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Approximation properties of *q*-Baskakov-Durrmeyer-Stancu operators

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Abstract

Purpose: The purpose of this paper is to introduce and study the Baskakov- Durrmeyer- Stancu operators based on q-integers.

Methods: First we use property of q-calculus to estimate moments of these operators. Also study some approximation properties, asymptotic formula including q-derivative and point-wise estimation for the operators $\mathcal{L}_{n,q}^{(\alpha,\beta)}$.

Results: We studied better error estimations for these operators using King type approach.

Conclusions: The results proposed here are new and have a better rate of convergence.

Keywords: Baskakov-Durrmeyer-Stancu operators; Point-wise convergent; q-calculus; Asymptotic formula; Rate of convergence

MSC 2000: Primary 41*A*25, 41*A*35.

Introduction

For $f \in C[0,\infty)$, a new type of Baskakov-Durrmeyer operators studied by Finta [1] is defined as

$$\mathcal{L}_n(f,x) = \sum_{k=1}^{\infty} p_{n,k}(x) \int_0^{\infty} b_{n,k}(t) f(t)dt + p_{n,0}(x) f(0), \tag{1}$$

where $p_{n,k}(x) = \binom{n+k-1}{k} \frac{x^k}{(1+x)^{n+k}}$ and $b_{n,k}(x) = \frac{1}{B(k,n+1)} \frac{x^{(k-1)}}{(1+x)^{n+k+1}}$.

Very recently in [2], Gupta introduce q analogue of (1) which is defined as

$$\mathcal{L}_n^q(f,x) = \sum_{k=1}^{\infty} p_{n,k}^q(x) \int_0^{\infty/A} q^{-k} b_{n,k}^q(t) f(t) d_q t + p_{n,0}^q(x) f(0), \tag{2}$$

where $p_{n,k}^q(x) = \binom{n+k-1}{k}_q q^{\frac{k^2}{2}} \frac{(qx)^k}{(1+qx)_q^{n+k}}$ and $b_{n,k}^q(x) = \frac{1}{B_q(k,n+1)} q^{\frac{k^2}{2}} \frac{x^{k-1}}{(1+x)_q^{n+k+1}}$.

Recently, Govil and Gupta [3] studied some approximation properties for the operators defined in (1) and estimated local result in terms of modulus of continuity. Also, further properties like point-wise convergence, asymptotic formula and inverse result in simultaneous approximation have been established in [4].

Starting with two parameters α, β satisfying the condition $0 \leq \alpha \leq \beta$ in 1983, the generalization of Stancu operators was given in [5] and studied the linear positive operators $S_n^{\alpha,\beta} : C[0,1] \to C[0,1]$ defined for any $f \in C[0,1]$ as follows:

$$S_n^{\alpha}(f,x) = \sum_{k=0}^{n} p_{n,k}(x) f\left(\frac{k+\alpha}{n+\beta}\right), 0 \leq x \leq 1. \tag{3}$$

where $p_{n,k}(x) = \binom{n}{k} x^k (1-x)^{n-k}$ is the Bernstein basis function.

In 2010, Ibrahim [6] introduced Stancu-Chlodowsky Polynomials and investigated convergence and approximation properties of these operators. Motivated by such type operators Verma *et al.* [7] introduce the Baskakov-Durrmeyer-Stancu operators $\mathcal{L}_n^{(\alpha,\beta)}$, which is a

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Stancu-type generalization of the Baskakov-Durrmeyer operators (1) as follows:

$$\mathcal{L}_n^{(\alpha,\beta)}(f,x)=\sum_{k=1}^{\infty}p_{n,k}(x)\int_0^{\infty}b_{n,k}(t)f\left(\frac{nt+\alpha}{n+\beta}\right)dt + p_{n,0}(x)f\left(\frac{\alpha}{n+\beta}\right), \tag{4}$$

where the Baskakov and beta basis functions are given in (1).

During last decade, q-calculus was extensively used for constructing various generalizations of many classical approximation operators. In 1987, Lupaş [8] introduced q-Bernstein polynomial whose approximation properties were studied in [9,10]. The recent work on such type of operators can be found in [11-13].

A Lupaş-Phillips-type q-analog of the operators $\mathcal{L}_n^{(\alpha,\beta)}$ is defined in (4) as follows:

$$\mathcal{L}_{n,q}^{(\alpha,\beta)}(f,x)=\sum_{k=1}^{\infty}p_{n,k}^q(x)\int_0^{\infty/A}q^{-k}b_{n,k}^q(t)f\left(\frac{[n]_q\,t+\alpha}{[n]_q+\beta}\right)d_qt + p_{n,0}^q(x)f\left(\frac{\alpha}{[n]_q+\beta}\right), \tag{5}$$

where $p_{n,k}^q(x)$ and $b_{n,k}^q(x)$ are given in (2).

Notice that

$$\int_o^{\infty/A}q^{-k}b_{n,k}^q(t)d_qt=q^{\frac{-k}{2}} \text{ and } \sum_{k=0}^{\infty}p_{n,k}^q(x)q^{\frac{-k}{2}}=1.$$

The aim of this paper is to study the approximation properties of a new generalization of the Baskakov-Durrmeyer operators with two parameter α and β based on q-integers. We estimated moments for these operators and also studied the asymptotic formula based on q-derivative for the operators defined in (5). Finally, we give point-wise estimation and better error estimations for operators $\mathcal{L}_{n,q}^{(\alpha,\beta)}$ using King's approach. First, we recall some definitions and notations of q-calculus. Such notations can be found in [14].

We consider q as a real number satisfying $0<q<1$.

We have

$$[n]_q=\begin{cases}\frac{1-q^n}{1-q}, & q\neq 1,\\ n, & q=1\end{cases}$$

and

$$[n]_q!=\begin{cases}[n]_q[n-1]_q[n-2]_q\dots[1]_q, & n=1,2,\dots,\\ 1, & n=0.\end{cases}$$

Furthermore,

$$(1+x)_q^n=(-x;q)_n =\begin{cases}(1+x)(1+qx)(1+q^2x)\dots(1+q^{n-1}x), & n=1,2,\dots,\\ 1, & n=0.\end{cases}$$

Also, for any real number α, we have

$$(1+x)_q^{\alpha}=\frac{(1+x)_q^{\infty}}{(1+q^{\alpha}x)_q^{\infty}}.$$

In special case when α is a whole number, this definition coincides with the above definition.

The q-binomial coefficients are given by

$$\binom{n}{k}_q=\frac{[n]_q\,!}{[k]_q\,![n-k]_q\,!}, 0\le k\le n.$$

The q-derivative D_qf of a function f is given by

$$D_q(f(x))=\frac{f(x)-f(qx)}{(1-q)x}.$$

The q-Jackson integral and q-improper integral defined as

$$\int_0^a f(x)d_qx=(1-q)a\sum_{n=0}^{\infty}f(aq^n)q^n$$

and

$$\int_0^{\infty/A}f(x)d_qx=(1-q)a\sum_{n=0}^{\infty}f\left(\frac{q^n}{A}\right)\frac{q^n}{A},$$

provided sum converges absolutely.

De Sole and Kac [15] defined q-analog of beta function of second kind $B(t,s)=\int_0^{\infty}\frac{x^{t-1}}{(1+x)^{t+s}}dx$ as follows:

$$B(t,s)=K(A,t)\int_0^{\infty/A}\frac{x^{t-1}}{(1+x)_q^{t+s}}d_qx,$$

where $K(x,t)=\frac{1}{1+x}x^t\left(1+\frac{1}{x}\right)_q^t(1+x)_q^{1-t}$. This function is q-constant in x i.e. $K(qx,t)=K(x,t)$.

In particular, for any positive integer n,

$$K(x,n)=q^{\frac{n(n-1)}{2}},\ K(x,0)=1, \text{ and } B_q(t,s)=\frac{[t-1]_q\,![s-1]_q\,!}{[t+s-1]_q\,!}.$$

Moment Estimation

Remark 1. Applying the product rule for differentiation, we easily obtain the relation

$$q^k\phi^{*2}(x)D_q\left(p_{n,k}^q(x)\right)=\left([k]_q-q^{k+1}[n]_q\,x\right)p_{n,k}^q(qx),$$
$$q^{k-1}\phi^2(x)D_q\left(b_{n,k}^q(x)\right)=\left([k-1]_q-q^{k-1}[n+2]_q\,x\right)b_{n,k}^q(qx),$$

where $\phi^{*2}(x)=x(1+qx)$ and $\phi^2(x)=x(1+x)$.

Lemma 1. *[2] The following hold:*

1. $\mathcal{L}_n^q(1,x) = 1,$
2. $\mathcal{L}_n^q(t,x) = x,$
3. $\mathcal{L}_n^q(t^2,x) = \frac{q[n]_q+1}{q^2[n-1]_q}x^2 + \frac{[2]_q}{q^2[n-1]_q}x,$ *for* $n > 1.$

Lemma 2. *Let us define* $S_{n,m}(x) = \mathcal{L}_n^q(e_m,x),\ e_m = t^m,$ $m = 0,1,2,\ldots$. *Then, we have*

$$S_{n,m+1}(qx) = \frac{\phi^{*2}(x)D_q(S_{n,m}(x)) + \{[m]_q + q[n]_q x\}S_{n,m}(qx)}{q^m[n-m]_q},$$
$$n \geq m+2.$$

From this recurrence relation, we obtain $\mathcal{L}_n^q((t-x)^m,x) = O\left(\frac{1}{[n]_q^{\left[\frac{m+1}{2}\right]}}\right)$, *where* $[\alpha]$ *denotes the integer part of* α.

