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The generalized χ^2 sequence spaces over *p*- metric spaces defined by Musielak

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Abstract

In this paper, we introduce generalized χ^2 sequence spaces over *p*- metric spaces defined by Musielak function $f = (f_{mn})$ and study some topological properties.

Keywords: Analytic sequence; Double sequences; χ^2 space; Difference sequence space; Musielak-modulus function; *p*- metric space; Duals

MSC: 40A05; 40C05; 40D05

Introduction

Throughout this paper, w, χ, and Λ denote the classes of all, gai, and analytic scalar valued single sequences, respectively. We write w^2 for the set of all complex sequences (x_{mn}), where $m, n \in \mathbb{N}$, the set of positive integers. Then, w^2 is a linear space under the coordinatewise addition and scalar multiplication.

Some initial works on double sequence spaces is found in Bromwich [1]. Later on, they were investigated by Hardy [2], Moricz [3], Moricz and Rhoades [4], Basarir and Solankan [5], Tripathy [6], Turkmenoglu [7], and many others. We procure the following sets of double sequences:

$$\mathcal{M}_u(t) := \left\{(x_{mn}) \in w^2 : sup_{m,n\in N} |x_{mn}|^{t_{mn}} < \infty\right\},$$

$$\mathcal{C}_p(t) := \left\{(x_{mn}) \in w^2 : p - lim_{m,n\to\infty} |x_{mn}-|^{t_{mn}} =1 \text{ for some } \in \mathbb{C}\right\},$$

$$\mathcal{C}_{0p}(t) := \left\{(x_{mn}) \in w^2 : p - lim_{m,n\to\infty} |x_{mn}|^{t_{mn}} = 1\right\},$$

$$\mathcal{L}_u(t) := \left\{(x_{mn}) \in w^2 : \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} |x_{mn}|^{t_{mn}} < \infty\right\},$$

$$\mathcal{C}_{bp}(t) := \mathcal{C}_p(t) \bigcap \mathcal{M}_u(t) \text{ and } \mathcal{C}_{0bp}(t) = \mathcal{C}_{0p}(t) \bigcap \mathcal{M}_u(t),$$

where $t = (t_{mn})$ is the sequence of strictly positive reals t_{mn} for all $m, n \in \mathbb{N}$ and $p - lim_{m,n\to\infty}$ denotes the limit in the Pringsheim's sense. In the case where $t_{mn} = 1$ for all $m, n \in \mathbb{N}, \mathcal{M}_u(t), \mathcal{C}_p(t), \mathcal{C}_{0p}(t), \mathcal{L}_u(t), \mathcal{C}_{bp}(t)$, and $\mathcal{C}_{0bp}(t)$ reduce to the sets $\mathcal{M}_u, \mathcal{C}_p, \mathcal{C}_{0p}, \mathcal{L}_u, \mathcal{C}_{bp}$, and $\mathcal{C}_{0bp}$, respectively. Now, we may summarize the knowledge given in some document related to the double sequence spaces. Gökhan and Colak [8,9] have proved that $\mathcal{M}_u(t)$ and $\mathcal{C}_p(t), \mathcal{C}_{bp}(t)$ are complete paranormed spaces of double sequences and gave the $\alpha-, \beta-, \gamma-$ duals of the spaces $\mathcal{M}_u(t)$ and $\mathcal{C}_{bp}(t)$. Quite recently, in her PhD thesis, Zeltser [10] has essentially studied both the theory of topological double sequence spaces and the theory of summability of double sequences. Mursaleen and Edely [11], and Tripathy [6] have independently introduced the statistical convergence and Cauchy for double sequences and given the relation between statistical convergent and strongly Cesàro summable double sequences. Altay and BaŞar [12] have defined the spaces $\mathcal{BS}, \mathcal{BS}(t), \mathcal{CS}_p, \mathcal{CS}_{bp}, \mathcal{CS}_r$, and $\mathcal{BV}$ of double sequences consisting of all double series whose sequence of partial sums is in the spaces $\mathcal{M}_u, \mathcal{M}_u(t), \mathcal{C}_p, \mathcal{C}_{bp}, \mathcal{C}_r$, and $\mathcal{L}_u$, respectively, and also examined some properties of those sequence spaces and determined the $\alpha-$ duals of the spaces $\mathcal{BS}, \mathcal{BV}, \mathcal{CS}_{bp}$, and the $\beta(\vartheta)-$ duals of the spaces $\mathcal{CS}_{bp}$ and $\mathcal{CS}_r$ of double series. Basar and Sever [13] have introduced the Banach space $\mathcal{L}_q$ of double sequences corresponding to the well-known space ℓ_q of single sequences and examined some properties of the space $\mathcal{L}_q$. Quite recently, Subramanian and Misra [14] have studied the space $\chi^2_M(p, q, u)$ of double sequences and gave some inclusion relations.

The class of sequences which is strongly Cesàro summable with respect to a modulus was introduced

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by Maddox [15] as an extension of the definition of strongly Cesàro summable sequences. Connor [16] further extended this definition to a definition of strong A-summability with respect to a modulus, where $A = (a_{n,k})$ is a non-negative regular matrix, and established some connections between strong A- summability, strong A-summability with respect to a modulus, and A- statistical convergence. In [17], the notion of convergence of double sequences was presented by Pringsheim. Also, in [18,19], and [20], the four-dimensional matrix transformation $(Ax)_{k,\ell} = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} a_{k\ell}^{mn} x_{mn}$ was studied extensively by Hamilton.

We need the following inequality in the sequel of the paper. For $a, b, \geq 0$ and $0 < p < 1$, we have

$$(a+b)^p \leq a^p + b^p. \tag{1}$$

The double series $\sum_{m,n=1}^{\infty} x_{mn}$ is called convergent if and only if the double sequence (s_{mn}) is convergent, where $s_{mn} = \sum_{i,j=1}^{m,n} x_{ij} (m, n \in \mathbb{N})$. A sequence $x = (x_{mn})$ is said to be double analytic if $sup_{mn} |x_{mn}|^{1/m+n} < \infty$. The vector space of all double analytic sequences will be denoted by Λ^2. A sequence $x = (x_{mn})$ is called double gai sequence if$((m+n)! |x_{mn}|)^{1/m+n} \to 0$ as $m, n \to \infty$. The double gai sequences will be denoted by χ^2. Let $\phi = \{$all finite sequences$\}$.

Consider a double sequence $x = (x_{ij})$. The $(m,n)^{\text{th}}$ section $x^{[m,n]}$ of the sequence is defined by $x^{[m,n]} = \sum_{i,j=0}^{m,n} x_{ij} \Im_{ij}$ for all $m, n \in \mathbb{N}$, where $\Im_{ij}$ denotes the double sequence whose only non-zero term is a $\frac{1}{(i+j)!}$ in the (i,j)th place for each $i, j \in \mathbb{N}$.

A Fréchet coordinate space (FK-space or a metric space) X is said to have an AK property if $(\Im_{mn})$ is a Schauder basis for X, or equivalently $x^{[m,n]} \to x$. An FDK-space is a double sequence space endowed with a complete metrizable space, locally convex topology under which the coordinate mappings $x = (x_k) \to (x_{mn}) (m, n \in \mathbb{N})$ are also continuous.

Let M and Φ be mutually complementary modulus functions. Then, we have

(1) For all $u, y \geq 0$,

$$uy \leq M(u) + \Phi(y), \text{(Young's inequality; see [21]).} \tag{2}$$

(2) For all $u \geq 0$,

$$u\eta(u) = M(u) + \Phi(\eta(u)). \tag{3}$$

(3) For all $u \geq 0$ and $0 < \lambda < 1$,

$$M(\lambda u) \leq \lambda M(u). \tag{4}$$

Lindenstrauss and Tzafriri [22] used the idea of Orlicz function to construct Orlicz sequence space

$$\ell_M = \left\{ x \in w : \sum_{k=1}^{\infty} M\left(\frac{|x_k|}{\rho}\right) < \infty, \text{for some } \rho > 0 \right\}.$$

The space ℓ_M with the norm

$$\|x\| = inf \left\{ \rho > 0 : \sum_{k=1}^{\infty} M\left(\frac{|x_k|}{\rho}\right) \leq 1 \right\}$$

becomes a Banach space which is called an Orlicz sequence space. For $M(t) = t^p \, (1 \leq p < \infty)$, the spaces ℓ_M coincide with the classical sequence space ℓ_p.

A sequence $f = (f_{mn})$ of modulus function is called a Musielak-modulus function. A sequence $g = (g_{mn})$ defined by

$$g_{mn}(v) = sup \left\{ |v| \, u - (f_{mn})(u) : u \geq 0 \right\}, m, n = 1, 2, \cdots$$

is called the complementary function of a Musielak-modulus function f. For a given Musielak modulus function f, the Musielak-modulus sequence space t_f and its subspace h_f are defined, respectively, as follows:

$$t_f = \left\{ x \in w^2 : I_f (|x_{mn}|)^{1/m+n} \to 0 \text{ as } m, n \to \infty \right\}$$

and

$$h_f = \left\{ x \in w^2 : I_f (|x_{mn}|)^{1/m+n} \to 0 \text{ as } m, n \to \infty \right\},$$

where I_f is a convex modular defined by

$$I_f(x) = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} f_{mn} (|x_{mn}|)^{1/m+n}, x = (x_{mn}) \in t_f.$$

We consider that t_f is equipped with the Luxemburg metric

$$d(x,y) = sup_{mn} \left\{ inf \left(\sum_{m=1}^{\infty} \sum_{n=1}^{\infty} f_{mn} \times \left(\frac{|x_{mn}|^{1/m+n}}{mn} \right) \right) \leq 1 \right\}.$$

If X is a sequence space, we give the following definitions:

(1) X' = the continuous dual of X;
(2) $X^{\alpha} = \left\{ a = (a_{mn}) : \sum_{m,n=1}^{\infty} |a_{mn} x_{mn}| < \infty, \right.$ for each $x \in X \}$;
(3) $X^{\beta} = \left\{ a = (a_{mn}) : \sum_{m,n=1}^{\infty} a_{mn} x_{mn} \right.$ is convergent, for each $x \in X \}$;
(4) $X^{\gamma} = \left\{ a = (a_{mn}) : sup_{mn} \geq 1 \left| \sum_{m,n=1}^{M,N} a_{mn} x_{mn} \right| < \infty, \right.$ for each $x \in X \}$;
(5) let X be an FK-space $\supset \phi$, then $X^f = \left\{ f(\Im_{mn}) : f \in X' \right\}$;
(6) $X^{\delta} = \left\{ a = (a_{mn}) : sup_{mn} |a_{mn} x_{mn}|^{1/m+n} < \infty, \right.$ for each $x \in X \}$,

where X^{α}, X^{β}, and X^{γ} are called $\alpha-$ (or Köthe-Toeplitz) dual of X, $\beta-$ (or generalized Köthe-Toeplitz) dual of X, $\gamma-$ dual of X, and $\delta-$ dual of X, respectively. X^{α} is defined by Kantham and Gupta [21]. It is clear that $X^{\alpha} \subset X^{\beta}$ and $X^{\alpha} \subset X^{\gamma}$, but $X^{\beta} \subset X^{\gamma}$ does not hold since the sequence of partial sums of a double convergent series needs not to be bounded.

