

$$\begin{aligned}S_{n,p}\left(f;z\right)&=e^{-(n+p)z}\sum_{j=0}^{\infty}\frac{((n+p)\,z)^j}{j!}f\left(\frac{j}{n}\right)\\&=\sum_{j=0}^{\infty}\frac{((n+p)\,z)^j}{j!}f\left(\frac{j}{n}\right)\left(\sum_{i=0}^{\infty}\frac{(-(n+p)\,z)^i}{i!}\right)\\&=\sum_{j=0}^{\infty}\sum_{i=0}^{\infty}(-1)^i\,\frac{((n+p)\,z)^{j+i}}{j!\,i!}f\left(\frac{j}{n}\right)\\&=\sum_{i=0}^{\infty}\left\{\frac{(n+p)^i}{i!}\sum_{j=0}^{i}\binom{i}{j}(-1)^{i-j}f\left(\frac{j}{n}\right)\right\}z^i\\&=\sum_{i=0}^{\infty}\frac{(n+p)^i}{i!}\,\triangle_{\frac{1}{n}}^{i}f\left(0\right)z^i\\&=\sum_{i=0}^{\infty}\left(\frac{n+p}{n}\right)^i\left[0,\frac{1}{n},\frac{2}{n},...,\frac{i}{n};f\right]z^i.\end{aligned}$$

□

As an important consequence of Lemma 2, we obtain the following inequality for the operators (1).

**Lemma 3.** *For arbitrary $|z|\leq r$, $k,n\in\mathbb{N}$ and $p\in\mathbb{N}_0$, then the following holds*

$$\left|S_{n,p}\left(e_k;z\right)\right|\leq\left(2\left(1+p\right)r\right)^k.$$

*Proof.* In view of Lemmas 1 and 2 and the relation between divided difference and derivative, we have

$$\begin{aligned}\left|S_{n,p}\left(e_k;z\right)\right|&=\left|\sum_{i=0}^{k}\left(\frac{n+p}{n}\right)^i\left[0,\frac{1}{n},\frac{2}{n},\ldots,\frac{i}{n};e_k\right]z^i\right|\\&\leq\sum_{i=0}^{k}\left(\frac{n+p}{n}\right)^i\left|\left[0,\frac{1}{n},\frac{2}{n},\ldots,\frac{i}{n};e_k\right]\right||z|^i\\&\leq\sum_{i=0}^{k}\left(\frac{n+p}{n}\right)^i\frac{k\left(k-1\right)\ldots\left(k-i+1\right)}{i!}r^{k-i}r^i\\&\leq(p+1)^k\sum_{i=0}^{k}\binom{k}{i}r^k\\&=\left(2\left(1+p\right)r\right)^k,\end{aligned}$$

and the proof is completed. □

**Lemma 4.** *For $z \in \mathbb{C}$, $n \in \mathbb{N}$ and $k, p \in \mathbb{N}_0$, let $T_{n,p,k}(z) = S_{n,p}(e_k; z)$. Then the recurrence formula*

$$T_{n,p,k+1}(z) = \frac{z}{n} T'_{n,p,k}(z) + \frac{(n+p)z}{n} T_{n,p,k}(z)$$

*is valid.*

*Proof.* Differentiating the function $T_{n,p,k}(z)$ with respect to $z \neq 0$, we can write

$$\begin{aligned} T'_{n,p,k}(z) &= -(n+p) T_{n,p,k}(z) + \frac{n}{z} e^{-(n+p)z} \\ &\quad \times \sum_{j=0}^{\infty} \frac{((n+p)z)^j}{j!} \left(\frac{j}{n}\right)^{k+1} \\ &= -(n+p) T_{n,p,k}(z) + \frac{n}{z} T_{n,p,k+1}(z). \end{aligned}$$