Proof. Using remark 1, we obtain

$$\begin{aligned}
&\phi^{*2}(x)D_q(S_{n,m}(x))\\
&= \sum_{k=1}^{\infty}\left(\phi^{*2}(x)\right)D_q(p_{n,k}^q(x))\int_0^{\infty/A} q^{-k}b_{n,k}^q(t)t^m d_qt\\
&= \sum_{k=1}^{\infty}\left\{q^{-k}[k]_q - q[n]_q x\right\}p_{n,k}^q(qx)\int_0^{\infty/A} q^{-k}b_{n,k}^q(t)t^m d_qt\\
&= \sum_{k=1}^{\infty}q^{-k}[k]_q p_{n,k}^q(qx)\int_0^{\infty/A} q^{-k}b_{n,k}^q(t)t^m d_qt - q[n]_q x S_{n,m}(qx)\\
&= \sum_{k=1}^{\infty}\left\{q^{-k}[k-1]_q + q^{-1}\right\}p_{n,k}^q(qx)\int_0^{\infty/A} q^{-k}b_{n,k}^q(t)t^m d_qt\\
&\quad - q[n]_q x S_{n,m}(qx)\\
&= \sum_{k=1}^{\infty}q^{-k}[k-1]_q\, p_{n,k}^q(qx)\int_0^{\infty/A} q^{-k}b_{n,k}^q(t)t^m d_qt\\
&\quad + (q^{-1} - q[n]_q x)S_{n,m}(qx)\\
&= q^{-2}\sum_{k=1}^{\infty}p_{n,k}^q(qx)\int_0^{\infty/A}\left\{q^{-k+2}[k-1]_q - [n+2]_q t + [n+2]_q t\right\}\\
&\quad \times q^{-k}b_{n,k}^q(t)t^m d_qt + (q^{-1} - q[n]_q x)S_{n,m}(qx)\\
&= q^{-2}\sum_{k=1}^{\infty}p_{n,k}^q(qx)\int_0^{\infty/A}\left\{q^{-k+2}[k-1]_q - [n+2]_q t\right\}\\
&\quad \times q^{-k}b_{n,k}^q(t)t^m d_qt + q^{-2}[n+2]_q S_{n,m+1}(qx)\\
&\quad + (q^{-1} - q[n]_q x)S_{n,m}(qx).
\end{aligned}$$

To simplify the integral, we make use of the chain rule (which is applicable only for this particular transformation) for the transformation $t = qz$, which gives $d_qt = qd_qz$ (see page 3-4, [14]). Thus, in view of above remark, we get

$$\begin{aligned}
&\phi^{*2}(x)D_q(S_{n,m}(x))\\
&= q^m\sum_{k=1}^{\infty}p_{n,k}^q(qx)\int_0^{\infty/A}\left\{q^{-k+1}[k-1]_q - [n+2]_q z\right\}\\
&\quad \times q^{-k}b_{n,k}^q(qz)z^m d_qz + q^{-2}[n+2]_q S_{n,m+1}(qx)\\
&\quad + (q^{-1} - q[n]_q x)S_{n,m}(qx)\\
&= q^m\sum_{k=1}^{\infty}p_{n,k}^q(qx)\int_0^{\infty/A} q^{-k}(z^{m+1} + z^{m+2})D_q\left(b_{n,k}^q(z)\right)d_qz\\
&\quad + q^{-2}[n+2]_q S_{n,m+1}(qx) + (q^{-1} - q[n]_q x)S_{n,m}(qx)\\
&= I_1 + I_2 + q^{-2}[n+2]_q S_{n,m+1}(qx) + (q^{-1} - q[n]_q x)\\
&\quad \times S_{n,m}(qx).
\end{aligned}$$

In order to obtain I_1 and I_2 we make use of the q integration by parts

$$\int_a^b u(t)D_q(v(t))d_qt = [u(t)v(t)]_a^b - \int_a^b v(qt)D_q(u(t))d_qt.$$

Therefore, we get $I_1 = -q^{-1}[m+1]_q S_{n,m}(qx)$ and $I_2 = -q^{-2}[m+2]_q S_{n,m+1}(qx)$. Combining the expression, we have

$$S_{n,m+1}(qx) = \frac{\phi^{*2}(x)D_q(S_{n,m}(x)) + \{[m]_q + q[n]_q x\}S_{n,m}(qx)}{q^m[n-m]_q}.$$

□

Lemma 3. *The following hold:*

1. $\mathcal{L}_{n,q}^{(\alpha,\beta)}(1,x) = 1$
2. $\mathcal{L}_{n,q}^{(\alpha,\beta)}(t,x) = \frac{[n]_q x + \alpha}{[n]_q + \beta}$
3. $\mathcal{L}_{n,q}^{(\alpha,\beta)}(t^2,x) = \frac{[n]_q^2(q[n]_q+1)}{q^2[n-1]_q([n]_q+\beta)^2}x^2 + \left(\frac{[n]_q^2[2]_q}{q^2[n-1]_q([n]_q+\beta)^2} + \frac{2[n]_q\alpha}{([n]_q+\beta)^2}\right)x + \frac{\alpha^2}{([n]_q+\beta)^2}$, *for* $n > 1$

Proof. Using Lemma 1, for every $n > 1$ and $x \in [0,\infty)$, we have

$$\begin{aligned}
\mathcal{L}_{n,q}^{(\alpha,\beta)}(1,x) &= \mathcal{L}_n^q(1,x) = 1,\\
\mathcal{L}_{n,q}^{(\alpha,\beta)}(t,x) &= \frac{[n]_q}{[n]_q + \beta}\mathcal{L}_n^q(t,x) + \frac{\alpha}{[n]_q + \beta}\mathcal{L}_n^q(1,x)\\
&= \frac{[n]_q x + \alpha}{[n]_q + \beta}.
\end{aligned}$$

Finally,

$$\begin{aligned}\mathcal{L}_{n,q}^{(\alpha,\beta)}(t^2,x) =&\frac{[n]_q^2}{([n]_q+\beta)^2}\mathcal{L}_n^q(t^2,x)+\frac{2[n]_q\,\alpha}{[n]_q+\beta^2}\mathcal{L}_n^q(t,x)\\&+\frac{\alpha^2}{[n]_q+\beta^2}\mathcal{L}_n^q(1,x)\\=&\frac{[n]_q^2\,(q[n]_q+1)}{q^2[n-1]_q\,([n]_q+\beta)^2}x^2\\&+\left(\frac{[n]_q^2\,[2]_q}{q^2[n-1]_q\,([n]_q+\beta)^2}+\frac{2[n]_q\,\alpha}{([n]_q+\beta)^2}\right)x\\&+\frac{\alpha^2}{([n]_q+\beta)^2}.\end{aligned}$$

□

Remark 2. If we put $q = 1$ and $\alpha = \beta = 0$, we get the moments of Baskakov-Durrmeyer operators (1) as $\mathcal{L}_n(t,x) = x$ and $\mathcal{L}_n(t^2,x) = \frac{(n+1)x^2+2x}{n-1}$.

Lemma 4. *If we define the central moments as* $\mu_{n,m}^q(x) = \mathcal{L}_{n,q}^{(\alpha,\beta)}((t-x)^m,x)$, $m \in \mathbf{N}$. *Then*

$$\begin{aligned}\mu_{n,1}^q(x) =&\mathcal{L}_{n,q}^{(\alpha,\beta)}(t-x,x)=\frac{\alpha-\beta x}{[n]_q+\beta},\\ \mu_{n,2}^q(x) =&\mathcal{L}_{n,q}^{(\alpha,\beta)}((t-x)^2,x)\\=&\left(\frac{[n]_q^2\,(q[n]_q+1)}{q^2([n]_q+\beta)^2[n-1]_q}+1-\frac{2[n]_q}{[n]_q+\beta}\right)x^2\\&+\left(\frac{[2]_q\,[n]_q^2}{q^2([n]_q+\beta)^2[n-1]_q}+\frac{2[n]_q\,\alpha}{([n]_q+\beta)^2}-\frac{2\alpha}{[n]_q+\beta}\right)x\\&+\frac{\alpha^2}{([n]_q+\beta)^2}.\end{aligned}$$

Proof. Notice that

$$\begin{aligned}\mu_{n,1}^q(x) =&\mathcal{L}_{n,q}^{(\alpha,\beta)}((t-x),x)\\=&\mathcal{L}_{n,q}^{(\alpha,\beta)}(t,x)-x\mathcal{L}_{n,q}^{(\alpha,\beta)}(1,x)=\frac{[n]_q\,x+\alpha}{[n]_q+\beta}-x\\=&\frac{\alpha-\beta x}{[n]_q+\beta}.\\ \mu_{n,2}^q(x) =&\mathcal{L}_{n,q}^{(\alpha,\beta)}((t-x)^2,x)=\mathcal{L}_{n,q}^{(\alpha,\beta)}(t^2,x)-2x\mathcal{L}_{n,q}^{(\alpha,\beta)}(t,x)\\&+x^2\mathcal{L}_{n,q}^{(\alpha,\beta)}(t,x)\\=&\frac{[n]_q^2\,(q[n]_q+1)}{q^2[n-1]_q\,([n]_q+\beta)^2}x^2\\&+\left(\frac{[n]_q^2\,(1+q)}{q^2[n-1]_q\,([n]_q+\beta)^2}+\frac{2[n]_q\,\alpha}{([n]_q+\beta)^2}\right)x\\&+\frac{\alpha^2}{([n]_q+\beta)^2}-2x\left(\frac{[n]_q\,x+\alpha}{[n]_q+\beta}\right)+x^2\\=&\left(\frac{[n]_q^2\,(q[n]_q+1)}{q^2([n]_q+\beta)^2[n-1]_q}+1-\frac{2[n]_q}{[n]_q+\beta}\right)x^2\\&+\left(\frac{[2]_q\,[n]_q^2}{q^2([n]_q+\beta)^2[n-1]_q}+\frac{2[n]_q\,\alpha}{([n]_q+\beta)^2}-\frac{2\alpha}{[n]_q+\beta}\right)x\\&+\frac{\alpha^2}{([n]_q+\beta)^2}.\end{aligned}$$