The notion of difference sequence spaces (for single sequences) was introduced by Kizmaz [23] as follows:

$$Z(\Delta) = \{x = (x_k) \in w : (\Delta x_k) \in Z\}$$

for $Z = c, c_0$ and ℓ_{∞}, where $\Delta x_k = x_k - x_{k+1}$ for all $k \in \mathbb{N}$.

Here, c, c_0, and ℓ_{∞} denote the classes of convergent, null, and bounded scalar valued single sequences, respectively. The difference sequence space bv_p of the classical space ℓ_p is introduced and studied in the case $1 \leq p \leq \infty$ and in the case $0 < p < 1$ by Altay and BaŞar in [12]. The spaces $c(\Delta)$, $c_0(\Delta)$, $\ell_{\infty}(\Delta)$, and bv_p are Banach spaces normed by

$$\|x\| = |x_1| + sup_{k\geq 1} |\Delta x_k| \text{ and } \|x\|_{bv_p} = \left(\sum_{k=1}^{\infty} |x_k|^p\right)^{1/p}, (1 \leq p < \infty).$$

Later on, the notion was further investigated by many others. We now introduce the following difference double sequence spaces defined by

$$Z(\Delta) = \left\{x = (x_{mn}) \in w^2 : (\Delta x_{mn}) \in Z\right\},$$

where $Z = \Lambda^2, \chi^2$ and $\Delta x_{mn} = (x_{mn} - x_{mn+1}) - (x_{m+1n} - x_{m+1n+1}) = x_{mn} - x_{mn+1} - x_{m+1n} + x_{m+1n+1}$ for all $m, n \in \mathbb{N}$.

Definition and preliminaries

Let $n \in \mathbb{N}$ and X be a real vector space of dimension w, where $n \leq w$. A real valued function $d_p(x_1, \ldots, x_n) = \|(d_1(x_1), \ldots, d_n(x_n))\|_p$ on X satisfying the following four conditions:

(1) $\|(d_1(x_1), \ldots, d_n(x_n))\|_p = 0$ if and and only if $d_1(x_1), \ldots, d_n(x_n)$ are linearly dependent,
(2) $\|(d_1(x_1), \ldots, d_n(x_n))\|_p$ is invariant under permutation,
(3) $\|(\alpha d_1(x_1), \ldots, d_n(x_n))\|_p = |\alpha| \, \|(d_1(x_1), \ldots, d_n(x_n))\|_p, \alpha \in \mathbb{R}$
(4) $d_p((x_1, y_1), (x_2, y_2) \cdots (x_n, y_n)) = (d_X(x_1, x_2, \cdots x_n)^p + d_Y(y_1, y_2, \cdots y_n)^p)^{1/p} for 1 \leq p < \infty$; (or)
(5) $d((x_1, y_1), (x_2, y_2), \cdots (x_n, y_n)) := \sup\left\{d_X(x_1, x_2, \cdots x_n), d_Y(y_1, y_2, \cdots y_n)\right\}$, for $x_1, x_2, \cdots x_n \in X, y_1, y_2, \cdots y_n \in Y$ which is called the p product metric of the Cartesian product of n metric spaces is the p norm of the n-vector of the norms of the n subspaces.

A trivial example of the p product metric of the n metric space is the p norm space which is $X = \mathbb{R}$ equipped with the following Euclidean metric in the product space:

$$\|(d_1(x_1), \ldots, d_n(x_n))\|_E = sup\left(|det(d_{mn}(x_{mn}))|\right) = sup\left(\left|\begin{matrix} d_{11}(x_{11}) & d_{12}(x_{12}) & \cdots & d_{1n}(x_{1n}) \\ d_{21}(x_{21}) & d_{22}(x_{22}) & \cdots & d_{2n}(x_{1n}) \\ \vdots & & & \\ d_{n1}(x_{n1}) & d_{n2}(x_{n2}) & \cdots & d_{nn}(x_{nn}) \end{matrix}\right|\right),$$

where $x_i = (x_{i1}, \cdots x_{in}) \in \mathbb{R}^n$ for each $i = 1, 2, \cdots n$.

If every Cauchy sequence in X converges to some $L \in X$, then X is said to be complete with respect to the p- metric. Any complete p- metric space is said to be p- Banach metric space.

Let X be a linear metric space. A function $w : X \to \mathbb{R}$ is called paranorm if

(1) $w(x) \geq 0$ for all $x \in X$;
(2) $w(-x) = w(x)$ for all $x \in X$,
(3) $w(x + y) \leq w(x) + w(y)$ for all $x, y \in X$;
(4) If (σ_{mn}) is a sequence of scalars with $\sigma_{mn} \to \sigma$ as $m, n \to \infty$, and (x_{mn}) is a sequence of vectors with $w(x_{mn} - x) \to 0$ as $m, n \to \infty$, then $w(\sigma_{mn} x_{mn} - \sigma x) \to 0$ as $m, n \to \infty$.

A paranorm w for which $w(x) = 0$ implies $x = 0$ is called a total paranorm, and the pair (X, w) is called a total paranormed space. It is well known that the metric of any linear metric space is given by some total paranorm (see [24], Theorem 10.4.2, p.183).

The notion of $\lambda-$ double gai and double analytic sequences is as follows: Let $\lambda = (\lambda_{mn})_{m,n=0}^{\infty}$ be a strictly increasing sequence of positive real numbers tending to infinity, that is,

$$0 < \lambda_0 < \lambda_1 < \cdots \text{ and } \lambda_{mn} \to \infty \text{ as } m, n \to \infty$$

and that a sequence $x = (x_{mn}) \in w^2$ is $\lambda-$ convergent to 0, called a the $\lambda-$ limit of x, if $\mu_{mn}(x) \to 0$ as $m, n \to \infty$, where

$$\mu_{mn}(x) = \frac{1}{\varphi_{rs}} \sum_{m\in\sigma,\sigma\in P_{rs}} \sum_{n\in\sigma,\sigma\in P_{rs}} \times \left(\lambda_{m,n} - \lambda_{m,n+1} - \lambda_{m+1,n} + \lambda_{m+1,n+1}\right) |x_{mn}|^{1/m+n}.$$

The sequence $x = (x_{mn}) \in w^2$ is $\lambda-$ double analytic if $sup_{uv} |\mu_{mn}(x)| < \infty$. If $lim_{mn} x_{mn} = 0$ in the ordinary sense of convergence, then

$$lim_{mn}\left(\frac{1}{\varphi_{rs}} \sum_{m\in\sigma,\sigma\in P_{rs}} \sum_{n\in\sigma,\sigma\in P_{rs}} \left(\lambda_{m,n} - \lambda_{m,n+1} - \lambda_{m+1,n} + \lambda_{m+1,n+1}\right) \left((m+n)!\,|x_{mn} - 0|\right)^{1/m+n}\right) = 0.$$

This implies that it yields $lim_{uv} \mu_{mn}(x) = 0$, and hence, $x = (x_{mn}) \in w^2$ is $\lambda-$ convergent to 0. Let $f = (f_{mn})$ be a Musielak-modulus function, $\left(X, \|(d(x_1), d(x_2), \cdots, d(x_{n-1}))\|_p\right)$ be a p-metric space, and $q = (q_{mn})$ be double analytic sequence of strictly positive real numbers. By $w^2(p - X)$, we denote the space of all sequences as $\left(X, \|(d(x_1), d(x_2), \cdots, d(x_{n-1}))\|_p\right)$. The following inequality will be used throughout the paper. If $0 \leq q_{mn} \leq sup q_{mn} = H, K = max\left(1, 2^{H-1}\right)$, then

$$|a_{mn} + b_{mn}|^{q_{mn}} \leq K\left\{|a_{mn}|^{q_{mn}} + |b_{mn}|^{q_{mn}}\right\} \quad (5)$$

for all m, n and $a_{mn}, b_{mn} \in \mathbb{C}$. Also, $|a|^{q_{mn}} \leq max\left(1, |a|^H\right)$ for all $a \in \mathbb{C}$.