So the desired result is obtained for $z \in \mathbb{C}$. □

**Lemma 5.** *For $z \in \mathbb{C}$, $k, n \in \mathbb{N}$ and $p \in \mathbb{N}_0$, we have*

$$\begin{aligned} T_{n,p,k}(z) - z^k &= \frac{z}{n}\left(T_{n,p,k-1}(z) - z^{k-1}\right)' \\ &\quad + \frac{(n+p)z}{n}\left(T_{n,p,k-1}(z) - z^{k-1}\right) \\ &\quad + \frac{p}{n} z^k + \frac{k-1}{n} z^{k-1}. \end{aligned}$$

*Proof.* This result is direct conclusion of Lemma 4. □

## 3 Quantitative results for the $S_{n,p}$ operators

In this section, we will get upper estimate in approximation by $S_{n,p}(f; z)$, simultaneous approximation by the operators (1), the Voronovskaja-type formula with a quantitative upper estimate and lastly the exact degree of approximation for $S_{n,p}(f; z)$.

Let us denote the disk $\mathbb{D}_R$ by

$$\mathbb{D}_R = \{z \in \mathbb{C} : |z| < R\}.$$

**Theorem 1.** *Let $p \in \mathbb{N}_0$, $r \geq 1$, $2 < R < \infty$ be such that $r(p+1) < \frac{R}{2}$. Assume that the function $f : \bar{\mathbb{D}}_R \cup [R, \infty) \to \mathbb{C}$ is analytic in $\mathbb{D}_R$ and bounded on $[0, \infty)$. Then, the following assertions hold:*

*(i) For arbitrary $|z| \leq r$ and $n \in \mathbb{N}$, we have*

$$\left|S_{n,p}(f; z) - f(z)\right| \leq \frac{1}{n} C_{r,p}(f),$$

*where*

$$\begin{aligned} C_{r,p}(f) = \sum_{k=1}^{\infty} |c_k| \Bigg\{ &\frac{p}{p+1}((1+p)r)^k \\ &+ \frac{3k(k-1)}{2}(2(1+p)r)^{k-1} \\ &+ \frac{k-1}{p+1}(r(p+1))^k \Bigg\}. \end{aligned}$$

*(ii) If $1 \leq r < r_1 \leq r_1(p+1) < \frac{R}{2}$, then for any $|z| \leq r$ and $n, m \in \mathbb{N}$*

$$\left|S^{(m)}_{n,p}(f; z) - f^{(m)}(z)\right| \leq \frac{1}{n} \frac{m!\, r_1}{(r_1 - r)^{m+1}} C_{r_1,p}(f),$$

*where $C_{r_1,p}(f)$ is mentioned as above.*

*Proof.* *(i)* Because the function $f$ is analytic in $\mathbb{D}_R$, for $z \in \bar{\mathbb{D}}_r$ we can write $f(z) = \sum_{k=0}^{\infty} c_k z^k$. From this fact, we easily get

$$S_{n,p}(f; z) = \sum_{k=0}^{\infty} c_k T_{n,p,k}(z).$$

Therefore,

$$\left|S_{n,p}(f; z) - f(z)\right| \leq \sum_{k=1}^{\infty} |c_k| \left|T_{n,p,k}(z) - z^k\right| \tag{3}$$

follows from the above facts.

Now, we are in a position to find upper bound for $\left|T_{n,p,k}(z) - z^k\right|$. Taking Bernstein inequality, Lemma 3 and Lemma 5 into consideration, by simple calculations we get

$$\begin{aligned} \left|T_{n,p,k}(z) - z^k\right| &\leq \frac{k-1}{n} \left\|T_{n,p,k-1} - e_{k-1}\right\|_r \\ &\quad + \frac{r(n+p)}{n}\left|T_{n,p,k-1}(z) - z^{k-1}\right| \\ &\quad + \frac{p}{n} r^k + \frac{k-1}{n} r^{k-1} \\ &\leq \frac{k-1}{n}\left[(2(1+p)r)^{k-1} + r^{k-1}\right] \\ &\quad + \frac{r(n+p)}{n}\left|T_{n,p,k-1}(z) - z^{k-1}\right| \\ &\quad + \frac{p}{n} r^k + \frac{k-1}{n} r^{k-1} \\ &\leq \frac{r(n+p)}{n}\left|T_{n,p,k-1}(z) - z^{k-1}\right| \\ &\quad + 3\frac{k-1}{n}(2(1+p)r)^{k-1} + \frac{p}{n} r^k \\ &\leq r(1+p)\left|T_{n,p,k-1}(z) - z^{k-1}\right| \\ &\quad + 3\frac{k-1}{n}(2(1+p)r)^{k-1} + \frac{p+1}{n} r^k. \end{aligned} \tag{4}$$