□

Remark 3. For all $m \in \mathbf{N}, 0 \le \alpha \le \beta$; we have the following recursive relation for the images of the monomials t^m under $\mathcal{L}_{n,q}^{\alpha,\beta}(t^m,x)$ in terms of $\mathcal{L}_n^q(t^j,x), j = 0,1,2,\ldots,m$ as

$$\mathcal{L}_{n,q}^{(\alpha,\beta)}(t^m,x)=\sum_{j=0}^{m}\binom{m}{j}\frac{[n]_q^j\,\alpha^{m-j}}{([n]_q+\beta)^m}\mathcal{L}_n^q(t^j,x).$$

Also, we have

$$\begin{aligned}\mathcal{L}_{n,q}^{(\alpha,\beta)}((t-x)^m,x)=&\sum_{k=0}^{m}\binom{m}{k}(-x)^{m-k}\mathcal{L}_{n,q}^{\alpha,\beta}(t^k,x).\\=&\sum_{k=0}^{m}\binom{m}{k}(-x)^{m-k}\sum_{j=0}^{k}\binom{k}{j}\frac{[n]_q^j\,\alpha^{k-j}}{([n]_q+\beta)^k}\\&\times\mathcal{L}_n^q(t^j,x).\end{aligned}$$

By Lemma 2 and above equality, we have $\mathcal{L}_{n,q}^{(\alpha,\beta)}((t-x)^m,x) = O\left(\frac{1}{[n]_q^{\left[\frac{m+1}{2}\right]}}\right)$, where $[\alpha]$ denotes the integer part of α.

Direct results

Let the space $C_B[0,\infty)$ of all continuous and bounded functions be endowed with the norm $\|f\| = \sup\{|f(x)| : x \in [0,\infty)\}$. Further let us consider the following K-functional:

$$K_2(f,\delta)=\inf_{g\in W^2}\{\|f-g\|+\delta\|g''\|\}, \tag{6}$$

where $\delta > 0$ and $W^2 = \{g \in C_B[0,\infty) : g', g'' \in C_B[0,\infty)\}$.

By the method as given in p. 177, Theorem 2.4 of [16], there exists an absolute constant $C > 0$ such that

$$K_2(f,\delta)\le C\omega_2(f,\sqrt{\delta}), \tag{7}$$

where

$$\omega_2(f,\sqrt{\delta})=\sup_{0<h<\sqrt{\delta}}\sup_{x\in[0,\infty)}|f(x+2h)-2f(x+h)+f(x)| \tag{8}$$

is the second order modulus of smoothness of $f \in C_B[0,\infty)$. Also, we set

$$\omega(f,\sqrt{\delta})=\sup_{0<h<\sqrt{\delta}}\sup_{x\in[0,\infty)}|f(x+h)-f(x)|. \tag{9}$$

In what follows, we shall use notation $\phi(x) = \sqrt{x(1+x)}$, where $x \in [0,\infty)$.

Theorem 1. *Let* $f \in C_B[0,\infty)$ *and* $0 < q < 1$. *Then for all* $x \in [0,\infty)$, *there exists an absolute constant* $M > 0$ *such that*

$$\left|\mathcal{L}_{n,q}^{\alpha,\beta}(f,x) - f(x)\right| \leq M\omega_2\left(f, \sqrt{\mu_{n,2}^q + (\mu_{n,1}^q)^2}\right) + \omega(f, \mu_{n,1}^q). \tag{10}$$

Proof. Let $g \in W^2$ and $x, t \in [0,\infty)$. By Taylor's expansion, we have

$$g(t) = g(x) + g'(x)(t-x) + \int_x^t (t-u)g''(u)du. \tag{11}$$

Let us define auxiliary operators $\widetilde{\mathcal{L}}_{n,q}^{(\alpha,\beta)}$ as follows

$$\widetilde{\mathcal{L}}_{n,q}^{(\alpha,\beta)}(f,x) = \mathcal{L}_{n,q}^{(\alpha,\beta)}(f,x) + f(x) - f\left(x + \frac{\alpha - \beta x}{[n]_q + \beta}\right). \tag{12}$$

Now, we have $\widetilde{\mathcal{L}}_{n,q}^{(\alpha,\beta)}(t-x,x) = 0$.
Applying $\widetilde{\mathcal{L}}_{n,q}^{\alpha,\beta}$ on both side of (11), we get

$$\begin{aligned}
\widetilde{\mathcal{L}}_{n,q}^{(\alpha,\beta)}(g,x) - g(x) &= g'(x)\widetilde{\mathcal{L}}_{n,q}^{(\alpha,\beta)}((t-x),x) + \widetilde{\mathcal{L}}_{n,q}^{(\alpha,\beta)} \\
&\quad \times \left(\int_x^t (t-u)g''(u)du, x\right) \\
&= \mathcal{L}_{n,q}^{(\alpha,\beta)}\left(\int_x^t (t-u)g''(u)du, x\right) \\
&\quad + \int_x^{x+\frac{\alpha-\beta x}{[n]_q+\beta}} \left(x + \frac{\alpha - \beta x}{[n]_q + \beta} - u\right) g''(u)du.
\end{aligned}$$

On the other hand, we obtain $\left|\int_x^t (x-u)g''(u)du\right| \leq \|g''\|(t-x)^2$ and

$$\begin{aligned}
\int_x^{x+\frac{\alpha-\beta x}{[n]_q+\beta}} \left(x + \frac{\alpha-\beta x}{[n]_q+\beta} - u\right) g''(u)du &\leq \left(x + \frac{\alpha - \beta x}{[n]_q + \beta} - x\right)^2 \|g''\| \\
&= \|g''\| \left(\mathcal{L}_{n,q}^{(\alpha,\beta)}(t-x,x)\right)^2 \\
&= \|g''\| \left(\mu_{n,1}^q(x)\right)^2.
\end{aligned}$$

Notice that,

$$\begin{aligned}
\left|\widetilde{\mathcal{L}}_{n,q}^{(\alpha,\beta)}(g,x) - g(x)\right| &\leq \left|\mathcal{L}_{n,q}^{(\alpha,\beta)}\left(\int_x^t (t-u)g''(u)du, x\right)\right| \\
&\quad + \left|\int_x^{x+\frac{\alpha-\beta x}{[n]_q+\beta}} \left(x + \frac{\alpha-\beta x}{[n]_q+\beta} - u\right) g''(u)du\right| \\
&\leq \|g''\| \mathcal{L}_{n,q}^{(\alpha,\beta)}\left((t-x)^2, x\right) + \|g''\| \left(\mu_{n,1}^q(x)\right)^2 \\
&\leq \|g''\| \left[\mu_{n,2}^q(x) + \left(\mu_{n,1}^q(x)\right)^2\right].
\end{aligned}$$

Observe that

$$\begin{aligned}
\left|\mathcal{L}_{n,q}^{(\alpha,\beta)}(f,x) - f(x)\right| &\leq \left|\widetilde{\mathcal{L}}_{n,q}^{(\alpha,\beta)}(f-g,x) - (f-g)(x)\right| \\
&\quad + \left|\widetilde{\mathcal{L}}_{n,q}^{(\alpha,\beta)}(g,x) - g(x)\right| \\
&\quad + \left|f(x) - f\left(x + \frac{\alpha - \beta x}{[n]_q + \beta}\right)\right| \\
&\leq 4\|f-g\| + \|g''\| \left[\mu_{n,2}^q(x) + \left(\mu_{n,1}^q(x)\right)^2\right] \\
&\quad + \omega(f, \mu_{n,1}^q(x)).
\end{aligned}$$

Now, taking infimum on the right-hand side over all $g \in C_B^2[0,\infty)$ and from (7), we get

$$\begin{aligned}
\left|\mathcal{L}_{n,q}^{(\alpha,\beta)}(f,x) - f(x)\right| &\leq 4K_2\left(f, \mu_{n,2}^q(x) + \left(\mu_{n,1}^q(x)\right)^2\right) \\
&\quad + \omega(f, \mu_{n,1}^q(x)) \\
&\leq M\omega_2\left(f, \sqrt{\mu_{n,2}^q(x) + \left(\mu_{n,1}^q(x)\right)^2}\right) \\
&\quad + \omega(f, \mu_{n,1}^q(x)).
\end{aligned}$$

This complete the proof of Theorem 1. □

Central moments and asymptotic formula

In this section, we observe that it is not possible to estimate recurrence formula $\mathcal{L}_{n,q_n}^{(\alpha,\beta)}$ in q calculus; however, there may be some techniques, but at the moment it can be considered as an open problem. Here we establish the recurrence relation for the central moments and obtain asymptotic formula.

Let $B_{x^2}[0,\infty) = \{f : \text{for every } x \in [0,\infty), |f(x)| \leq M_f(1+x^2)\}$, M_f being a constant depending on f. By $C_{x^2}[0,\infty)$, we denote the subspace of all continuous functions belonging to $B_{x^2}[0,\infty)$. Also, $C_{x^2}^*[0,\infty)$ is subspace of all functions $f \in C_{x^2}[0,\infty)$, for which $\lim_{x\to\infty} \frac{f(x)}{1+x^2}$ is finite. The norm on $C_{x^2}^*[0,\infty)$ is $\|f\|_{x^2} = \sup_{x\in[0,\infty)} \frac{|f(x)|}{1+x^2}$.