In the present paper, we define the following sequence spaces:

$$\left[\chi^{2q}_{f\mu}, \|(d(x_1), d(x_2), \cdots, d(x_{n-1}))\|^{\varphi}_p\right] = \lim_{mn}\left\{\left[f_{mn}\left(\|\mu_{mn}(x), (d(x_1), d(x_2), \cdots, d(x_{n-1}))\|_p\right)\right]^{q_{mn}} = 0\right\},$$

$$\left[\Lambda^{2q}_{f\mu}, \|(d(x_1), d(x_2), \cdots, d(x_{n-1}))\|^{\varphi}_p\right] = sup_{mn}\left\{\left[f_{mn}\left(\|\mu_{mn}(x), (d(x_1), d(x_2), \cdots, d(x_{n-1}))\|_p\right)\right]^{q_{mn}} < \infty\right\}.$$

If we take $f_{mn}(x) = x$, we get

$$\left[\chi^{2q}_{\mu}, \|(d(x_1), d(x_2), \cdots, d(x_{n-1}))\|^{\varphi}_p\right] = \lim_{mn}\left\{\left[\left(\|\mu_{mn}(x), (d(x_1), d(x_2), \cdots, d(x_{n-1}))\|_p\right)\right]^{q_{mn}} = 0\right\},$$

$$\left[\Lambda^{2q}_{\mu}, \|(d(x_1), d(x_2), \cdots, d(x_{n-1}))\|^{\varphi}_p\right] = sup_{mn}\left\{\left[\left(\|\mu_{mn}(x), (d(x_1), d(x_2), \cdots, d(x_{n-1}))\|_p\right)\right]^{q_{mn}} < \infty\right\}.$$

If we take $q = (q_{mn}) = 1$

$$\left[\chi^{2}_{f\mu}, \|(d(x_1), d(x_2), \cdots, d(x_{n-1}))\|^{\varphi}_p\right] = \lim_{mn}\left\{\left[f_{mn}\left(\|\mu_{mn}(x), (d(x_1), d(x_2), \cdots, d(x_{n-1}))\|_p\right)\right] = 0\right\},$$

$$\left[\Lambda^{2}_{f\mu}, \|(d(x_1), d(x_2), \cdots, d(x_{n-1}))\|^{\varphi}_p\right] = sup_{mn}\left\{\left[f_{mn}\left(\|\mu_{mn}(x), (d(x_1), d(x_2), \cdots, d(x_{n-1}))\|_p\right)\right] < \infty\right\}.$$

In the present paper, we plan to study some topological properties and inclusion relation between the above defined sequence spaces, $\left[\chi^{2q}_{f\mu}, \|(d(x_1), d(x_2), \cdots, d(x_{n-1}))\|^{\varphi}_p\right]$ and $\left[\Lambda^{2q}_{f\mu}, \|(d(x_1), d(x_2), \cdots, d(x_{n-1}))\|^{\varphi}_p\right]$, which we shall discuss in this paper.

Main results

Theorem 1. *Let $f = \left(f_{mn}\right)$ be a Musielak-modulus function and $q = (q_{mn})$ be a double analytic sequence of strictly positive real numbers; the sequence spaces $\left[\chi^{2q}_{f\mu}, \|(d(x_1), d(x_2), \cdots, d(x_{n-1}))\|^{\varphi}_p\right]$ and $\left[\Lambda^{2q}_{f\mu}, \|(d(x_1), d(x_2), \cdots, d(x_{n-1}))\|^{\varphi}_p\right]$ are linear spaces.*

Proof. It is routine verification. Therefore, the proof is omitted. □

Theorem 2. *Let $f = \left(f_{mn}\right)$ be a Musielak-modulus function and $q = (q_{mn})$ be a double analytic sequence of strictly positive real numbers; the sequence space $\left[\chi^{2q}_{f\mu}, \|(d(x_1), d(x_2), \cdots, d(x_{n-1}))\|^{\varphi}_p\right]$ is a paranormed space with respect to the paranorm defined by*

$$g(x) = inf\left\{\left(\left[f_{mn}\left(\|\mu_{mn}(x), (d(x_1), d(x_2), \cdots, d(x_{n-1}))\|^{\varphi}_p\right)\right]^{q_{mn}}\right)^{1/H} \leq 1\right\},$$

where $H = max\,(1, sup_{mn} q_{mn} < \infty)$.

Proof. Clearly, $g(x) \geq 0$ for $x = (x_{mn}) \in \left[\chi^{2q}_{f\mu}, \|(d(x_1), d(x_2), \cdots, d(x_{n-1}))\|^{V_2}_p\right]$. Since $f_{mn}(0) = 0$, we get $g(0) = 0$.

Conversely, suppose that $g(x) = 0$, then

$$inf\left\{\left(\left[f_{mn}\left(\|\mu_{mn}(x), (d(x_1), d(x_2), \cdots, d(x_{n-1}))\|^{\varphi}_p\right)\right]^{q_{mn}}\right)^{1/H}\right\} \leq 1 = 0.$$

Suppose that $\mu_{mn}(x) \neq 0$ for each $m, n \in \mathbb{N}$. Then, $\|\mu_{mn}(x), (d(x_1), d(x_2), \cdots, d(x_{n-1}))\|^{\varphi}_p \to \infty$. It follows that $\left(\left[f_{mn}\left(\|\mu_{mn}(x), (d(x_1), d(x_2), \cdots, d(x_{n-1}))\|^{V_2}_p\right)\right]^{q_{mn}}\right)^{1/H} \to \infty$ which is a contradiction. Therefore, $\mu_{mn}(x) = 0$. Let

$$\left(\left[f_{mn}\left(\|\mu_{mn}(x), (d(x_1), d(x_2), \cdots, d(x_{n-1}))\|^{\varphi}_p\right)\right]^{q_{mn}}\right)^{1/H} \leq 1$$

and

$$\left(\left[f_{mn}\left(\|\mu_{mn}(y), (d(x_1), d(x_2), \cdots, d(x_{n-1}))\|^{\varphi}_p\right)\right]^{q_{mn}}\right)^{1/H} \leq 1.$$

Then, by using Minkowski's inequality, we have

$$\left(\left[f_{mn}\left(\left\|\mu_{mn}(x+y),(d(x_1),d(x_2),\cdots,d(x_{n-1}))\right\|_p^{\varphi}\right)\right]^{q_{mn}}\right)^{1/H}$$
$$\leq\left(\left[f_{mn}\left(\left\|\mu_{mn}(x),(d(x_1),d(x_2),\cdots,d(x_{n-1}))\right\|_p^{\varphi}\right)\right]^{q_{mn}}\right)^{1/H}$$
$$+\left(\left[f_{mn}\left(\left\|\mu_{mn}(y),(d(x_1),d(x_2),\cdots,d(x_{n-1}))\right\|_p^{\varphi}\right)\right]^{q_{mn}}\right)^{1/H}.$$

So, we have

$$g(x+y)=inf\left\{\left(\left[f_{mn}\left(\left\|\mu_{mn}(x+y),(d(x_1),d(x_2),\cdots,d(x_{n-1}))\right\|_p^{\varphi}\right)\right]^{q_{mn}}\right)^{1/H}\leq 1\right\}$$
$$\leq inf\left\{\left(\left[f_{mn}\left(\left\|\mu_{mn}(x),(d(x_1),d(x_2),\cdots,d(x_{n-1}))\right\|_p^{\varphi}\right)\right]^{q_{mn}}\right)^{1/H}\leq 1\right\}$$
$$+inf\left\{\left(\left[f_{mn}\left(\left\|\mu_{mn}(y),(d(x_1),d(x_2),\cdots,d(x_{n-1}))\right\|_p^{\varphi}\right)\right]^{q_{mn}}\right)^{1/H}\leq 1\right\}.$$

Therefore,

$$g(x+y)\leq g(x)+g(y).$$

Finally, to prove that the scalar multiplication is continuous, let λ be any complex number. By definition,

$$g(\lambda x)=inf\left\{\left(\left[f_{mn}\left(\left\|\mu_{mn}(\lambda x),(d(x_1),d(x_2),\cdots,d(x_{n-1}))\right\|_p^{\varphi}\right)\right]^{q_{mn}}\right)^{1/H}\leq 1\right\}.$$

Then,

$$g(\lambda x)=inf\left\{((|\lambda|\,t)^{q_{mn}/H}:\left(\left[f_{mn}\left(\left\|\mu_{mn}(x),(d(x_1),d(x_2),\cdots,d(x_{n-1}))\right\|_p^{\varphi}\right)\right]^{q_{mn}}\right)^{1/H}\leq 1\right\},$$

where $t=\frac{1}{|\lambda|}$. Since $|\lambda|^{q_{mn}}\leq max\left(1,|\lambda|^{supp_{mn}}\right)$, we have

$$g(\lambda x)\leq max\left(1,|\lambda|^{supp_{mn}}\right)$$
$$\times inf\left\{t^{q_{mn}/H}:\left(\left[f_{mn}\left(\left\|\mu_{mn}(\lambda x),(d(x_1),d(x_2),\cdots,d(x_{n-1}))\right\|_p^{\varphi}\right)\right]^{q_{mn}}\right)^{1/H}\leq 1\right\}.$$

□

Theorem 3. *The* $\beta-$ *dual space of* $\left[\chi^{2q}_{f\mu},\|\mu_{mn}(x),(d(x_1),d(x_2),\cdots,d(x_{n-1}))\|_p^{\varphi}\right]^{\beta}=\left[\Lambda^{2q}_{f\mu},\|\mu_{mn}(x),(d(x_1),d(x_2),\cdots,d(x_{n-1}))\|_p^{\varphi}\right]$.

Proof. First, we observe that

$$\left[\chi^{2q}_{f\mu},\|\mu_{mn}(x),(d(x_1),d(x_2),\cdots,d(x_{n-1}))\|_p^{\varphi}\right]^{\beta}\subset\left[\Gamma^{2q}_{f\mu},\|\mu_{mn}(x),(d(x_1),d(x_2),\cdots,d(x_{n-1}))\|_p^{\varphi}\right].$$

Therefore,

$$\left[\Gamma^{2q}_{f\mu},\|\mu_{mn}(x),(d(x_1),d(x_2),\cdots,d(x_{n-1}))\|_p^{\varphi}\right]^{\beta}\subset\left[\chi^{2q}_{f\mu},\|\mu_{mn}(x),(d(x_1),d(x_2),\cdots,d(x_{n-1}))\|_p^{\varphi}\right]^{\beta}.$$

But

$$\left[\Gamma^{2q}_{f\mu}\right]^{\beta}\underset{\neq}{\subset}\left[\Lambda^{2q}_{f\mu},\|\mu_{mn}(x),(d(x_1),d(x_2),\cdots,d(x_{n-1}))\|_p^{\varphi}\right].$$

Hence,

$$\left[\Lambda^{2q}_{f\mu},\|\mu_{mn}(x),(d(x_1),d(x_2),\cdots,d(x_{n-1}))\|_p^{\varphi}\right]\subset\left[\chi^{2q}_{f\mu},\|\mu_{mn}(x),(d(x_1),d(x_2),\cdots,d(x_{n-1}))\|_p^{\varphi}\right]^{\beta}. \tag{6}$$