On the other hand, the following inequality

$$\left|T_{n,p,1}(z) - z\right| \leq \frac{p}{n} r$$

is satisfied. If we put $k = 2$ in the inequality (4), we find

$$\begin{aligned} \left|T_{n,p,2}(z) - z^2\right| &\leq \frac{p}{n}(p+1) r^2 + \frac{3}{n}(2(1+p)r)^{2-1} \\ &\quad + \frac{p+1}{n} r^2. \end{aligned}$$

By using the above inequality in (4) for $k=3$, we obtain

$$\left|T_{n,p,3}(z)-z^3\right| \le \frac{p}{n}(p+1)^2 r^3 + \frac{3}{n}(2(1+p)r)^{3-1}(1+2) + \frac{2}{n}r^3(p+1)^2.$$

A similar procedure to that applied for arbitrary natural number $k$ in (4) allows us to show that

$$\begin{aligned}\left|T_{n,p,k}(z)-z^k\right| &\le \frac{p}{n}(p+1)^{k-1}r^k + \frac{3}{n}(2(1+p)r)^{k-1} \\ &\quad\times(1+2+\ldots+(k-1)) \\ &\quad+\frac{k-1}{n}r^k(p+1)^{k-1} \\ &= \frac{p}{n}(p+1)^{k-1}r^k + \frac{3k(k-1)}{2n} \\ &\quad\times(2(1+p)r)^{k-1} + \frac{k-1}{n}r^k(p+1)^{k-1}.\end{aligned} \tag{5}$$

By considering the expression (5) in (3), we see that

$$\begin{aligned}\left|S_{n,p}(f;z)-f(z)\right| &\le \sum_{k=1}^{\infty}|c_k|\left|T_{n,p,k}(z)-z^k\right| \\ &= \frac{1}{n}\sum_{k=1}^{\infty}|c_k|\left\{\frac{p}{p+1}((1+p)r)^k \right.\\ &\quad+\frac{3k(k-1)}{2}(2(1+p)r)^{k-1} \\ &\quad\left.+\frac{k-1}{p+1}(r(p+1))^k\right\} = \frac{1}{n}C_{r,p}(f).\end{aligned}$$

So the proof of ($i$) of Theorem 1 is completed.

($ii$) Let us denote the circle of radius $r_1>r$ centered at origin by $\gamma$. For any $|z|\le r$ and $\vartheta\in\gamma$, we have $|\vartheta-z|\ge r_1-r$. By using Cauchy integral formula, we deduce

$$\begin{aligned}\left|S_{n,p}^{(m)}(f;z)-f^{(m)}(z)\right| &\le \frac{m!}{2\pi}\int_{\gamma}\frac{\left|S_{n,p}(f;\vartheta)-f(\vartheta)\right|}{|\vartheta-z|^{m+1}}|d\vartheta| \\ &\le \frac{1}{n}\frac{m!}{2\pi}C_{r_1,p}(f)\int_{\gamma}\frac{|d\vartheta|}{|\vartheta-z|^{m+1}} \\ &\le \frac{1}{n}\frac{m!\,r_1}{(r_1-r)^{m+1}}C_{r_1,p}(f),\end{aligned}$$

for arbitrary $|z|\le r$ and $n,m\in\mathbb{N}$. □

*Remark 1.* Since by the hypothesis of Theorem 1, $f(z)=\sum_{k=0}^{\infty}c_k z^k$ is absolutely and uniformly convergent $|z|\le r(p+1)<\frac{R}{2}$, it is clear that $C_{r,p}(f)$ is finite. So, the mentioned elementary idea is valid on the following discussion.