Lemma 5. *If we define the central moments as*

$$\begin{aligned}
T_{n,m}(x) &= \mathcal{L}_{n,q_n}^{(\alpha,\beta)}((t-x)_q^m, x) = \sum_{k=1}^{\infty} p_{n,k}^q(x) \int_0^{\infty} q^{-k} b_{n,k}^q(t) \\
&\quad \times \left(\frac{[n]_q t + \alpha}{[n]_q + \beta} - x\right)_q^m d_q t + p_{n,0}^q(x) \left(\frac{\alpha}{[n]_q + \beta} - x\right)_q^m,
\end{aligned}$$

then $\mathcal{L}_{n,q_n}^{(\alpha,\beta)}(1,x) = 1$, $\mathcal{L}_{n,q_n}^{(\alpha,\beta)}(t-x,x) = \frac{\alpha-\beta x}{[n]_q+\beta}$ *and for* $n > m+2$, *we have the following recurrence relation:*

$$\begin{aligned}
&\frac{[n]_q + \beta}{[n]_q}\left(q^{-3}\frac{([n]_q + \beta)[m+2]_q}{[n]_q} - q^{-1}[n+2]_q\right) T_{n,m+1}(qx) \\
&\quad = -x(1+qx)\left[D_q(T_{n,m}(x)) + [m]_q T_{n,m-1}(qx)\right]
\end{aligned}$$

$$+\left[-q^{-3}\frac{([n]_q+\beta)}{[n]_q}\left\{\frac{([n]_q+\beta)}{[n]_q}[3]_q q^m x+q-\frac{2\alpha}{[n]_q}-x\frac{([n]_q+\beta)}{[n]_q}\right\}\right.$$
$$\times[m+1]_q+q^{-1}-q[n]_q x+q^{-1}[n+2]_q$$
$$\left.\times\left\{[2]_q q^m x-x-\frac{\alpha}{[n]_q+\beta}\right\}\right]T_{n,m}(qx)+\left[-\left\{\frac{([n]_q+\beta)}{[n]_q}\right.\right.$$
$$\times\left([2]_q q^m x\left(\frac{([n]_q+\beta)}{[n]_q}[3]_q q^m x+q-\frac{2\alpha}{[n]_q}-x\frac{([n]_q+\beta)}{[n]_q}\right)\right.$$
$$\left.-[3]_q q^{2m+1}x^2\frac{([n]_q+\beta)}{[n]_q}\right)-\left(x\left(q-\frac{2\alpha}{[n]_q}\right)\frac{([n]_q+\beta)}{[n]_q}\right.$$
$$\left.\left.+\frac{\alpha^2+q\alpha[n]_q}{[n]_q^2}\right)\right\}q^{-3}[m]_q+\left(1-q[n]_q x+q^{-1}[n+2]_q\right.$$
$$\left.\left.\times\left(q^m x-\frac{\alpha}{[n]_q+\beta}\right)\right)(q^m-1)x\right]T_{n,m-1}(qx)+\left[\left\{q^m x\right.\right.$$
$$\times\left\{\frac{([n]_q+\beta)}{[n]_q}\left([2]_q q^m x\left(\frac{([n]_q+\beta)}{[n]_q}[3]_q q^m x+q-\frac{2\alpha}{[n]_q}\right.\right.\right.$$
$$\left.\left.-x\frac{([n]_q+\beta)}{[n]_q}\right)-[3]_q q^{2m+1}x^2\frac{([n]_q+\beta)}{[n]_q}\right)$$
$$\left.\left.-\left(x\left(q-\frac{2\alpha}{[n]_q}\right)\frac{([n]_q+\beta)}{[n]_q}+\frac{\alpha^2+q\alpha[n]_q}{[n]_q^2}\right)\right\}\right\}$$
$$-\left\{q^{2m+1}x^2\frac{([n]_q+\beta)}{[n]_q}\left(q^{m+2}x\frac{([n]_q+\beta)}{[n]_q}+q-\frac{2\alpha}{[n]_q}-x\frac{([n]_q+\beta)}{[n]_q}\right)\right.$$
$$\left.\left.-x\frac{\alpha^2+q\alpha[n]_q}{[n]_q^2}\right\}\right\}q^{-3}[m-1]_q\Bigg]T_{q,m-2}(qx)$$
$$+(1+q^2x)_q^{-n}\left\{\frac{[n]_q+\beta}{[n]_q}\left(q^{-3}\frac{([n]_q+\beta)[m+2]_q}{[n]_q}-q^{-1}[n+2]\right)\right.$$
$$\times\left(\frac{\alpha}{[n]_q+\beta}-qx\right)_q^{m+1}+\left[q^{-3}\frac{([n]_q+\beta)}{[n]_q}\left\{\frac{([n]_q+\beta)}{[n]_q}[3]_q q^m x\right.\right.$$
$$\left.+q-\frac{2\alpha}{[n]_q}-x\frac{([n]_q+\beta)}{[n]_q}\right\}[m+1]_q-q^{-1}-q^{-1}[n+2]_q$$
$$\left.\times\left\{[2]_q q^m x-x-\frac{\alpha}{[n]_q+\beta}\right\}\right]\left(\frac{\alpha}{[n]_q+\beta}-qx\right)_q^m$$
$$+\left[\left\{\frac{([n]_q+\beta)}{[n]_q}\left([2]_q q^m x\left(\frac{([n]_q+\beta)}{[n]_q}[3]_q q^m x+q-\frac{2\alpha}{[n]_q}\right.\right.\right.\right.$$
$$\left.\left.-x\frac{([n]_q+\beta)}{[n]_q}\right)-[3]_q q^{2m+1}x^2\frac{([n]_q+\beta)}{[n]_q}\right)-\left(x\left(q-\frac{2\alpha}{[n]_q}\right)\right.$$
$$\left.\left.\times\frac{([n]_q+\beta)}{[n]_q}+\frac{\alpha^2+q\alpha[n]_q}{[n]_q^2}\right)\right\}q^{-3}[m]_q-\left(1+q^{-1}[n+2]_q\right.$$
$$\left.\left.\times\left(q^m x\frac{\alpha}{[n]_q+\beta}\right)\right)(q^m-1)x\right]\left(\frac{\alpha}{[n]_q+\beta}-qx\right)_q^{m-1}$$
$$+\left[\left\{q^m x\left\{\frac{([n]_q+\beta)}{[n]_q}\left([2]_q q^m x\right.\right.\right.\right.$$
$$\times\left(\frac{([n]_q+\beta)}{[n]_q}[3]_q q^m x+q-\frac{2\alpha}{[n]_q}-x\frac{([n]_q+\beta)}{[n]_q}\right)$$
$$\left.-[3]_q q^{2m+1}x^2\frac{([n]_q+\beta)}{[n]_q}\right)-\left(x\left(q-\frac{2\alpha}{[n]_q}\right)\frac{([n]_q+\beta)}{[n]_q}\right.$$
$$\left.\left.+\frac{\alpha^2+q\alpha[n]_q}{[n]_q^2}\right)\right\}-\left\{q^{2m+1}x^2\frac{([n]_q+\beta)}{[n]_q}\left(q^{m+2}x\frac{([n]_q+\beta)}{[n]_q}\right.\right.$$
$$\left.\left.\left.+q-\frac{2\alpha}{[n]_q}-x\frac{([n]_q+\beta)}{[n]_q}\right)-x\frac{\alpha^2+q\alpha[n]_q}{[n]_q^2}\right\}\right\}$$
$$\left.\times q^{-3}[m-1]_q\right]\left(\frac{\alpha}{[n]_q+\beta}-qx\right)_q^{m-2}\Bigg\}.$$

Proof. Applying q derivatives of product rule, we have

$$D_q(T_{n,m}(x))=-[m]_q\sum_{k=1}^{\infty}p_{n,k}^q(qx)\int_0^{\infty}q^{-k}b_{n,k}^q(t)$$
$$\times\left(\frac{[n]_q t+\alpha}{[n]_q+\beta}-qx\right)_q^{m-1}d_qt-[m]_q p_{n,0}^q(qx)$$
$$\times\left(\frac{\alpha}{[n]_q+\beta}-qx\right)_q^{m-1}$$
$$+\sum_{k=1}^{\infty}D_q(p_{n,k}^q(x))\int_0^{\infty}q^{-k}b_{n,k}^q(t)\left(\frac{[n]_q t+\alpha}{[n]_q+\beta}-x\right)_q^m$$
$$\times d_t+D_q(p_{n,0}^q(x))\left(\frac{\alpha}{[n]_q+\beta}-x\right)_q^m.$$