Next, we show that

$$\left[\chi^{2q}_{f\mu},\|\mu_{mn}(x),(d(x_1),d(x_2),\cdots,d(x_{n-1}))\|_p^{\varphi}\right]^{\beta}\subset\left[\Lambda^{2q}_{f\mu},\|\mu_{mn}(x),(d(x_1),d(x_2),\cdots,d(x_{n-1}))\|_p^{\varphi}\right].$$

Let $y=(y_{mn})\in\left[\chi^{2q}_{f\mu},\|\mu_{mn}(x),(d(x_1),d(x_2),\cdots,d(x_{n-1}))\|_p^{\varphi}\right]^{\beta}$. Consider $f(x)=\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}x_{mn}y_{mn}$ with

$$x=(x_{mn})\in\left[\chi^{2q}_{f\mu},\|\mu_{mn}(x),(d(x_1),d(x_2),\cdots,d(x_{n-1}))\|_p^{\varphi}\right]$$

$$x=[(\lambda_{mn}-\lambda_{mn+1})-(\lambda_{m+1n}-\lambda_{m+1n+1})]$$
$$=\begin{pmatrix}0&0&\ldots 0&0&\ldots&0\\0&0&\ldots 0&0&\ldots&0\\ \vdots&&&&&\\0&0\ldots&\frac{\varphi_{rs}}{\Delta\lambda_{mn}(m+n)!}&\frac{-\varphi_{rs}}{\Delta\lambda_{mn}(m+n)!}&\ldots&0\\0&0&\ldots 0&0&\ldots&0\end{pmatrix}$$
$$-\begin{pmatrix}0&0&\ldots 0&0&\ldots&0\\0&0&\ldots 0&0&\ldots&0\\ \vdots&&&&&\\0&0\ldots&\frac{\varphi_{rs}}{\Delta\lambda_{mn}(m+n)!}&\frac{-\varphi_{rs}}{\Delta\lambda_{mn}(m+n)!}&\ldots&0\\0&0&\ldots 0&0&\ldots&0\end{pmatrix}$$

$$\left[f_{mn}\left(\|\mu_{mn}(x),(d(x_1),d(x_2),\cdots,d(x_{n-1}))\|_p^{\varphi}\right)\right]$$
$$=\begin{pmatrix} 0 & 0 & \ldots 0 & 0 & \ldots 0 \\ 0 & 0 & \ldots 0 & 0, & \ldots 0 \\ \vdots & & & & \\ 0 & 0 & \ldots f_{mn}\left(\frac{\varphi_{rs}}{\Delta\lambda_{mn}(m+n)!}\right) & f_{mn}\left(\frac{-\varphi_{rs}}{\Delta\lambda_{mn}(m+n)!}\right) & \ldots 0 \\ 0 & 0 & \ldots f_{mn}\left(\frac{-\varphi_{rs}}{\Delta\lambda_{mn}(m+n)!}\right) & f_{mn}\left(\frac{\varphi_{rs}}{\Delta\lambda_{mn}(m+n)!}\right) & \ldots 0 \\ 0 & 0 & \ldots 0 & 0, & \ldots 0 \end{pmatrix}.$$

Hence, it converges to zero.

Therefore,

$$[(\lambda_{mn}-\lambda_{mn+1})-(\lambda_{m+1n}-\lambda_{m+1n+1})] \in \left[\chi^{2q}_{f\mu},\|\mu_{mn}(x),(d(x_1),d(x_2),\cdots,d(x_{n-1}))\|_p^{\varphi}\right].$$

Hence, $d((\lambda_{mn}-\lambda_{mn+1})-(\lambda_{m+1n}-\lambda_{m+1n+1}),0)=1$. But

$$\left|y_{mn}\right| \le \|f\|\, d((\lambda_{mn}-\lambda_{mn+1})-(\lambda_{m+1n}-\lambda_{m+1n+1}),0) \le \|f\|\cdot 1<\infty$$

for each m,n. Thus, (y_{mn}) is a p- metric paranormed space of double analytic sequence and, hence, an p- metric double analytic sequence.

In other words. $y\in\left[\Lambda^{2q}_{f\mu},\|\mu_{mn}(x),(d(x_1),d(x_2),\cdots,d(x_{n-1}))\|_p^{\varphi}\right]$. But $y=(y_{mn})$ is arbitrary in $\left[\chi^{2q}_{f\mu},\|\mu_{mn}(x),(d(x_1),d(x_2),\cdots,d(x_{n-1}))\|_p^{\varphi}\right]^{\beta}$.Therefore,

$$\left[\chi^{2q}_{f\mu},\|\mu_{mn}(x),(d(x_1),d(x_2),\cdots,d(x_{n-1}))\|_p^{\varphi}\right]^{\beta} \subset \left[\Lambda^{2q}_{f\mu},\|\mu_{mn}(x),(d(x_1),d(x_2),\cdots,d(x_{n-1}))\|_p^{\varphi}\right]. \tag{7}$$

From (6) and (7), we get

$$\left[\chi^{2q}_{f\mu},\|\mu_{mn}(x),(d(x_1),d(x_2),\cdots,d(x_{n-1}))\|_p^{\varphi}\right]^{\beta} = \left[\Lambda^{2q}_{f\mu},\|\mu_{mn}(x),(d(x_1),d(x_2),\cdots,d(x_{n-1}))\|_p^{\varphi}\right].$$

□

Theorem 4. *The dual space of* $\left[\chi^{2q}_{f\mu},\|\mu_{mn}(x),(d(x_1),d(x_2),\cdots,d(x_{n-1}))\|_p^{\varphi}\right]$ *is* $\left[\Lambda^{2q}_{f\mu},\|\mu_{mn}(x),(d(x_1),d(x_2),\cdots,d(x_{n-1}))\|_p^{\varphi}\right]$. *In other words.* $\left[\chi^{2q}_{f\mu},\|\mu_{mn}(x),(d(x_1),d(x_2),\cdots,d(x_{n-1}))\|_p^{\varphi}\right]^{*}=\left[\Lambda^{2q}_{f\mu},\|\mu_{mn}(x),(d(x_1),d(x_2),\cdots,d(x_{n-1}))\|_p^{\varphi}\right]$.

Proof. We recall that

$$\lambda_{mn}=\begin{pmatrix} 0 & 0 & \ldots 0 & 0 & \ldots \\ 0 & 0 & \ldots 0 & 0 & \ldots \\ \vdots & & & & \\ 0 & 0 & \ldots \frac{\varphi_{rs}}{\Delta\lambda_{mn}(m+n)!} & 0 & \ldots \\ 0 & 0 & \ldots 0 & 0 & \ldots \end{pmatrix}$$

with $\frac{\varphi_{rs}}{\Delta\lambda_{mn}(m+n)!}$ in the (m,n)th position and zeros elsewhere,

$$\left[\chi^{2q}_{f\mu},\|\mu_{mn}(x),(d(x_1),d(x_2),\cdots,d(x_{n-1}))\|_p^{\varphi}\right]$$
$$=\begin{pmatrix} 0. & . & .\ 0 \\ . & & \\ . & & \\ . & & \\ 0 & f\left(\frac{\varphi_{rs}}{\Delta\lambda_{mn}(m+n)!}\right)^{1/m+n} & .\ 0 \\ & (m,n)^{\text{th}} & \\ 0 & . & .\ 0 \end{pmatrix}$$

which is a p- metric of double gai sequence. Hence,

$$x\in\left[\chi^{2q}_{f\mu},\|\mu_{mn}(x),(d(x_1),d(x_2),\cdots,d(x_{n-1}))\|_p^{\varphi}\right] f(x) = \sum_{m,n=1}^{\infty} x_{mn}y_{mn}$$

with $x\in\left[\chi^{2q}_{f\mu},\|\mu_{mn}(x),(d(x_1),d(x_2),\cdots,d(x_{n-1}))\|_p^{\varphi}\right]$ and $f\in\left[\chi^{2q}_{f\mu},\|\mu_{mn}(x),(d(x_1),d(x_2),\cdots,d(x_{n-1}))\|_p^{\varphi}\right]^{*}$, where $\left[\chi^{2q}_{f\mu},\|\mu_{mn}(x),(d(x_1),d(x_2),\cdots,d(x_{n-1}))\|_p^{\varphi}\right]^{*}$ is the dual space of $\left[\chi^{2q}_{f\mu},\|\mu_{mn}(x),(d(x_1),d(x_2),\cdots,d(x_{n-1}))\|_p^{\varphi}\right]$.

Take $x=(x_{mn})\in\left[\chi^{2q}_{f\mu},\|\mu_{mn}(x),(d(x_1),d(x_2),\cdots,d(x_{n-1}))\|_p^{\varphi}\right]$. Then,

$$\left|y_{mn}\right|\le\|f\|\,d(\varphi_{rs},0)<\infty\forall m,n. \tag{8}$$

Thus, (y_{mn}) is a p- metric of the double analytic sequence and an p- metric of double analytic sequence. In other words, $y\in\left[\Lambda^{2q}_{f\mu},\|\mu_{mn}(x),(d(x_1),d(x_2),\cdots,d(x_{n-1}))\|_p^{\varphi}\right]$. Therefore,

$$\left[\chi^{2q}_{f\mu},\|\mu_{mn}(x),(d(x_1),d(x_2),\cdots,d(x_{n-1}))\|_p^{\varphi}\right]^{*} = \left[\Lambda^{2q}_{f\mu},\|\mu_{mn}(x),(d(x_1),d(x_2),\cdots,d(x_{n-1}))\|_p^{\varphi}\right].$$

This completes the proof. □

Theorem 5. *(1) If the sequence (f_{mn}) satisfies uniform Δ_2- condition, then*

$$\left[\chi^{2q}_{f\mu}, \|\mu_{mn}(x), (d(x_1), d(x_2), \cdots, d(x_{n-1}))\|^{\varphi}_p\right]^{\alpha} = \left[\chi^{2q\mu}_{g}, \|\mu_{uv}(x), (d(x_1), d(x_2), \cdots, d(x_{n-1}))\|^{\varphi}_p\right].$$

(2) If the sequence (g_{mn}) satisfies uniform Δ_2- condition, then

$$\left[\chi^{2q\mu}_{g}, \|\mu_{mn}(x), (d(x_1), d(x_2), \cdots, d(x_{n-1}))\|^{\varphi}_p\right]^{\alpha} = \left[\chi^{2q}_{f\mu}, \|\mu_{mn}(x), (d(x_1), d(x_2), \cdots, d(x_{n-1}))\|^{\varphi}_p\right].$$