**Theorem 2.** *Let be $p\in\mathbb{N}_0$, $2<R<\infty$ such that $p+1<\frac{R}{2}$. Also suppose that the function $f:\bar{\mathbb{D}}_R\cup[R,\infty)\to\mathbb{C}$ is analytic in $\mathbb{D}_R$ and bounded on $[0,\infty)$. Then the following is true for any $z\in\bar{\mathbb{D}}_1$ and $n\in\mathbb{N}$*

$$\left|S_{n,p}(f;z)-f(z)-\frac{pz}{n}f'(z)-\frac{z}{2n}f''(z)\right| \le \frac{H_p(f)}{n^2},$$

*where*

$$H_p(f)=\sum_{k=2}^{\infty}|c_k|\left\{(6+p)(k-1)A_k(2(p+1))^{k-2} + (k-1)B_{p,k}(p+1)^{k-2}\right\}<\infty$$

*and $A_k=(k-1)^2(k-2)$, $B_{p,k}=(k-1)\left(4p(k-1)+p^2\right)$.*

*Proof.* Since

$$T_{n,p,0}(z)=1,\quad T_{n,p,1}(z)=\left(1+\frac{p}{n}\right)z,$$

the above identities yield that

$$\begin{aligned}&\left|S_{n,p}(f;z)-f(z)-\frac{pz}{n}f'(z)-\frac{z}{2n}f''(z)\right| \\ &\quad\le\sum_{k=2}^{\infty}|c_k|\left|T_{n,p,k}(z)-z^k-\frac{kp}{n}z^k-\frac{k(k-1)}{2n}z^{k-1}\right|.\end{aligned} \tag{6}$$

Let us define the function

$$E_{n,p,k}(z)=T_{n,p,k}(z)-z^k-\frac{kp}{n}z^k-\frac{k(k-1)}{2n}z^{k-1},$$

by Lemma 4 we get

$$\begin{aligned}E_{n,p,k}(z) &= \frac{z}{n}E'_{n,p,k-1}(z)+\frac{z(n+p)}{n}E_{n,p,k-1}(z) \\ &\quad-\frac{(k-1)p}{n}z^k+\frac{(k-1)(n+p)p}{n^2}z^k \\ &\quad+\frac{k-1}{2n^2}\left((3k-4)pz+(k-2)^2\right)z^{k-2} \\ &= \frac{z}{n}E'_{n,p,k-1}(z)+\frac{z(n+p)}{n}E_{n,p,k-1}(z) \\ &\quad+\frac{k-1}{2n^2}\left((3k-4)pz+(k-2)^2\right)z^{k-2} \\ &\quad+\frac{(k-1)p^2}{n^2}z^k.\end{aligned}$$

From the above equality by using the Bernstein inequality, we have for $|z| \leq 1$

$$\begin{aligned}\left|E_{n,p,k}(z)\right| &\leq \frac{1}{n}\left|E'_{n,p,k-1}(z)\right| + \frac{(n+p)}{n}\left|E_{n,p,k-1}(z)\right| \\ &\quad + \frac{k-1}{2n^2}\left((3k-4)p+(k-2)^2\right) + \frac{(k-1)p^2}{n^2} \\ &\leq (p+1)\left|E_{n,p,k-1}(z)\right| + \frac{k-1}{n}\left\|E_{n,p,k-1}\right\|_1 \\ &\quad + \frac{k-1}{2n^2}\left((3k-4)p+(k-2)^2\right) + \frac{(k-1)p^2}{n^2} \\ &\leq (p+1)\left|E_{n,p,k-1}(z)\right| \\ &\quad + \frac{k-1}{n}\left(\left\|T_{n,p,k-1}-e_{k-1}\right\|_1 + \frac{(k-1)p}{n} \right.\\ &\quad \left. + \frac{(k-1)(k-2)}{2n}\right) \\ &\quad + \frac{k-1}{2n^2}\left((3k-4)p+(k-2)^2\right) + \frac{(k-1)p^2}{n^2}.\end{aligned} \tag{7}$$