Using the identity

$$q^k x(1+qx)D_q\left(p_{n,k}^q(x)\right)=\left([k]_q-q^{k+1}[n]_q x\right)p_{n,k}^q(qx),$$

$$x(1+qx)\left[D_q(T_{n,m}(x))+[m]_q T_{n,m-1}(qx)\right]$$
$$=\sum_{k=1}^{\infty}x(1+qx)D_q(p_{n,k}^q(x))\int_0^{\infty}q^{-k}b_{n,k}^q(t)$$
$$\times\left(\frac{[n]_q t+\alpha}{[n]_q+\beta}-x\right)_q^m dt+x(1+qx)D_q(p_{n,0}^q(x))$$
$$\times\left(\frac{\alpha}{[n]_q+\beta}-x\right)_q^m$$
$$=\sum_{k=1}^{\infty}\left(q^{-k}[k]_q-q[n]_q x\right)p_{n,k}^q(qx)\int_0^{\infty}q^{-k}b_{n,k}^q(t)$$
$$\times\left(\frac{[n]_q t+\alpha}{[n]_q+\beta}-x\right)_q^m dt+\left(-q[n]_q x\right)p_{n,0}^q(qx)$$
$$\times\left(\frac{\alpha}{[n]_q+\beta}-x\right)_q^m,$$

$$=\left[\sum_{k=1}^{\infty} q^{-k}[k]_q\, p_{n,k}^q(qx)\int_0^{\infty} q^{-k} b_{n,k}^q(t)\left(\frac{[n]_q\, t+\alpha}{[n]_q+\beta}-x\right)_q^m d_qt\right]$$
$$-\, q[n]_q\, x\left[\sum_{k=1}^{\infty} p_{n,k}^q(qx)\int_0^{\infty} q^{-k} b_{n,k}^q(t)\left(\frac{[n]_q\, t+\alpha}{[n]_q+\beta}-x\right)_q^m\right.$$
$$\left.\times\, d_qt + p_{n,0}^q(qx)\left(\frac{\alpha}{[n]_q+\beta}-x\right)_q^m\right],$$
$$:= I_1 - q[n]_q\, x I_2. \tag{13}$$

We can write I_1 as

$$I_1 = \sum_{k=1}^{\infty} q^{-k}\left([k-1]_q + q^{k-1}\right) p_{n,k}^q(qx)\int_0^{\infty} q^{-k} b_{n,k}^q(t)$$
$$\times\left(\frac{[n]_q\, t+\alpha}{[n]_q+\beta}-x\right)_q^m d_qt$$
$$= \sum_{k=1}^{\infty} q^{-k} p_{n,k}^q(qx)\int_0^{\infty} q^{-k}\left([k-1]_q - q^{k-1}[n+2]_q\, t\right.$$
$$\left. + q^{k-1}[n+2]_q t + q^{k-1}\right) b_{n,k}^q(t)\left(\frac{[n]_q\, t+\alpha}{[n]_q+\beta}-x\right)_q^m d_qt$$
$$= \sum_{k=1}^{\infty} q^{-k} p_{n,k}^q(qx)\int_0^{\infty} q^{-k}\left([k-1]_q - q^{k-1}[n+2]_q\, t\right)$$
$$\times\, b_{n,k}^q(t)\left(\frac{[n]_q\, t+\alpha}{[n]_q+\beta}-x\right)_q^m d_qt$$
$$+ \sum_{k=1}^{\infty} q^{-1} p_{n,k}^q(qx)\int_0^{\infty} q^{-k}\left([n+2]_q\, t+1\right) b_{n,k}^q(t)$$
$$\times\left(\frac{[n]_q\, t+\alpha}{[n]_q+\beta}-x\right)_q^m d_qt.$$

Now using the identity

$$q^{k-1}x(x+1)D_q\left(b_{n,k}^q(x)\right)$$
$$= \left([k-1]_q - q^{k-1}[n+2]_q\, x\right) b_{n,k}^q(qx),$$

we have

$$I_1 = \sum_{k=1}^{\infty} q^{-k} p_{n,k}^q(qx)\int_0^{\infty} q^{-k} q^{k-1}\frac{t}{q}\left(1+\frac{t}{q}\right)$$
$$\times\, D_q\left(b_{n,k}^q\left(\frac{t}{q}\right)\right)\left(\frac{[n]_q\, t+\alpha}{[n]_q+\beta}-x\right)_q^m d_qt$$
$$+ \sum_{k=1}^{\infty} q^{-1} p_{n,k}^q(qx)\int_0^{\infty} q^{-k}\left([n+2]_q\, t+1\right) b_{n,k}^q(t)$$
$$\times\left(\frac{[n]_q\, t+\alpha}{[n]_q+\beta}-x\right)_q^m d_qt$$

$$= \sum_{k=1}^{\infty} q^{-3} p_{n,k}^q(qx)\int_0^{\infty} q^{-k}\left(tq+t^2\right) D_q\left(b_{n,k}^q\left(\frac{t}{q}\right)\right)$$
$$\times\left(\frac{[n]_q\, t+\alpha}{[n]_q+\beta}-x\right)_q^m d_qt + [n+2]_q\sum_{k=1}^{\infty} q^{-1} p_{n,k}^q(qx)$$
$$\times\int_0^{\infty} q^{-k} t\, b_{n,k}^q(t)\left(\frac{[n]_q\, t+\alpha}{[n]_q+\beta}-x\right)_q^m d_qt$$
$$+ \sum_{k=1}^{\infty} q^{-1} p_{n,k}^q(qx)\int_0^{\infty} q^{-k} b_{n,k}^q(t)\left(\frac{[n]_q\, t+\alpha}{[n]_q+\beta}-x\right)_q^m d_qt$$
$$= I_3 + I_4 + I_5. \tag{14}$$

Notice that

$$\left(\frac{[n]_q\, t+\alpha}{[n]_q+\beta}-q^m x\right)\left(\frac{[n]_q\, t+\alpha}{[n]_q+\beta}-q^{m+1}x\right)$$
$$= \left(\frac{[n]_q\, t+\alpha}{[n]_q+\beta}\right)^2 - [2]_q\, q^m x\left(\frac{[n]_q\, t+\alpha}{[n]_q+\beta}\right) + q^{2m+1}x^2$$

and

$$\left(\frac{[n]_q\, t+\alpha}{[n]_q+\beta}-q^m x\right)\left(\frac{[n]_q\, t+\alpha}{[n]_q+\beta}-q^{m+1}x\right)$$
$$\times\left(\frac{[n]_q\, t+\alpha}{[n]_q+\beta}-q^{m+2}x\right)$$
$$= \left(\frac{[n]_q\, t+\alpha}{[n]_q+\beta}\right)^3 - [3]_q\, q^m x\left(\frac{[n]_q\, t+\alpha}{[n]_q+\beta}\right)^2$$
$$+ [3]_q\, q^{2m+1}x^2\left(\frac{[n]_q\, t+\alpha}{[n]_q+\beta}\right) - q^{3m+3}x^3.$$

We obtain the following identity after some computation

$$(tq+t^2)\left(\frac{[n]_q\, t+\alpha}{[n]_q+\beta}-x\right)_q^m$$
$$= (tq+t^2)\left(\frac{[n]_q\, t+\alpha}{[n]_q+\beta}-x\right)\left(\frac{[n]_q\, t+\alpha}{[n]_q+\beta}-qx\right)_q^{m-1}$$
$$= \left(\frac{([n]_q+\beta)^2}{[n]_q^2}\left(\frac{[n]_q\, t+\alpha}{[n]_q+\beta}\right)^2 + \left(q-\frac{2\alpha}{[n]_q}\right)\right.$$
$$\left.\times\,\frac{([n]_q+\beta)}{[n]_q}\left(\frac{[n]_q\, t+\alpha}{[n]_q+\beta}\right) - \frac{\alpha^2+q\alpha[n]_q}{[n]_q^2}\right)$$
$$\times\left(\frac{[n]_q\, t+\alpha}{[n]_q+\beta}-x\right)\left(\frac{[n]_q\, t+\alpha}{[n]_q+\beta}-qx\right)_q^{m-1}$$
$$= \left\{\frac{([n]_q+\beta)^2}{[n]_q^2}\left(\frac{[n]_q\, t+\alpha}{[n]_q+\beta}\right)^3 + \left(q-\frac{2\alpha}{[n]_q}\right.\right.$$
$$\left. -\, x\frac{([n]_q+\beta)}{[n]_q}\right)\frac{([n]_q+\beta)}{[n]_q}\left(\frac{[n]_q\, t+\alpha}{[n]_q+\beta}\right)^2$$