Proof. Let the sequence (f_{mn}) satisfies uniform Δ_2- condition; we get

$$\left[\chi^{2q\mu}_{g}, \|\mu_{mn}(x), (d(x_1), d(x_2), \cdots, d(x_{n-1}))\|^{\varphi}_p\right] \subset \left[\chi^{2q}_{f\mu}, \|\mu_{mn}(x), (d(x_1), d(x_2), \cdots, d(x_{n-1}))\|^{\varphi}_p\right]^{\alpha}. \quad (9)$$

To prove the inclusion

$$\left[\chi^{2q}_{f\mu}, \|\mu_{mn}(x), (d(x_1), d(x_2), \cdots, d(x_{n-1}))\|^{\varphi}_p\right]^{\alpha} \subset \left[\chi^{2q\mu}_{g}, \|\mu_{mn}(x), (d(x_1), d(x_2), \cdots, d(x_{n-1}))\|^{\varphi}_p\right],$$

let $a \in \left[\chi^{2q}_{f\mu}, \|\mu_{mn}(x), (d(x_1), d(x_2), \cdots, d(x_{n-1}))\|^{\varphi}_p\right]^{\alpha}$. Then, for all $\{x_{mn}\}$ with $(x_{mn}) \in \left[\chi^{2q}_{f\mu}, \|\mu_{mn}(x), (d(x_1), d(x_2), \cdots, d(x_{n-1}))\|^{\varphi}_p\right]$, we have

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty} |x_{mn}a_{mn}| < \infty. \quad (10)$$

Since the sequence (f_{mn}) satisfies the uniform Δ_2- condition and then

$$(y_{mn}) \in \left[\chi^{2q}_{f\mu}, \|\mu_{mn}(x), (d(x_1), d(x_2), \cdots, d(x_{n-1}))\|^{\varphi}_p\right],$$

we get $\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\left|\frac{\varphi_{rs}y_{mn}a_{mn}}{\Delta\lambda_{mn}(m+n)!}\right| < \infty$. by (10). Thus, $(\varphi_{rs}a_{mn}) \in \left[\chi^{2q}_{f\mu}, \|\mu_{mn}(x), (d(x_1), d(x_2), \cdots, d(x_{n-1}))\|^{\varphi}_p\right] = \left[\chi^{2q\mu}_{g}, \|\mu_{mn}(x), (d(x_1), d(x_2), \cdots, d(x_{n-1}))\|^{\varphi}_p\right]$, and hence, $(a_{mn}) \in \left[\chi^{2q\mu}_{g}, \|\mu_{mn}(x), (d(x_1), d(x_2), \cdots, d(x_{n-1}))\|^{\varphi}_p\right]$. This gives that

$$\left[\chi^{2q}_{f\mu}, \|\mu_{mn}(x), (d(x_1), d(x_2), \cdots, d(x_{n-1}))\|^{\varphi}_p\right]^{\alpha} \subset \left[\chi^{2q\mu}_{g}, \|\mu_{mn}(x), (d(x_1), d(x_2), \cdots, d(x_{n-1}))\|^{\varphi}_p\right]. \quad (11)$$

We are granted with (9) and (11) that

$$\left[\chi^{2q}_{f\mu}, \|\mu_{mn}(x), (d(x_1), d(x_2), \cdots, d(x_{n-1}))\|^{\varphi}_p\right]^{\alpha} = \left[\chi^{2q\mu}_{g}, \|\mu_{mn}(x), (d(x_1), d(x_2), \cdots, d(x_{n-1}))\|^{\varphi}_p\right].$$

(3) Similarly, one can prove that

$$\left[\chi^{2q\mu}_{g}, \|\mu_{mn}(x), (d(x_1), d(x_2), \cdots, d(x_{n-1}))\|^{\varphi}_p\right]^{\alpha} \subset \left[\chi^{2q}_{f\mu}, \|\mu_{mn}(x), (d(x_1), d(x_2), \cdots, d(x_{n-1}))\|^{\varphi}_p\right]$$

if the sequence (g_{mn}) satisfies the uniform Δ_2- condition. □

Proposition 1. *If $0 < q_{mn} < p_{mn} < \infty$ for each m and m, then*

$$\left[\Lambda^{2q}_{f\mu}, \|\mu_{mn}(x), (d(x_1), d(x_2), \cdots, d(x_{n-1}))\|^{\varphi}_p\right] \subseteq \left[\Lambda^{2p}_{f\mu}, \|\mu_{mn}(x), (d(x_1), d(x_2), \cdots, d(x_{n-1}))\|^{\varphi}_p\right].$$

Proof. Let $x = (x_{mn}) \in \left[\Lambda^{2q}_{f\mu}, \|\mu_{mn}(x), (d(x_1), d(x_2), \cdots, d(x_{n-1}))\|^{\varphi}_p\right]$. We have

$$sup_{mn}\left[\Lambda^{2q}_{f\mu}, \|\mu_{mn}(x), (d(x_1), d(x_2), \cdots, d(x_{n-1}))\|^{\varphi}_p\right] < \infty.$$

This implies that

$$\left[\Lambda^{2q}_{f\mu}, \|\mu_{mn}(x), (d(x_1), d(x_2), \cdots, d(x_{n-1}))\|^{\varphi}_p\right] < 1$$

for sufficiently large value of m and n. Since f_{mn}s are non-decreasing, we get

$$sup_{mn}\left[\Lambda^{2p}_{f\mu}, \|\mu_{mn}(x), (d(x_1), d(x_2), \cdots, d(x_{n-1}))\|^{\varphi}_p\right] \le sup_{mn}\left[\Lambda^{2q}_{f\mu}, \|\mu_{mn}(x), (d(x_1), d(x_2), \cdots, d(x_{n-1}))\|^{\varphi}_p\right].$$

Thus, $x = (x_{mn}) \in \left[\Lambda^{2p}_{f\mu}, \|\mu_{mn}(x), (d(x_1), d(x_2), \cdots, d(x_{n-1}))\|^{\varphi}_p\right]$. □

Proposition 2. *(1) If $0 < infq_{mn} \le q_{mn} < 1$, then*

$$\left[\Lambda^{2q}_{f\mu}, \|\mu_{mn}(x), (d(x_1), d(x_2), \cdots, d(x_{n-1}))\|^{\varphi}_p\right] \subset \left[\Lambda^{2}_{f\mu}, \|\mu_{mn}(x), (d(x_1), d(x_2), \cdots, d(x_{n-1}))\|^{\varphi}_p\right].$$

(2) If $1 \le q_{mn} \le supq_{mn} < \infty$, then $\left[\Lambda^{2}_{f\mu}, \|\mu_{mn}(x), (d(x_1), d(x_2), \cdots, d(x_{n-1}))\|^{\varphi}_p\right] \subset \left[\Lambda^{2q}_{f\mu}, \|\mu_{mn}(x), (d(x_1), d(x_2), \cdots, d(x_{n-1}))\|^{\varphi}_p\right]$.

Proof. Let $x = (x_{mn}) \in \left[\Lambda^2_{f\mu}, \|\mu_{mn}(x), (d(x_1), d(x_2), \cdots, d(x_{n-1}))\|^{\varphi}_p\right]$. Since $0 < \inf q_{mn} \leq 1$, we have

$$sup_{uv}\left[\Lambda^2_{f\mu}, \|\mu_{mn}(x), (d(x_1), d(x_2), \cdots, d(x_{n-1}))\|^{\varphi}_p\right] \leq \left[\Lambda^{2q}_{f\mu}, \|\mu_{mn}(x), (d(x_1), d(x_2), \cdots, d(x_{n-1}))\|^{\varphi}_p\right],$$

and hence

$$x = (x_{mn}) \in \left[\Lambda^2_{f\mu}, \|\mu_{mn}(x), (d(x_1), d(x_2), \cdots, d(x_{n-1}))\|^{\varphi}_p\right].$$

(3) Let q_{mn} for each (m, n) and $sup_{mn}q_{mn} < \infty$.

Let $x = (x_{mn}) \in \left[\Lambda^2_{f\mu}, \|\mu_{mn}(x), (d(x_1), d(x_2), \cdots, d(x_{n-1}))\|^{\varphi}_p\right]$. Then, for each $0 < \epsilon < 1$, there exists a positive integer $\mathbb{N}$ such that

$$sup_{uv}\left[\Lambda^2_{f\mu}, \|\mu_{mn}(x), (d(x_1), d(x_2), \cdots, d(x_{n-1}))\|^{\varphi}_p\right] \leq \epsilon < 1,$$

for all $m, n \geq N$. This implies that

$$sup_{mn}\left[\Lambda^{2q}_{f\mu}, \|\mu_{mn}(x), (d(x_1), d(x_2), \cdots, d(x_{n-1}))\|^{\varphi}_p\right] \leq sup_{mn}\left[\Lambda^2_{f\mu}, \|\mu_{mn}(x), (d(x_1), d(x_2), \cdots, d(x_{n-1}))\|^{\varphi}_p\right].$$

Thus, $x = (x_{mn}) \in \left[\Lambda^{2q}_{f\mu}, \|\mu_{mn}(x), (d(x_1), d(x_2), \cdots, d(x_{n-1}))\|^{\varphi}_p\right]$. □

Proposition 3. *Let $f' = (f'_{mn})$ and $f'' = (f''_{mn})$ be sequences of Musielak functions; we have*

$$\left[\Lambda^{2q}_{f'\mu}, \|\mu_{mn}(x), (d(x_1), d(x_2), \cdots, d(x_{n-1}))\|^{\varphi}_p\right] \times \bigcap \left[\Lambda^{2q}_{f''\mu}, \|\mu_{mn}(x), (d(x_1), d(x_2), \cdots, d(x_{n-1}))\|^{\varphi}_p\right] \times \subseteq \left[\Lambda^{2q}_{f'+f''\mu}, \|\mu_{mn}(x), (d(x_1), d(x_2), \cdots, d(x_{n-1}))\|^{\varphi}_p\right].$$