By comparing (5) with (7), we find that

$$\begin{aligned}\left|E_{n,p,k}(z)\right| &\leq (p+1)\left|E_{n,p,k-1}(z)\right| + \frac{k-1}{n}\left\{\frac{p(p+1)^{k-2}}{n}\right. \\ &\quad + \frac{3(k-1)(k-2)}{2n}(2(p+1))^{k-2} \\ &\quad + \frac{k-2}{n}(p+1)^{k-2} + \frac{(k-1)p}{n} \\ &\quad \left. + \frac{(k-1)(k-2)}{2n}\right\} \\ &\quad + \frac{k-1}{2n^2}\left((3k-4)p+(k-2)^2\right) + \frac{(k-1)p^2}{n^2} \\ &\leq (p+1)\left|E_{n,p,k-1}(z)\right| \\ &\quad + \frac{k-1}{n^2}\left\{p(p+1)^{k-2}\right. \\ &\quad + 3(k-1)(k-2)(2(p+1))^{k-2} \\ &\quad + (k-2)(p+1)^{k-2} + (k-1)p \\ &\quad \left. + (k-1)(k-2)\right\} \\ &\quad + \frac{k-1}{n^2}\left((3k-4)p+(k-2)^2+p^2\right) \\ &\leq (p+1)\left|E_{n,p,k-1}(z)\right| \\ &\quad + \frac{k-1}{n^2}\left\{p(p+1)^{k-2} + 4(k-1)(k-2)\right. \\ &\quad \times (2(p+1))^{k-2} \\ &\quad + (k-2)(p+1)^{k-2} + p(4k-5) + (k-2)^2 \\ &\quad \left. + p^2\right\} \\ &\leq (p+1)\left|E_{n,p,k-1}(z)\right| \\ &\quad + \frac{k-1}{n^2}\left((6+p)(k-1)(k-2)(2(p+1))^{k-2}\right. \\ &\quad \left. + 4p(k-1)+p^2\right) \\ &= (p+1)\left|E_{n,p,k-1}(z)\right| + \frac{1}{n^2}(6+p)A_k(2(p+1))^{k-2} \\ &\quad + \frac{1}{n^2}B_{p,k},\end{aligned} \tag{8}$$

for $z \in \bar{\mathbb{D}}_1$ and $k \geq 2$. On the other hand, if we consider $E_{n,p,0}(z) = E_{n,p,1}(z) = 0$ in (8) for $k=2$, then we obtain

$$\left|E_{n,p,2}(z)\right| \leq \frac{1}{n^2}(6+p)(2(p+1))^0 A_2 + \frac{1}{n^2}B_{p,2}.$$

Taking account of the above inequality in (8) for $k=3$, we find

$$\begin{aligned}\left|E_{n,p,3}(z)\right| &\leq (p+1)\left(\frac{1}{n^2}(6+p)(2(p+1))^0 A_2 + \frac{1}{n^2}B_{p,2}\right) \\ &\quad + \frac{1}{n^2}(6+p)(2(p+1))^1 A_3 + \frac{1}{n^2}B_{p,3} \\ &\leq \frac{1}{n^2}(6+p)(2(p+1))^1(A_2+A_3) + \frac{p+1}{n^2} \\ &\quad \times \left(B_{p,2}+B_{p,3}\right).\end{aligned}$$

By the same discussion, for $k \geq 2$ we deduce

$$\begin{aligned}\left|E_{n,p,k}(z)\right| &\leq \frac{1}{n^2}(6+p)(2(p+1))^{k-2}\left(\sum_{j=2}^{k} A_j\right) \\ &\quad + \frac{(p+1)^{k-2}}{n^2}\left(\sum_{j=2}^{k} B_{p,j}\right).\end{aligned}$$