$$
\begin{aligned}
&-\left(x\left(q-\frac{2\alpha}{[n]_q}\right)\frac{([n]_q+\beta)}{[n]_q}+\frac{\alpha^2+q\alpha[n]_q}{[n]_q^2}\right)\\
&\times\left(\frac{[n]_q\,t+\alpha}{[n]_q+\beta}\right)+x\frac{\alpha^2+q\alpha[n]_q}{[n]_q^2}\Bigg\}\left(\frac{[n]_q\,t+\alpha}{[n]_q+\beta}-qx\right)_q^{m-1}\\
=&\frac{([n]_q+\beta)^2}{[n]_q^2}\left(\frac{[n]_q\,t+\alpha}{[n]_q+\beta}-qx\right)_q^{m+2}\\
&+\frac{([n]_q+\beta)}{[n]_q}\left\{\frac{([n]_q+\beta)}{[n]_q}[3]_q\,q^m x+q-\frac{2\alpha}{[n]_q}\right.\\
&\left.-x\frac{([n]_q+\beta)}{[n]_q}\right\}\left(\frac{[n]_q\,t+\alpha}{[n]_q+\beta}-qx\right)_q^{m+1}\\
&+\left\{\frac{([n]_q+\beta)}{[n]_q}\left([2]_q\,q^m x\left(\frac{([n]_q+\beta)}{[n]_q}[3]_q\,q^m x+q\right.\right.\right.\\
&\left.\left.-\frac{2\alpha}{[n]_q}-x\frac{([n]_q+\beta)}{[n]_q}\right)-[3]_q\,q^{2m+1}x^2\frac{([n]_q+\beta)}{[n]_q}\right)\\
&\left.-\left(x\left(q-\frac{2\alpha}{[n]_q}\right)\frac{([n]_q+\beta)}{[n]_q}+\frac{\alpha^2+q\alpha[n]_q}{[n]_q^2}\right)\right\}\\
&\times\left(\frac{[n]_q\,t+\alpha}{[n]_q+\beta}-qx\right)_q^{m}\\
&+\left[q^m x\left\{\frac{([n]_q+\beta)}{[n]_q}\left([2]_q\,q^m x\left(\frac{([n]_q+\beta)}{[n]_q}[3]_q\,q^m x\right.\right.\right.\right.\\
&\left.\left.+q-\frac{2\alpha}{[n]_q}-x\frac{([n]_q+\beta)}{[n]_q}\right)-[3]_q\,q^{2m+1}x^2\frac{([n]_q+\beta)}{[n]_q}\right)\\
&\left.-\left(x\left(q-\frac{2\alpha}{[n]_q}\right)\frac{([n]_q+\beta)}{[n]_q}+\frac{\alpha^2+q\alpha[n]_q}{[n]_q^2}\right)\right\}\\
&-\left\{q^{2m+1}x^2\frac{([n]_q+\beta)}{[n]_q}\left(q^{m+2}x\frac{([n]_q+\beta)}{[n]_q}+q-\frac{2\alpha}{[n]_q}\right.\right.\\
&\left.\left.\left.-x\frac{([n]_q+\beta)}{[n]_q}\right)-x\frac{\alpha^2+q\alpha[n]_q}{[n]_q^2}\right\}\right]\\
&\times\left(\frac{[n]_q\,t+\alpha}{[n]_q+\beta}-qx\right)_q^{m-1}.
\end{aligned}
$$

Using the above identity and q integral by parts

$$\int_a^b u(t)D_q[\,v(t)]\,d_qt=[\,u(t)v(t)]_a^b-\int_a^b v(qt)D_q[\,u(t)]\,d_qt,$$

we obtain

$$
\begin{aligned}
q^3I_3=&-[\,m+2]_q\,\frac{([n]_q+\beta)^2}{[n]_q^2}\left(T_{n,m+1}(qx)-(1+q^2x)_q^{-n}\right.\\
&\left.\times\left(\frac{\alpha}{[n]_q+\beta}-qx\right)_q^{m+1}\right)-\frac{([n]_q+\beta)}{[n]_q}\\
&\times\left\{\frac{([n]_q+\beta)}{[n]_q}[3]_q q^m x+q-\frac{2\alpha}{[n]_q}-x\frac{([n]_q+\beta)}{[n]_q}\right\}\\
&\times[m+1]_q\left(T_{n,m}(qx)-(1+q^2x)_q^{-n}\left(\frac{\alpha}{[n]_q+\beta}-qx\right)_q^{m}\right)\\
&-\left\{\frac{([n]_q+\beta)}{[n]_q}\left([2]_q\,q^m x\left(\frac{([n]_q+\beta)}{[n]_q}[3]_q\,q^m x+q-\frac{2\alpha}{[n]_q}\right.\right.\right.\\
&\left.\left.-x\frac{([n]_q+\beta)}{[n]_q}\right)-[3]_q\,q^{2m+1}x^2\frac{([n]_q+\beta)}{[n]_q}\right)\\
&\left.-\left(x\left(q-\frac{2\alpha}{[n]_q}\right)\frac{([n]_q+\beta)}{[n]_q}+\frac{\alpha^2+q\alpha[n]_q}{[n]_q^2}\right)\right\}[\,m]_q\\
&\times\left(T_{n,m-1}(qx)-(1+q^2x)_q^{-n}\left(\frac{\alpha}{[n]_q+\beta}-qx\right)_q^{m-1}\right)\\
&-\left[q^m x\left\{\frac{([n]_q+\beta)}{[n]_q}\left([2]_q\,q^m x\left(\frac{([n]_q+\beta)}{[n]_q}[3]_q\,q^m x+q\right.\right.\right.\right.\\
&\left.\left.-\frac{2\alpha}{[n]_q}-x\frac{([n]_q+\beta)}{[n]_q}\right)-[3]_q\,q^{2m+1}x^2\frac{([n]_q+\beta)}{[n]_q}\right)\\
&\left.-\left(x\left(q-\frac{2\alpha}{[n]_q}\right)\frac{([n]_q+\beta)}{[n]_q}+\frac{\alpha^2+q\alpha[n]_q}{[n]_q^2}\right)\right\}\\
&-\left\{q^{2m+1}x^2\frac{([n]_q+\beta)}{[n]_q}\left(q^{m+2}x\frac{([n]_q+\beta)}{[n]_q}+q-\frac{2\alpha}{[n]_q}\right.\right.\\
&\left.\left.\left.-x\frac{([n]_q+\beta)}{[n]_q}\right)-x\frac{\alpha^2+q\alpha[n]_q}{[n]_q^2}\right\}\right][m-1]_q\\
&\times\left(T_{n,m-2}(qx)-(1+q^2x)_q^{-n}\left(\frac{\alpha}{[n]_q+\beta}-qx\right)_q^{m-2}\right).
\end{aligned}
\tag{15}
$$

Finally, using

$$
\begin{aligned}
&\left(\frac{[n]_q\,t+\alpha}{[n]_q+\beta}-x\right)_q^{m}\\
&\qquad=\left(\frac{[n]_q\,t+\alpha}{[n]_q+\beta}-x\right)\left(\frac{[n]_q\,t+\alpha}{[n]_q+\beta}-qx\right)_q^{m-1}\\
&\qquad=\left(\frac{[n]_q\,t+\alpha}{[n]_q+\beta}-q^m x+q^m x-x\right)\\
&\qquad\quad\times\left(\frac{[n]_q\,t+\alpha}{[n]_q+\beta}-qx\right)_q^{m-1}\\
&\qquad=\left(\frac{[n]_q\,t+\alpha}{[n]_q+\beta}-qx\right)_q^{m}+(q^m-1)x\\
&\qquad\quad\times\left(\frac{[n]_q\,t+\alpha}{[n]_q+\beta}-qx\right)_q^{m-1},
\end{aligned}
$$

we get

$$\begin{aligned}I_5 =&q^{-1}\sum_{k=1}^{\infty}p_{n,k}^q(qx)\int_0^{\infty}q^{-k}b_{n,k}^q(t)\left(\frac{[n]_q\,t+\alpha}{[n]_q+\beta}-x\right)_q^m d_qt\\ =&q^{-1}\left\{T_{n,m}(qx)-(1+q^2x)_q^{-n}\left(\frac{\alpha}{[n]_q+\beta}-qx\right)_q^m\right.\\ &+(q^m-1)x\left(T_{n,m-1}(qx)-(1+q^2x)_q^{-n}\right.\\ &\left.\left.\times\left(\frac{\alpha}{[n]_q+\beta}-qx\right)_q^{m-1}\right)\right\}.\end{aligned} \tag{16}$$

Also,

$$-q[n]_q\,xI_2=-q[n]_q\,x\left\{T_{n,m}(qx)+(q^m-1)x\left(T_{n,m-1}(qx)\right)\right\}.$$

To estimate I_4 we use

$$\begin{aligned}&t\left(\frac{[n]_q\,t+\alpha}{[n]_q+\beta}-x\right)_q^m\\ &=\frac{[n]_q+\beta}{[n]_q}\left\{\frac{[n]_q\,t+\alpha}{[n]_q+\beta}-q^mx+q^mx-\frac{\alpha}{[n]_q+\beta}\right\}\\ &\quad\times\left(\frac{[n]_q\,t+\alpha}{[n]_q+\beta}-x\right)_q^m\\ &=\frac{[n]_q+\beta}{[n]_q}\left\{\left(\frac{[n]_q\,t+\alpha}{[n]_q+\beta}-x\right)_q^{m+1}\right.\\ &\quad\left.+\left(q^mx-\frac{\alpha}{[n]_q+\beta}\right)\left(\frac{[n]_q\,t+\alpha}{[n]_q+\beta}-x\right)_q^m\right\}\\ &=\frac{[n]_q+\beta}{[n]_q}\left\{\left(\frac{[n]_q\,t+\alpha}{[n]_q+\beta}-qx\right)_q^{m+1}\right.\\ &\quad+(q^{m+1}-1)x\left(\frac{[n]_q\,t+\alpha}{[n]_q+\beta}-qx\right)_q^m\\ &\quad+\left(q^mx-\frac{\alpha}{[n]_q+\beta}\right)\left[\left(\frac{[n]_q\,t+\alpha}{[n]_q+\beta}-qx\right)_q^m\right.\\ &\quad\left.\left.+\ (q^m-1)x\left(\frac{[n]_q\,t+\alpha}{[n]_q+\beta}-qx\right)_q^{m-1}\right]\right\}\\ &=\frac{[n]_q+\beta}{[n]_q}\left\{\left(\frac{[n]_q\,t+\alpha}{[n]_q+\beta}-qx\right)_q^{m+1}\right.\\ &\quad+\left([2]_q\,q^mx-x-\frac{\alpha}{[n]_q+\beta}\right)\left(\frac{[n]_q\,t+\alpha}{[n]_q+\beta}-qx\right)_q^m\\ &\quad+\left(q^mx-\frac{\alpha}{[n]_q+\beta}\right)(q^m-1)x\\ &\quad\left.\times\left(\frac{[n]_q\,t+\alpha}{[n]_q+\beta}-qx\right)_q^{m-1}\right\}.\end{aligned}$$