Proof. The proof is easy, so we omit it. □

Proposition 4. *For any sequence of Musielak functions $f = (f_{mn})$ and $q = (q_{mn})$ be double analytic sequence of strictly positive real numbers. Then,*

$$\left[\chi^{2q}_{f\mu}, \|\mu_{mn}(x), (d(x_1), d(x_2), \cdots, d(x_{n-1}))\|^{\varphi}_p\right] \subset \left[\Lambda^{2q}_{f\mu}, \|\mu_{mn}(x), (d(x_1), d(x_2), \cdots, d(x_{n-1}))\|^{\varphi}_p\right].$$

Proof. The proof is easy, so we omit it. □

Proposition 5. *The sequence space $\left[\Lambda^{2q}_{f\mu}, \|\mu_{mn}(x), (d(x_1), d(x_2), \cdots, d(x_{n-1}))\|^{\varphi}_p\right]$ is solid.*

Proof. Let $x = (x_{mn}) \in \left[\Lambda^{2q}_{f\mu}, \|\mu_{mn}(x), (d(x_1), d(x_2), \cdots, d(x_{n-1}))\|^{\varphi}_p\right]$, i.e.,

$$sup_{mn}\left[\Lambda^{2q}_{f\mu}, \|\mu_{mn}(x), (d(x_1), d(x_2), \cdots, d(x_{n-1}))\|^{\varphi}_p\right] < \infty.$$

Let (α_{mn}) be double sequence of scalars such that $|\alpha_{mn}| \leq 1$ for all $m, n \in N \times N$. Then, we get

$$sup_{mn}\left[\Lambda^{2q}_{f\mu}, \|\mu_{mn}(\alpha x), (d(x_1), d(x_2), \cdots, d(x_{n-1}))\|^{\varphi}_p\right] \leq sup_{mn}\left[\Lambda^{2q}_{f\mu}, \|\mu_{mn}(x), (d(x_1), d(x_2), \cdots, d(x_{n-1}))\|^{\varphi}_p\right].$$

□

Proposition 6. *The sequence space $\left[\Lambda^{2q}_{f\mu}, \|\mu_{mn}(x), (d(x_1), d(x_2), \cdots, d(x_{n-1}))\|^{\varphi}_p\right]$ is monotone.*

Proof. The proof follows from Proposition 5. □

Proposition 7. *If $f = (f_{mn})$ is any Musielak function, then*

$$\left[\Lambda^{2q}_{f\mu}, \|\mu_{mn}(x), (d(x_1), d(x_2), \cdots, d(x_{n-1}))\|^{\varphi^*}_p\right] \subset \left[\Lambda^{2q}_{f\mu}, \|\mu_{mn}(x), (d(x_1), d(x_2), \cdots, d(x_{n-1}))\|^{\varphi^{**}}_p\right]$$

if and only if $sup_{r,s\geq 1}\frac{\varphi^*_{rs}}{\varphi^{**}_{rs}} < \infty$.

Proof. Let $x \in \left[\Lambda^{2q}_{f\mu}, \|\mu_{mn}(x), (d(x_1), d(x_2), \cdots, d(x_{n-1}))\|^{\varphi^*}_p\right]$ and $N = sup_{r,s\geq 1}\frac{\varphi^*_{rs}}{\varphi^{**}_{rs}} < \infty$. Then, we get

$$\left[\Lambda^{2q}_{f\mu}, \|\mu_{mn}(x), (d(x_1), d(x_2), \cdots, d(x_{n-1}))\|^{\varphi^{**}_{rs}}_p\right] = N\left[\Lambda^{2q}_{f\mu}, \|\mu_{mn}(x), (d(x_1), d(x_2), \cdots, d(x_{n-1}))\|^{\varphi^*_{rs}}_p\right] = 0.$$

Thus, $x \in \left[\Lambda^{2q}_{f\mu}, \|\mu_{mn}(x), (d(x_1), d(x_2), \cdots, d(x_{n-1}))\|^{\varphi^{**}}_p\right]$. Conversely, suppose that

$$\left[\Lambda^{2q}_{f\mu}, \|\mu_{mn}(x), (d(x_1), d(x_2), \cdots, d(x_{n-1}))\|^{\varphi^*}_p\right] \subset \left[\Lambda^{2q}_{f\mu}, \|\mu_{mn}(x), (d(x_1), d(x_2), \cdots, d(x_{n-1}))\|^{\varphi^{**}}_p\right]$$

and $x \in \left[\Lambda^{2q}_{f\mu}, \|\mu_{mn}(x), (d(x_1), d(x_2), \cdots, d(x_{n-1}))\|^{\varphi^*}_p\right]$.

Then, $\left[\Lambda^{2q}_{f\mu}, \|\mu_{mn}(x), (d(x_1), d(x_2), \cdots, d(x_{n-1}))\|^{\varphi^*}_p\right] < \epsilon$ for every $\epsilon > 0$. Suppose that $sup_{r,s\geq 1}\frac{\varphi^*_{rs}}{\varphi^{**}_{rs}} = \infty$, then there exists a sequence of members (rs_{jk}) such that $lim_{j,k\to\infty}\frac{\varphi^*_{jk}}{\varphi^{**}_{jk}} = \infty$. Hence, we have $\left[\Lambda^{2q}_{f\mu}, \|\mu_{mn}(x), (d(x_1), d(x_2), \cdots, d(x_{n-1}))\|^{\varphi^*_{rs}}_p\right] = \infty$. Therefore, $x \notin \left[\Lambda^{2q}_{f\mu}, \|\mu_{mn}(x), (d(x_1), d(x_2), \cdots, d(x_{n-1}))\|^{\varphi^{**}}_p\right]$, which is a contradiction. □

Proposition 8. *If* $f = (f_{mn})$ *is any Musielak function, then*

$$\left[\Lambda^{2q}_{f\mu}, \|\mu_{mn}(x), (d(x_1), d(x_2), \cdots, d(x_{n-1}))\|^{\varphi^*}_p\right] = \left[\Lambda^{2q}_{f\mu}, \|\mu_{mn}(x), (d(x_1), d(x_2), \cdots, d(x_{n-1}))\|^{\varphi^{**}}_p\right]$$

if and only if $sup_{r,s\geq 1}\frac{\varphi^*_{rs}}{\varphi^{**}_{rs}} < \infty, sup_{r,s\geq 1}\frac{\varphi^{**}_{rs}}{\varphi^*_{rs}} > \infty.$

Proof. It is easy to prove, so we omit it. □

Proposition 9. *The sequence space* $\left[\chi^{2q}_{f\mu}, \|\mu_{mn}(x), (d(x_1), d(x_2), \cdots, d(x_{n-1}))\|^{\varphi}_p\right]$ *is not solid.*

Proof. The result follows from the following example. Consider

$$x = (x_{mn}) = \begin{pmatrix} 1 & 1 & \ldots & 1 \\ 1 & 1 & \ldots & 1 \\ \vdots \\ 1 & 1 & \ldots & 1 \end{pmatrix} \in \left[\chi^{2q}_{f\mu}, \|\mu_{mn}(x), (d(x_1), d(x_2), \cdots, d(x_{n-1}))\|^{\varphi}_p\right].$$

Let

$$\alpha_{mn} = \begin{pmatrix} -1^{m+n} & -1^{m+n} & \ldots & -1^{m+n} \\ -1^{m+n} & -1^{m+n} & \ldots & -1^{m+n} \\ \vdots \\ -1^{m+n} & -1^{m+n} & \ldots & -1^{m+n} \end{pmatrix},$$

for all $m, n \in \mathbb{N}$. Then, $\alpha_{mn}x_{mn} \notin \left[\chi^{2q}_{f\mu}, \|\mu_{mn}(x), (d(x_1), d(x_2), \cdots, d(x_{n-1}))\|^{\varphi}_p\right]$. Hence, $\left[\chi^{2q}_{f\mu}, \|\mu_{mn}(x), (d(x_1), d(x_2), \cdots, d(x_{n-1}))\|^{\varphi}_p\right]$ is not solid. □

Proposition 10. *The sequence space* $\left[\chi^{2q}_{f\mu}, \|\mu_{mn}(x), (d(x_1), d(x_2), \cdots, d(x_{n-1}))\|^{\varphi}_p\right]$ *is not monotone.*

Proof. The proof follows from Proposition 9. □

Generalized four-dimensional infinite matrix sequence spaces

Let $A = \left(a^{mn}_{k\ell}\right)$ be a four-dimensional infinite matrix of complex numbers. Then, we have $A(x) = (Ax)_{k\ell} = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} a^{mn}_{k\ell}x_{mn}$ which converges for each k, ℓ.

In this section, we introduce the following sequence spaces:

$$\left[\chi^{2qA}_{f\mu}, \|(d(x_1), d(x_2), \cdots, d(x_{n-1}))\|^{\varphi}_p\right] = lim_{mn}\left\{\left[f_{mn}\left(\|A_{mn}\mu_{mn}(x), (d(x_1), d(x_2), \cdots, d(x_{n-1}))\|_p\right)\right]^{q_{mn}} = 0\right\},$$

$$\left[\Lambda^{2qA}_{f\mu}, \|(d(x_1), d(x_2), \cdots, d(x_{n-1}))\|^{\varphi}_p\right] = sup_{mn}\left\{\left[f_{mn}\left(\|A_{mn}\mu_{mn}(x), (d(x_1), d(x_2), \cdots, d(x_{n-1}))\|_p\right)\right]^{q_{mn}} < \infty\right\}.$$

If we take $f_{mn}(x) = x$, we get

$$\left[\chi^{2qA}_{\mu}, \|(d(x_1), d(x_2), \cdots, d(x_{n-1}))\|^{\varphi}_p\right] = lim_{mn}\left\{\left[\left(\|A_{mn}\mu_{mn}(x), (d(x_1), d(x_2), \cdots, d(x_{n-1}))\|_p\right)\right]^{q_{mn}} = 0\right\},$$

$$\left[\Lambda^{2qA}_{\mu}, \|(d(x_1), d(x_2), \cdots, d(x_{n-1}))\|^{\varphi}_p\right] = sup_{mn}\left\{\left[\left(\|A_{mn}\mu_{mn}(x), (d(x_1), d(x_2), \cdots, d(x_{n-1}))\|_p\right)\right]^{q_{mn}} < \infty\right\}.$$