Due to the fact that the sequences $\left(A_j\right)$ and $\left(B_{p,j}\right)$ are increasing, one can write for any $z \in \bar{\mathbb{D}}_1$ and $k \geq 2$

$$\begin{aligned}\left|E_{n,p,k}(z)\right| &\leq \frac{1}{n^2}(6+p)(2(p+1))^{k-2}(k-1)A_k \\ &\quad + \frac{(p+1)^{k-2}}{n^2}(k-1)B_{p,k}.\end{aligned} \tag{9}$$

By substituting (9) in (6), it follows that

$$\begin{aligned}&\left|S_{n,p}(f;z) - f(z) - \frac{pz}{n}f'(z) - \frac{z}{2n}f''(z)\right| \\ &\quad \leq \frac{1}{n^2}\sum_{k=2}^{\infty}|c_k|\left\{(6+p)(k-1)A_k(2(p+1))^{k-2}\right. \\ &\qquad \left. + (k-1)B_{p,k}(p+1)^{k-2}\right\} \\ &\quad = \frac{1}{n^2}H_p(f).\end{aligned}$$

So, we arrive at an estimate as in theorem. □

Following the same process in the proof of Theorem 2, we can easily get the below general result.

*Remark 2.* Assume that for $p \in \mathbb{N}_0$, $r \geq 1$ and $2 < R < \infty$ the following condition holds $r(p+1) < \frac{R}{2}$. If the function $f$ satisfies the same assumptions in Theorem 2, then for $z \in \bar{\mathbb{D}}_r$ and $n \in \mathbb{N}$

$$\left| S_{n,p}\left(f;z\right) - f\left(z\right) - \frac{pz}{n} f'\left(z\right) - \frac{z}{2n} f''\left(z\right) \right| \leq \frac{H_{r,p}\left(f\right)}{n^2},$$

where

$$H_{r,p}\left(f\right) = \sum_{k=2}^{\infty} |c_k| \left(k-1\right) \Big\{ \left(5 + r\left(p+1\right)\right) A_k \left(2r\left(p+1\right)\right)^{k-2} + B_{r,p,k} \left(r\left(p+1\right)\right)^{k-2} \Big\} < \infty$$

and $A_k = (k-1)^2 (k-2)$, $B_{r,p,k} = (k-1) \left[ \left( (4k-5) r^{k-1} + r \right) p + p^2 r^k \right]$.

**Theorem 3.** *Let assumptions $p \in \mathbb{N}_0$, $2 < R < \infty$ and $r(p+1) < \frac{R}{2}$ hold and suppose that the function $f : \bar{\mathbb{D}}_R \cup [R, \infty) \to \mathbb{C}$ is analytic in $\mathbb{D}_R$ and bounded on $[0, \infty)$. If $f$ is not a function of the form*

$$a_1 + a_2 e^{-2pz},$$

*with arbitrary complex constants $a_1$ and $a_2$, then for $r \geq 1$*

$$\left\| S_{n,p}\left(f;.\right) - f \right\|_r \geq \frac{1}{n} M_{r,p}\left(f\right)$$

*where the constant $M_{r,p}\left(f\right)$ depends only on $f$, $r$ and $p$.*

*Proof.* The following identity

$$S_{n,p}\left(f;z\right) - f\left(z\right) = \frac{1}{n} \Big\{ pzf'\left(z\right) + \frac{z}{2} f''\left(z\right) + \frac{1}{n} \left[ n^2 \left( S_{n,p}\left(f;z\right) - f\left(z\right) - \frac{pz}{n} f'\left(z\right) - \frac{z}{2n} f''\left(z\right) \right) \right] \Big\}$$

is quite obvious for any $p \in \mathbb{N}_0$, $n \in \mathbb{N}$ and $z \in \mathbb{C}$. Let $K_{n,p}\left(f;z\right)$ denote the function

$$K_{n,p}\left(f;z\right) := S_{n,p}\left(f;z\right) - f\left(z\right) - \frac{pz}{n} f'\left(z\right) - \frac{z}{2n} f''\left(z\right).$$