Therefore,

$$\begin{aligned}I_4 =&q^{-1}[\,n+2]_q\left\{\frac{[n]_q+\beta}{[n]_q}\left(T_{n,m+1}(qx)-(1+q^2x)_q^{-n}\right.\right.\\ &\left.\times\left(\frac{\alpha}{[n]_q+\beta}-qx\right)_q^{m+1}\right)+\left([2]_q\,q^mx-x-\frac{\alpha}{[n]_q+\beta}\right)\\ &\times\left(T_{n,m}(qx)-(1+q^2x)_q^{-n}\left(\frac{\alpha}{[n]_q+\beta}-qx\right)_q^m\right)\\ &+\left(q^mx-\frac{\alpha}{[n]_q+\beta}\right)(q^m-1)x\left(T_{n,m-1}(qx)\right.\\ &\left.\left.-\ (1+q^2x)_q^{-n}\left(\frac{\alpha}{[n]_q+\beta}-qx\right)_q^{m-1}\right)\right\}.\end{aligned} \tag{17}$$

Combining (13)−(17), we get required result. □

Theorem 2. *Let $f\in C[0,\infty)$ be a bounded function and (q_n) denote a sequence such that $0<q_n<1$ and $q_n\to 1$ as $n\to\infty$. Then we have for a point $x\in(0,\infty)$*

$$\begin{aligned}&\lim_{n\to\infty}[n]_q\left(\mathcal{L}_{n,q_n}^{(\alpha,\beta)}(f,x)-f(x)\right)\\ &=(\alpha-\beta x)D_{q_n}f(x)+x(1+x)D_{q_n}^2f(x).\end{aligned}$$

Proof. Using q-Taylor's expansion [15] of f, we can write

$$f(t)-f(x)=(t-x)D_qf(x)+\frac{(t-x)_q^2}{[2]_q}D_q^2f(x)+\Phi_q(x,t)(t-x)_q^2,$$

for $0<q<1$, where

$$\Phi_q(x,t)=\begin{cases}\dfrac{f(t)-f(x)-D_qf(x)(t-x)-\frac{1}{[2]_q}D_q^2f(x)(t-x)_q^2}{(t-x)_q^2}, & \text{if } x\neq t,\\ 0, & \text{if } x=t.\end{cases} \tag{18}$$

We know that for n large enough

$$\lim_{t\to x}\Phi_{q_n}(x,t)=0, \tag{19}$$

that is for any $\epsilon>0$, there exists a $\delta>0$ such that

$$|\Phi_{q_n}(x,t)|<\epsilon \tag{20}$$

for $|t-x|<\delta$ and n sufficiently large. Using (18), we can write

$$\begin{aligned}\mathcal{L}_{n,q_n}^{(\alpha,\beta)}(f,x)-f(x)=&D_{q_n}f(x)\mathcal{L}_{n,q_n}^{(\alpha,\beta)}((t-x)_{q_n},x)\\ &+\frac{D_{q_n}^2}{[2]_q}\mathcal{L}_{n,q_n}^{(\alpha,\beta)}((t-x)_{q_n}^2,x)+E_n^{q_n}(x),\end{aligned}$$

where

$$E_n^{q_n}(x)=\mathcal{L}_{n,q_n}^{(\alpha,\beta)}(\Phi_{q_n}(t,x)(t-x)_{q_n}^2,x).$$

We can easily see that $\lim_{n\to\infty}[n]_{q_n}\mathcal{L}_{n,q_n}^{(\alpha,\beta)}((t-x)_{q_n},x)=\alpha-\beta x$ and $\lim_{n\to\infty}[n]_{q_n}\mathcal{L}_{n,q_n}^{(\alpha,\beta)}((t-x)_{q_n}^2,x)=2x(1+x)$.

In order to complete the proof of the theorem, it is sufficient to show that $\lim_{n\to\infty}[n]_{q_n} E_n^{q_n}(x) = 0$. We proceed as follows:

Let

$$R_{n,1}^{q_n}(x) = [n]_{q_n}\mathcal{L}_{n,q_n}^{(\alpha,\beta)}(\Phi_{q_n}(x,t)(t-x)_{q_n}^2\chi_x(t),t)$$

and

$$R_{n,2}^{q_n}(x) = [n]_{q_n}\mathcal{L}_{n,q_n}^{(\alpha,\beta)}(\Phi_{q_n}(x,t)(t-x)_{q_n}^2(1-\chi_x(t)),t),$$

so that

$$[n]_{q_n}E_n^{q_n}(x) = R_{n,1}^{q_n}(x) + R_{n,2}^{q_n}(x),$$

where $\chi_x(t)$ is the characteristic function of the interval $\left\{t : \left|\frac{[n]_{q_n}t+\alpha}{[n]_{q_n}+\beta} - x\right| < \delta\right\}$.

It follows from (18)

$$R_{n,1}^{q_n}(x) < \epsilon 2x(x+1) \text{ as } n \to \infty.$$

If $\left|\frac{[n]_{q_n}t+\alpha}{[n]_{q_n}+\beta} - x\right| \geq \delta$, then $|\Phi_{q_n}(x,t)| \leq \frac{M}{\delta^2}\left(\frac{[n]_{q_n}t+\alpha}{[n]_{q_n}+\beta} - x\right)^2$, where $M > 0$ is a constant. Since

$$\begin{aligned}\left(\frac{[n]_{q_n}t+\alpha}{[n]_{q_n}+\beta}-x\right)^2 &= \left(\frac{[n]_{q_n}t+\alpha}{[n]_{q_n}+\beta}-x\right)\left(\frac{[n]_{q_n}t+\alpha}{[n]_{q_n}+\beta}-x\right)\\ &= \left(\frac{[n]_{q_n}t+\alpha}{[n]_{q_n}+\beta} - {q_n}^2x + {q_n}^2x - x\right)\\ &\quad\times\left(\frac{[n]_{q_n}t+\alpha}{[n]_{q_n}+\beta} - {q_n}^3x + {q_n}^3x - x\right)\\ &= \left(\frac{[n]_{q_n}t+\alpha}{[n]_{q_n}+\beta} - {q_n}^2x\right)\left(\frac{[n]_{q_n}t+\alpha}{[n]_{q_n}+\beta} - {q_n}^3x\right)\\ &\quad + x({q_n}^3-1)\left(\frac{[n]_{q_n}t+\alpha}{[n]_{q_n}+\beta} - {q_n}^2x\right)\\ &\quad + x({q_n}^2-1)\left(\frac{[n]_{q_n}t+\alpha}{[n]_{q_n}+\beta} - {q_n}^3x\right)\\ &\quad + x^2({q_n}^2-1)({q_n}^3-1),\end{aligned}$$

we have

$$\begin{aligned}R_{n,2}^{q_n}(x) = &\frac{M}{\delta^2}[n]_{q_n}\mathcal{L}_{n,q_n}^{(\alpha,\beta)}((t-x)_{q_n}^4,t) + \frac{M}{\delta^2}x[n]_{q_n}\\ &\times\left(|(q_n^3-1)+(q_n^2-1)|\right)\mathcal{L}_{n,q_n}^{(\alpha,\beta)}((t-x)_{q_n}^3,t)\\ &+\frac{M}{\delta^2}x^2[n]_{q_n}\left(q_n^2-1\right)^2\mathcal{L}_{n,q_n}^{(\alpha,\beta)}((t-x)_{q_n}^2,t)\end{aligned}$$

and

$$\begin{aligned}|R_{n,2}^{q_n}(x)| \leq &\frac{M}{\delta^2}\Big\{[n]_{q_n}\mathcal{L}_{n,q_n}^{(\alpha,\beta)}((t-x)_{q_n}^4,t) + x[n]_{q_n}\\ &\times\left(2-q_n^3-q_n^2\right)\mathcal{L}_{n,q_n}^{(\alpha,\beta)}((t-x)_{q_n}^3,t)\\ &+x^2[n]_{q_n}\left(q_n^2-1\right)^2\mathcal{L}_{n,q_n}^{(\alpha,\beta)}((t-x)_{q_n}^2,t)\Big\}.\end{aligned}$$

Using Lemma 5, we have

$$\mathcal{L}_{n,q_n}^{(\alpha,\beta)}((t-x)_{q_n}^4,t) \leq \frac{C_{1,x}}{[n]_{q_n}^2},\quad \mathcal{L}_{n,q_n}^{(\alpha,\beta)}((t-x)_{q_n}^3,t) \leq \frac{C_{2,x}}{[n]_{q_n}^2},$$

$$\mathcal{L}_{n,q_n}^{(\alpha,\beta)}((t-x)_{q_n}^2,t) \leq \frac{C_{3,x}}{[n]_{q_n}}.$$

Thus, for n sufficiently large $R_{n,2}^{q_n} \to 0$. This completes the proof of theorem. □

Corollary 1. *[17] Let $f \in C[0,\infty)$ be a bounded function and (q_n) denote a sequence such that $0 < q_n < 1$ and $q_n \to 1$ as $n \to \infty$. Suppose that the first and second derivative $f'(x)$ and $f''(x)$ exist at a point $x \in [0,\infty)$, we have*

$$\begin{aligned}&\lim_{n\to\infty}[n]_{q_n}\left(\mathcal{L}_{n,q_n}^{(\alpha,\beta)}(f,x) - f(x)\right)\\ &= (\alpha-\beta x)f'(x) + x(1+x)f''(x).\end{aligned}$$

Point-wise Estimation

Now, we establish some point-wise estimxates of the rate of convergence of the q-Baskakov-Durrmeyer-Stancu Operators. First we give the relationship between the local smoothness of f and local approximation.