If we take $q = (q_{mn}) = 1$,

$$\left[\chi^{2A}_{f\mu}, \|(d(x_1), d(x_2), \cdots, d(x_{n-1}))\|^{\varphi}_p\right] = lim_{mn}\left\{\left[f_{mn}\left(\|A_{mn}\mu_{mn}(x), (d(x_1), d(x_2), \cdots, d(x_{n-1}))\|_p\right)\right] = 0\right\},$$

$$\left[\Lambda^{2A}_{f\mu}, \|(d(x_1), d(x_2), \cdots, d(x_{n-1}))\|^{\varphi}_p\right] = sup_{mn}\left\{\left[f_{mn}\left(\|A_{mn}\mu_{mn}(x), (d(x_1), d(x_2), \cdots, d(x_{n-1}))\|_p\right)\right] < \infty\right\}.$$

Theorem 6. *For a Musielak-modulus function,* $f = (f_{mn})$*. Then, the sequence spaces* $\left[\chi^{2qA}_{f\mu}, \|(d(x_1), d(x_2), \cdots, d(x_{n-1}))\|^{\varphi}_p\right]$ *and* $\left[\Lambda^{2qA}_{f\mu}, \|(d(x_1), d(x_2), \cdots, d(x_{n-1}))\|^{\varphi}_p\right]$ *are linear spaces over the set of complex numbers* $\mathbb{C}$*.*

Proof. It is routine verification. Therefore, the proof is omitted. □

Theorem 7. *For any Musielak-modulus function $f = (f_{mn})$ and a double analytic sequence $q = (q_{mn})$ of strictly positive real numbers, the space $\left[\chi^{2qA}_{f\mu}, \|(d(x_1), d(x_2), \cdots, d(x_{n-1}))\|^{\varphi}_p\right]$ is a topological linear space paranormed by*

$$g(x) = inf\left\{\left(\left[f_{mn}\left(\|A_{mn}\mu_{mn}(x), (d(x_1), d(x_2), \cdots, d(x_{n-1}))\|^{\varphi}_p\right)\right]^{q_{mn}}\right)^{1/H} \le 1\right\},$$

where $H = max\,(1, sup_{mn}q_{mn} < \infty)$.

Proof. Clearly, $g(x) \ge 0$ for $x = (x_{mn}) \in \left[\chi^{2qA}_{f\mu}, \|(d(x_1), d(x_2), \cdots, d(x_{n-1}))\|^{V_2}_p\right]$. Since $f_{mn}(0) = 0$, we get $g(0) = 0$. Conversely, suppose that $g(x) = 0$, then $inf\left\{\left(\left[f_{mn}\left(\|A_{mn}\mu_{mn}(x), (d(x_1), d(x_2), \cdots, d(x_{n-1}))\|^{\varphi}_p\right)\right]^{q_{mn}}\right)^{1/H}\right\} \le 1 = 0$. Suppose that $A_{mn}\mu_{mn}(x) \ne 0$ for each $m, n \in \mathbb{N}$, then

$$\|A_{mn}\mu_{mn}(x), (d(x_1), d(x_2), \cdots, d(x_{n-1}))\|^{\varphi}_p \to \infty. \tag{12}$$

It follows that $\left(\left[f_{mn}\left(\|A_{mn}\mu_{mn}(x), (d(x_1), d(x_2), \cdots, d(x_{n-1}))\|^{V_2}_p\right)\right]^{q_{mn}}\right)^{1/H} \to \infty$ which is a contradiction. Therefore, $A_{mn}\mu_{mn}(x) = 0$. Let $\left(\left[f_{mn}\left(\|A_{mn}\mu_{mn}(x), (d(x_1), d(x_2), \cdots, d(x_{n-1}))\|^{\varphi}_p\right)\right]^{q_{mn}}\right)^{1/H} \le 1$ and $\left(\left[f_{mn}\left(\|A_{mn}\mu_{mn}(y), (d(x_1), d(x_2), \cdots, d(x_{n-1}))\|^{\varphi}_p\right)\right]^{q_{mn}}\right)^{1/H} \le 1$.

Then, by using Minkowski's inequality, we have

$$\begin{aligned}&\left(\left[f_{mn}\left(\|A_{mn}\mu_{mn}(x+y), (d(x_1), d(x_2), \cdots, d(x_{n-1}))\|^{\varphi}_p\right)\right]^{q_{mn}}\right)^{1/H}\\ &\le \left(\left[f_{mn}\left(\|A_{mn}\mu_{mn}(x), (d(x_1), d(x_2), \cdots, d(x_{n-1}))\|^{\varphi}_p\right)\right]^{q_{mn}}\right)^{1/H}\\ &+ \left(\left[f_{mn}\left(\|A_{mn}\mu_{mn}(y), (d(x_1), d(x_2), \cdots, d(x_{n-1}))\|^{\varphi}_p\right)\right]^{q_{mn}}\right)^{1/H}.\end{aligned}$$

So, we have

$$\begin{aligned}g(x+y) &= inf\left\{\left(\left[f_{mn}\left(\|A_{mn}\mu_{mn}(x+y), (d(x_1), d(x_2), \cdots, d(x_{n-1}))\|^{\varphi}_p\right)\right]^{q_{mn}}\right)^{1/H} \le 1\right\}\\ &\le inf\left\{\left(\left[f_{mn}\left(\|A_{mn}\mu_{mn}(x), (d(x_1), d(x_2), \cdots, d(x_{n-1}))\|^{\varphi}_p\right)\right]^{q_{mn}}\right)^{1/H} \le 1\right\}\\ &+ inf\left\{\left(\left[f_{mn}\left(\|A_{mn}\mu_{mn}(y), (d(x_1), d(x_2), \cdots, d(x_{n-1}))\|^{\varphi}_p\right)\right]^{q_{mn}}\right)^{1/H} \le 1\right\}.\end{aligned}$$

Therefore,

$$g(x+y) \le g(x) + g(y).$$

Finally, to prove that the scalar multiplication is continuous, let λ be any complex number. By definition,

$$g(\lambda x) = inf\left\{\left(\left[f_{mn}\left(\|A_{mn}\mu_{mn}(\lambda x), (d(x_1), d(x_2), \cdots, d(x_{n-1}))\|^{\varphi}_p\right)\right]^{q_{mn}}\right)^{1/H} \le 1\right\}.$$

Then,

$$g(\lambda x) = inf\left\{((|\lambda|\,t)^{q_{mn}/H} : \left(\left[f_{mn}\left(\|A_{mn}\mu_{mn}(x), (d(x_1), d(x_2), \cdots, d(x_{n-1}))\|^{\varphi}_p\right)\right]^{q_{mn}}\right)^{1/H} \le 1\right\},$$

where $t = \frac{1}{|\lambda|}$. Since $|\lambda|^{q_{mn}} \le max\left(1, |\lambda|^{supp_{mn}}\right)$, we have

$$g(\lambda x) \le max\left(1, |\lambda|^{supp_{mn}}\right) inf\left\{t^{q_{mn}/H} : \left(\left[f_{mn}\left(\|A_{mn}\mu_{mn}(\lambda x), (d(x_1), d(x_2), \cdots, d(x_{n-1}))\|^{\varphi}_p\right)\right]^{q_{mn}}\right)^{1/H} \le 1\right\}.$$

□

Theorem 8. *The $\beta-$ dual space of* $\left[\chi^{2qA}_{f\mu}, \|A_{mn}\mu_{mn}(x), (d(x_1), d(x_2), \cdots, d(x_{n-1}))\|^{\varphi}_p\right]^{\beta} = \left[\Lambda^{2qA}_{f\mu}, \|A_{mn}\mu_{mn}(x), (d(x_1), d(x_2), \cdots, d(x_{n-1}))\|^{\varphi}_p\right]$.

Proof. First, we observe that

$$\left[\chi^{2qA}_{f\mu}, \|A_{mn}\mu_{mn}(x), (d(x_1), d(x_2), \cdots, d(x_{n-1}))\|^{\varphi}_{p}\right]^{\beta} \subset \left[\Gamma^{2qA}_{f\mu}, \|A_{mn}\mu_{mn}(x), (d(x_1), d(x_2), \cdots, d(x_{n-1}))\|^{\varphi}_{p}\right].$$

Therefore,

$$\left[\Gamma^{2qA}_{f\mu}, \|A_{mn}\mu_{mn}(x), (d(x_1), d(x_2), \cdots, d(x_{n-1}))\|^{\varphi}_{p}\right]^{\beta} \subset \left[\chi^{2qA}_{f\mu}, \|A_{mn}\mu_{mn}(x), (d(x_1), d(x_2), \cdots, d(x_{n-1}))\|^{\varphi}_{p}\right]^{\beta}.$$

But

$$\left[\Gamma^{2qA}_{f\mu}\right]^{\beta} \overset{\subset}{\neq} \left[\Lambda^{2qA}_{f\mu}, \|A_{mn}\mu_{mn}(x), (d(x_1), \; d(x_2), \cdots, d(x_{n-1}))\|^{\varphi}_{p}\right].$$

Hence,

$$\left[\Lambda^{2qA}_{f\mu}, \|A_{mn}\mu_{mn}(x), (d(x_1), d(x_2), \cdots, d(x_{n-1}))\|^{\varphi}_{p}\right] \subset \left[\chi^{2qA}_{f\mu}, \|A_{mn}\mu_{mn}(x), (d(x_1), d(x_2), \cdots, d(x_{n-1}))\|^{\varphi}_{p}\right]^{\beta}.$$

Next, we show that

$$\left[\chi^{2qA}_{f\mu}, \|A_{mn}\mu_{mn}(x), (d(x_1), d(x_2), \cdots, d(x_{n-1}))\|^{\varphi}_{p}\right]^{\beta} \subset \left[\Lambda^{2qA}_{f\mu}, \|A_{mn}\mu_{mn}(x), (d(x_1), d(x_2), \cdots, d(x_{n-1}))\|^{\varphi}_{p}\right].$$