So, it follows that

$$\left\| S_{n,p}(f;.) - f \right\|_r \geq \frac{1}{n} \left\{ \left\| pe_1 f' + \frac{e_1}{2} f'' \right\|_r - \frac{1}{n} \left[ n^2 \left\| K_{n,p}(f;.) \right\|_r \right] \right\}. \tag{10}$$

Then we claim that

$$\left\| pe_1 f' + \frac{e_1}{2} f'' \right\|_r > 0.$$

Suppose that for arbitrary $z \in \bar{\mathbb{D}}_r$

$$pzf'\left(z\right) + \frac{z}{2} f''\left(z\right) = 0.$$

Solving the above complex differential equation by means of series method, we obtain for any complex numbers $a_1$ and $a_2$

$$f\left(z\right) = a_1 + a_2 e^{-2pz},$$

but this is a contradiction. On the other hand, Remark 2 allows us to write

$$n^2 \left\| K_{n,p}\left(f;.\right) \right\|_r \leq H_{r,p}\left(f\right).$$

Considering this fact in (10), then there exists a natural number $n_0 \in \mathbb{N}$ such that for arbitrary $n \geq n_0$

$$\left\| S_{n,p}\left(f;.\right) - f \right\|_r \geq \frac{1}{2n} \left\| pe_1 f' + \frac{e_1}{2} f'' \right\|_r. \tag{11}$$

In the case of for $n \in \{1, 2, \ldots, n_0 - 1\}$, we estimate

$$\left\| S_{n,p}\left(f;.\right) - f \right\|_r \geq \frac{A_{r,p,n}\left(f\right)}{n}, \tag{12}$$

where $A_{r,p,n}\left(f\right) = n \left\| S_{n,p}\left(f;.\right) - f \right\|_r > 0$. Finally, from (11) and (12) we derive the estimation for any $n \in \mathbb{N}$

$$\left\| S_{n,p}\left(f;.\right) - f \right\|_r \geq \frac{M_{r,p}\left(f\right)}{n},$$

where $M_{r,p}(f) = \min \left\{ A_{r,p,1}(f), \ldots, A_{r,p,n_0-1}(f), \frac{1}{2} \left\| pe_1 f' + \frac{e_1}{2} f'' \right\|_r \right\}$. □

Combining Theorem 1 with the above result we have:

**Corollary 1.** *Let be $p \in \mathbb{N}_0$, $2 < R < \infty$ and $r(p+1) < \frac{R}{2}$ and suppose that the function $f : \bar{\mathbb{D}}_R \cup [R, \infty) \to \mathbb{C}$ is analytic in $\mathbb{D}_R$ and bounded on $[0, \infty)$. If $f$ is not a function of the form*

$$a_1 + a_2 e^{-2pz},$$

*with arbitrary complex constants $a_1$ and $a_2$, then for $r \geq 1$*

$$\left\| S_{n,p}\left(f;.\right) - f \right\|_r \sim \frac{1}{n}$$

*where the constants in the equivalence depend only on $f$, $r$ and $p$.*

## 4 Quantitative results for the $\mathcal{K}_{n,p}$ operators

This section is based on the connection between the complex Szasz-Schurer operator given by (1) and the complex Kantorovich type generalization of the Szasz-Schurer operator given (2), presenting upper estimates in simultaneous approximation and also Voronovskaja's result with a quantitative estimate for them. Let us define the function $F$ as follows:

$$F\left(z\right) := \int_0^z f\left(t\right) dt.$$

**Theorem 4.** *For arbitrary $n \in \mathbb{N}$ and $z \in \mathbb{C}$, the relationship*

$$\mathcal{K}_{n,p}\left(f;z\right) = S'_{n+1,p}\left(F;z\right) \tag{13}$$

*holds.*

*Proof.* Relationship (13) can be directly obtained from the definition of $S_{n,p}$, that is more clearly

$$S'_{n+1,p}(F;z) = (n+p+1)\,e^{-(n+p+1)z}\sum_{j=0}^{\infty}\frac{((n+p+1)z)^j}{j!}\times\left(F\left(\frac{j+1}{n+1}\right)-F\left(\frac{j}{n+1}\right)\right)$$
$$= (n+p+1)\,e^{-(n+p+1)z}\sum_{j=0}^{\infty}\frac{((n+p+1)z)^j}{j!}\times\int_{\frac{j}{n+1}}^{\frac{j+1}{n+1}} f(t)\,dt.$$

□

**Theorem 5.** *Let $p \in \mathbb{N}_0$, $r \geq 1$, $2 < R < \infty$ be such that $r < r_1 \leq r_1(p+1) < \frac{R}{2}$. Assume that the function $f : \bar{\mathbb{D}}_R \cup [R,\infty) \to \mathbb{C}$ is analytic in $\mathbb{D}_R$ and bounded on $[0,\infty)$. Then the following are true:*

*(i)* *For any $|z| \leq r$ and $m, n \in \mathbb{N}$*

$$\left|\mathcal{K}^{(m)}_{n,p}(f;z) - f^{(m)}(z)\right| \leq \frac{1}{n+1}\frac{(m+1)!\,r_1}{(r_1-r)^{m+2}}C_{r_1,p}(F),$$

*where $C_{r_1,p}(F)$ is defined as in Theorem 1.*

*(ii)* *For arbitrary $|z| \leq r$ and $n \in \mathbb{N}$*

$$\left|\mathcal{K}_{n,p}(f;z) - \frac{n+p+1}{n+1}f(z) - \frac{2pz+1}{2(n+1)}f'(z) - \frac{z}{2(n+1)}f''(z)\right| \leq \frac{1}{(n+1)^2}\frac{r_1}{(r_1-r)^2}H_{r_1,p}(F),$$

*where $H_{r_1,p}(F)$ is defined as in Remark 2.*

*Proof.* $(i)$ Considering Theorem 1 and Theorem 4, we get

$$\left|\mathcal{K}^{(m)}_{n,p}(f;z) - f^{(m)}(z)\right| = \left|S^{(m+1)}_{n+1,p}(F;z) - F^{(m+1)}(z)\right| \leq \frac{1}{n+1}\frac{(m+1)!\,r_1}{(r_1-r)^{m+2}}C_{r_1,p}(F).$$

Keeping in mind that

$$F(z) = \int_0^z\left(\sum_{k=0}^{\infty}c_k t^k\right)dt = \sum_{k=1}^{\infty}\frac{c_{k-1}}{k}z^k = \sum_{k=1}^{\infty}\tilde{c}_k z^k.$$

$(ii)$ From Remark 2, we can write

$$\left|S_{n+1,p}(F;z) - F(z) - \frac{pz}{n+1}F'(z) - \frac{z}{2(n+1)}F''(z)\right| \leq \frac{1}{(n+1)^2}H_{r,p}(F). \tag{14}$$

Put

$$\tilde{S}_n(F;z) := S_{n+1,p}(F;z) - F(z) - \frac{pz}{n+1}F'(z) - \frac{z}{2(n+1)}F''(z),$$

and let us denote the circle of radius $r_1 > r$ centered at origin by $\Gamma$. For any $|z| \leq r$ and $\vartheta \in \Gamma$, we have $|\vartheta - z| \geq r_1 - r$. By using Cauchy integral formula and (14), we deduce

$$\left|\tilde{S}'_n(F;z)\right| \leq \frac{1}{2\pi}\int_{\Gamma}\frac{\left|\tilde{S}_n(F;\vartheta)\right|}{|\vartheta - z|^2}|d\vartheta| \leq \frac{1}{(n+1)^2}\frac{r_1}{(r_1-r)^2}H_{r_1,p}(F).$$

Hence, from the definition of $\tilde{S}_n(F;z)$ we obtain immediately the desired result. □

#### Competing interests
Both authors declare that they have no competing interests.

#### Authors' contributions
SS and Eİ contributed equally. Both authors read and approved the final manuscript.

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