Let $y = (y_{mn}) \in \left[\chi^{2qA}_{f\mu}, \|A_{mn}\mu_{mn}(x), (d(x_1), d(x_2), \cdots, d(x_{n-1}))\|^{\varphi}_{p}\right]^{\beta}$. Consider $f(x)=\sum_{m=1}^{\infty}\sum_{n=1}^{\infty} x_{mn}y_{mn}$ with

$$x=(x_{mn}) \in \left[\chi^{2qA}_{f\mu}, \|A_{mn}\mu_{mn}(x), (d(x_1), d(x_2), \cdots, d(x_{n-1}))\|^{\varphi}_{p}\right]$$

$$x = [(\lambda_{mn} - \lambda_{mn+1}) - (\lambda_{m+1n} - \lambda_{m+1n+1})]$$

$$= \begin{pmatrix} 0 & 0 & \cdots 0 & 0 & \cdots 0 \\ 0 & 0 & \cdots 0 & 0 & \cdots 0 \\ \cdots & & & & \\ 0 & 0 & \cdots \frac{\varphi_{rs}}{a^{mn}_{k\ell}\Delta\lambda_{mn}(m+n)!} & \frac{-\varphi_{rs}}{a^{mn}_{k\ell}\Delta\lambda_{mn}(m+n)!} & \dots 0 \\ 0 & 0 & \cdots 0 & 0 & \cdots 0 \end{pmatrix}$$

$$- \begin{pmatrix} 0 & 0 & \cdots 0 & 0 & \cdots 0 \\ 0 & 0 & \cdots 0 & 0 & \cdots 0 \\ \vdots & & & & \\ 0 & 0 & \cdots \frac{\varphi_{rs}}{a^{mn}_{k\ell}\Delta\lambda_{mn}(m+n)!} & \frac{-\varphi_{rs}}{a^{mn}_{k\ell}\Delta\lambda_{mn}(m+n)!} & \dots 0 \\ 0 & 0 & \cdots 0 & 0 & \cdots 0 \end{pmatrix}$$

$$\left[f_{mn}\left(\|A_{mn}\mu_{mn}(x), (d(x_1), d(x_2), \cdots, d(x_{n-1}))\|^{\varphi}_{p}\right)\right]$$

$$= \begin{pmatrix} 0 & 0 & \dots 0 & 0 & \dots 0 \\ 0 & 0 & \dots 0 & 0, & \dots 0 \\ . & & & & \\ . & & & & \\ . & & & & \\ 0 & 0 & \dots f_{mn}\left(\frac{\varphi_{rs}}{a^{mn}_{k\ell}\Delta\lambda_{mn}(m+n)!}\right) & f_{mn}\left(\frac{-\varphi_{rs}}{a^{mn}_{k\ell}\Delta\lambda_{mn}(m+n)!}\right) & \dots 0 \\ 0 & 0 & \dots f_{mn}\left(\frac{-\varphi_{rs}}{a^{mn}_{k\ell}\Delta\lambda_{mn}(m+n)!}\right) & f_{mn}\left(\frac{\varphi_{rs}}{a^{mn}_{k\ell}\Delta\lambda_{mn}(m+n)!}\right) & \dots 0 \\ 0 & 0 & \dots 0 & 0, & \dots 0 \end{pmatrix}.$$

Hence, converges to zero.

Therefore,

$$[(\lambda_{mn} - \lambda_{mn+1}) - (\lambda_{m+1n} - \lambda_{m+1n+1})] \times \in \left[\chi^{2qA}_{f\mu}, \|A_{mn}\mu_{mn}(x), (d(x_1), d(x_2), \cdots, d(x_{n-1}))\|^{\varphi}_{p}\right].$$

Hence, $d\left(a^{mn}_{k\ell}(\lambda_{mn} - \lambda_{mn+1}) - (\lambda_{m+1n} - \lambda_{m+1n+1}), 0\right)=1$. However, $\left|y_{mn}\right| \leq \|f\| d\left(a^{mn}_{k\ell}(\lambda_{mn} - \lambda_{mn+1}) - (\lambda_{m+1n} - \lambda_{m+1n+1}), 0\right) \leq \|f\| \cdot 1 < \infty$ for each m, n. Thus, (y_{mn}) is a p- metric paranormed space of double analytic sequence and, hence, an p- metric double analytic sequence.

In other words, $y \in \left[\Lambda^{2qA}_{f\mu}, \|\mu_{mn}(x), (d(x_1), d(x_2), \cdots, d(x_{n-1}))\|^{\varphi}_{p}\right]$. However, $y = (y_{mn})$ is arbitrary in $\left[\chi^{2q}_{f\mu}, \|A_{mn}\mu_{mn}(x), (d(x_1), d(x_2), \cdots, d(x_{n-1}))\|^{\varphi}_{p}\right]^{\beta}$. Therefore,

$$\left[\chi^{2qA}_{f\mu}, \|A_{mn}\mu_{mn}(x), (d(x_1), d(x_2), \cdots, d(x_{n-1}))\|^{\varphi}_{p}\right]^{\beta} \subset \left[\Lambda^{2qA}_{f\mu}, \|A_{mn}\mu_{mn}(x), (d(x_1), d(x_2), \cdots, d(x_{n-1}))\|^{\varphi}_{p}\right]. \tag{13}$$

From (12) and (13), we get

$$\left[\chi^{2qA}_{f\mu}, \|A_{mn}\mu_{mn}(x), (d(x_1), d(x_2), \cdots, d(x_{n-1}))\|^{\varphi}_{p}\right]^{\beta} = \left[\Lambda^{2qA}_{f\mu}, \|A_{mn}\mu_{mn}(x), (d(x_1), d(x_2), \cdots, d(x_{n-1}))\|^{\varphi}_{p}\right].$$

□

Theorem 9. *The dual space of* $\left[\chi^{2qA}_{f\mu}, \|A_{mn}\mu_{mn}(x), (d(x_1), d(x_2), \cdots, d(x_{n-1}))\|^{\varphi}_{p}\right]$ *is* $\left[\Lambda^{2qA}_{f\mu}, \|A_{mn}\mu_{mn}(x), (d(x_1), d(x_2), \cdots, d(x_{n-1}))\|^{\varphi}_{p}\right]$. *In other words,*

$$\left[\chi^{2qA}_{f\mu}, \|A_{mn}\mu_{mn}(x), (d(x_1), d(x_2), \cdots, d(x_{n-1}))\|^{\varphi}_{p}\right]^{*} = \left[\Lambda^{2qA}_{f\mu}, \|A_{mn}\mu_{mn}(x), (d(x_1), d(x_2), \cdots, d(x_{n-1}))\|^{\varphi}_{p}\right].$$

Proof. We recall that

$$\lambda_{mn} = \begin{pmatrix} 0 & 0 & \cdots 0 & 0 & \cdots \\ 0 & 0 & \cdots 0 & 0 & \cdots \\ \vdots & & & & \\ 0 & 0 & \cdots \frac{\varphi_{rs}}{a^{mn}_{k\ell}\Delta\lambda_{mn}(m+n)!} & 0 & \cdots \\ 0 & 0 & \cdots 0 & 0 & \cdots \end{pmatrix}$$

with $\frac{\varphi_{rs}}{a^{mn}_{k\ell}\Delta\lambda_{mn}(m+n)!}$ in the (m,n)th position and zero elsewhere,

$$\left[\chi^{2qA}_{f\mu}, \|A_{mn}\mu_{mn}(x), (d(x_1), d(x_2), \cdots, d(x_{n-1}))\|^{\varphi}_{p}\right] = \begin{pmatrix} 0. & . & .\ 0 \\ \vdots & & \\ 0 & f\left(\frac{\varphi_{rs}}{a^{mn}_{k\ell}\Delta\lambda_{mn}(m+n)!}\right)^{1/m+n} & .\ 0 \\ & (m,n)^{\text{th}} & \\ 0 & . & .\ 0 \end{pmatrix}$$

which is a p- metric of double gai sequence. Hence,

$$x \in \left[\chi^{2qA}_{f\mu}, \|A_{mn}\mu_{mn}(x), (d(x_1), d(x_2), \cdots, d(x_{n-1}))\|^{\varphi}_{p}\right] f(x) = \sum_{m,n=1}^{\infty} x_{mn}y_{mn}$$

with

$$x \in \left[\chi^{2qA}_{f\mu}, \|A_{mn}\mu_{mn}(x), (d(x_1), d(x_2), \cdots, d(x_{n-1}))\|^{\varphi}_{p}\right]$$

and

$$f \in \left[\chi^{2qA}_{f\mu}, \|A_{mn}\mu_{mn}(x), (d(x_1), d(x_2), \cdots, d(x_{n-1}))\|^{\varphi}_{p}\right]^{*},$$

where

$$\left[\chi^{2qA}_{f\mu}, \|A_{mn}\mu_{mn}(x), (d(x_1), d(x_2), \cdots, d(x_{n-1}))\|^{\varphi}_{p}\right]^{*}$$

is the dual space of $\left[\chi^{2qA}_{f\mu}, \|A_{mn}\mu_{mn}(x), (d(x_1), d(x_2), \cdots, d(x_{n-1}))\|^{\varphi}_{p}\right]$.

Take $x = (x_{mn}) \in \left[\chi^{2qA}_{f\mu}, \|A_{mn}\mu_{mn}(x), (d(x_1), d(x_2), \cdots, d(x_{n-1}))\|^{\varphi}_{p}\right]$. Then,

$$|y_{mn}| \le \|f\|\, d(\varphi_{rs}, 0) < \infty \forall m, n. \tag{14}$$

Thus, (y_{mn}) is a p- metric of double analytic sequence and, hence, an p- metric of double analytic sequence. In other words, $y \in \left[\Lambda^{2qA}_{f\mu}, \|A_{mn}\mu_{mn}(x), (d(x_1), d(x_2), \cdots, d(x_{n-1}))\|^{\varphi}_{p}\right]$. Therefore,

$$\left[\chi^{2qA}_{f\mu}, \|A_{mn}\mu_{mn}(x), (d(x_1), d(x_2), \cdots, d(x_{n-1}))\|^{\varphi}_{p}\right]^{*} = \left[\Lambda^{2q}_{f\mu}, \|A_{mn}\mu_{mn}(x), (d(x_1), d(x_2), \cdots, d(x_{n-1}))\|^{\varphi}_{p}\right].$$

□

Competing interests

The authors declare that they have no competing interests.

Authors' contributions

All authors contributed equally in introducing gai-2 sequence spaces generalized over p-metric defined by Musielak modulus function and in studying topological properties. All authors read and approved the final manuscript.

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