We know that a function $f \in C[0,\infty)$ is *Lip* γ on D, $\gamma \in (0,1]$, $D \subset [0,\infty)$ if it satisfies the condition

$$|f(t)-f(x)| \leq M_f|t-x|^\gamma,\ t \in [0,\infty) \text{ and } x \in D, \quad (21)$$

where M_f is a constant depending on γ and f.

Theorem 3. *Let $f \in Lip\gamma$, $D \subset [0,\infty)$ and $\gamma \in [0,1]$. We have*

$$|\mathcal{L}_{n,q}^{(\alpha,\beta)}(f,x) - f(x)| \leq M_f\left(\left[\mu_{n,2}^q(x)\right]^{\gamma/2} + 2d^\gamma(x,D)\right),\quad x \in [0,\infty)$$

where $d(x,D)$ represents the distance between x and D.

Proof. For $x_0 \in \overline{D}$, the closure of the set D in $[0,\infty)$, we have

$$|f(t)-f(x)| \leq |f(t)-f(x_0)| + |f(x_0)-f(x)|,\quad x \in [0,\infty).$$

Using (21) we get

$$\begin{aligned}|\mathcal{L}_{n,q}^{(\alpha,\beta)}(f,x) - f(x)| &\leq \mathcal{L}_{n,q}^{(\alpha,\beta)}(|f(t)-f(x_0)|,x) + |f(x_0)-f(x)|\\ &\leq M_f\mathcal{L}_{n,q}^{(\alpha,\beta)}(|t-x_0|^\gamma,x) + M_f|x_0-x|^\gamma.\end{aligned} \quad (22)$$

Then by the Holder's inequality with $p = \frac{2}{\gamma}$ and $\frac{1}{r} = 1 - \frac{1}{p}$, we have

$$\begin{aligned}\mathcal{L}_{n,q}^{(\alpha,\beta)}(|t-x_0|^\gamma,x) &\leq \left(\mathcal{L}_{n,q}^{(\alpha,\beta)}(|t-x_0|^2,x)\right)^{\gamma/2}\\ &\quad\times\left(\mathcal{L}_{n,q}^{(\alpha,\beta)}(1,x)\right)^{1-\gamma/2}.\end{aligned} \quad (23)$$

Also, $\mathcal{L}_{n,q}^{(\alpha,\beta)}$ is monotone,i.e.

$$\mathcal{L}_{n,q}^{(\alpha,\beta)}(|t-x_0|^\gamma,x) \le \left(\mathcal{L}_{n,q}^{(\alpha,\beta)}(|t-x|^2,x)\right)^{\gamma/2} + |x_0-x|^\gamma. \tag{24}$$

Using (22)-(24), we have desired result. □

Now, we give local direct estimate for the q-Baskakov-Durrmeyer-Stancu operators using the Lipschitz type maximal function of order γ introduced by B. Lenze [18] as

$$\tilde{\omega}_\gamma(f,x) = \sup_{t\neq x,\ t\in[0,\infty)} \frac{|f(t)-f(x)|}{|t-x|^\gamma},\ x\in[0,\infty),\quad \gamma\in(0,1] \tag{25}$$

Theorem 4. *Let $\gamma \in (0,1]$ and $f \in C_B(0,\infty)$. Then for all $x \in [0,\infty)$, we have*

$$|\mathcal{L}_{n,q}^{(\alpha,\beta)}(f,x) - f(x)| \le \tilde{\omega}_\gamma(f,x)\left(\mu_{n,2}^q\right)^{\gamma/2} \tag{26}$$

Proof. Form (25), we have

$$|f(t)-f(x)| \le \tilde{\omega}_\gamma(f,x)|t-x|^\gamma$$

and

$$\left|\mathcal{L}_{n,q}^{(\alpha,\beta)}(f,x) - f(x)\right| \le \mathcal{L}_{n,q}^{(\alpha,\beta)}(|f(t)-f(x)|,x) \le \tilde{\omega}_\gamma(f,x) \times \mathcal{L}_{n,q}^{(\alpha,\beta)}(|t-x|^\gamma,x)$$

Applying Holder's inequality with $p := \frac{2}{\gamma}$ and $\frac{1}{r} = 1-\frac{1}{p}$, we have

$$\left|\mathcal{L}_{n,q}^{(\alpha,\beta)}(f,x) - f(x)\right| \le \tilde{\omega}_\gamma(f,x)(\mathcal{L}_{n,q}^{(\alpha,\beta)}(|t-x|^2,x))^{\gamma/2},$$

which is required result. □

Better estimation

It is well known that the operators preserve constant as well as linear functions. To make the convergence faster, King [19] proposed an approach to modify the classical Bernstein polynomials so that this sequence preserves two test functions e_0 and e_2. After this, several researchers have studied that many approximating operators L possess these properties i.e., $L(e_i,x) = e_i(x)$, where $e_i(x) = x^i (i = 0,1)$, like Bernstein, Baskakov, and Baskakov-Durrmeyer-Stancu-type operators.

As the operators $\mathcal{L}_{n,q}^{(\alpha,\beta)}$ introduced in (5) preserve only the constant functions, further modification of these operators is proposed to be made so that the modified operators preserve the constant as well as linear functions. For this purpose, the modification of $\mathcal{L}_{n,q}^{(\alpha,\beta)}$ will be as follows:

$$\mathcal{L}^{*(\alpha,\beta)}_{n,q}(f,x) = \sum_{k=1}^{\infty} p_{n,k}^q(r_n(x)) \int_0^\infty b_{n,k}(t) f\left(\frac{[n]_q t+\alpha}{[n]_q+\beta}\right) \times d_q t + p_{n,0}(r_n(x)) f\left(\frac{\alpha}{[n]_q+\beta}\right),$$

where $r_n(x) = \frac{([n]_q+\beta)x-\alpha}{[n]_q}$ and $x \in I_n = \left[\frac{\alpha}{[n]_q+\beta},\infty\right)$.

Lemma 6. *For each $x \in I_n$, we have*

$$\mathcal{L}^{*(\alpha,\beta)}_{n,q}(1,x) = 1,\ \ \mathcal{L}^{*(\alpha,\beta)}_{n,q}(t,x) = \frac{[n]_q r_n(x)+\alpha}{[n]_q+\beta},$$
$$\mathcal{L}^{*(\alpha,\beta)}_{n,q}(t^2,x) = \frac{[n+1]_q}{q^2[n-1]_q}x^2 + \frac{[2]_q([n]_q-2\alpha)}{([n]_q+\beta)[n-1]_q}x + \frac{[2]_q(\alpha^2-[n]_q\alpha)}{([n]_q+\beta)^2[n-1]_q}.$$

Lemma 7. *For $x \in I_n$, the following hold,*

$$\tilde{\mu}_{n,1}^q(x) = \mathcal{L}^{*(\alpha,\beta)}_{n,q}(t-x,x) = 0,$$
$$\tilde{\mu}_{n,2}^q(x) = \mathcal{L}^{*(\alpha,\beta)}_{n,q}((t-x)^2,x) = \frac{[2]_q}{q^2[n-1]_q}x^2 + \frac{[2]_q([n]_q-2\alpha)}{([n]_q+\beta)[n-1]_q}x + \frac{[2]_q(\alpha^2-[n]_q\alpha)}{([n]_q+\beta)^2[n-1]_q}.$$

Theorem 5. *Let $f \in C_B(I_n)$, $x \in I_n$ and $0 < q < 1$. Then, for $n > 1$, there exist an absolute constant $C > 0$ such that*

$$\left|\mathcal{L}^{*(\alpha,\beta)}_{n,q}(f,x) - f(x)\right| \le C\omega_2\left(f,\sqrt{\tilde{\mu}_{n,2}^q(x)}\right).$$

Proof. Let $g \in C_B(I_n)$ and $x,t \in I_n$. By Taylor's expansion, we have

$$g(t) = g(x) + (t-x)g'(x) + \int_x^t (t-u)g''(u)du. \tag{27}$$

Applying $\mathcal{L}^{*(\alpha,\beta)}_{n,q}$, we get

$$\mathcal{L}^{*(\alpha,\beta)}_{n,q}(g,x) - g(x) = g'(x)\mathcal{L}^{*(\alpha,\beta)}_{n,q}((t-x),x) + \mathcal{L}^{*(\alpha,\beta)}_{n,q} \times \left(\int_x^t (t-u)g''(u)du, x\right).$$

Obviously, we have $\left|\int_x^t (t-x)g''(u)du\right| \le (t-x)^2\|g''\|$,

$$\left|\mathcal{L}^{*(\alpha,\beta)}_{n,q}(g,x) - g(x)\right| \le \mathcal{L}^{*(\alpha,\beta)}_{n,q}((t-x)^2,x)\|g''\| = \tilde{\mu}_{n,2}^q\|g''\|.$$

Since $\left|\mathcal{L}{*}_{n,q}^{(\alpha,\beta)}(f,x)\right| \leq \|f\|$,

$$\left|\mathcal{L}{*}_{n,q}^{(\alpha,\beta)}(f,x)-f(x)\right| \leq \left|\mathcal{L}{*}_{n,q}^{(\alpha,\beta)}(f-g,x)-(f-g)(x)\right| + \left|\mathcal{L}{*}_{n,q}^{(\alpha,\beta)}(g,x)-g(x)\right| \leq 2\|f-g\| + \tilde{\mu}_{n,2}^{q}\|g''\|.$$

Taking infimum overall $g \in C^2(I_n)$, we obtain

$$\left|\mathcal{L}{*}_{n,q}^{(\alpha,\beta)}(f,x)-f(x)\right| \leq K_2(f,\tilde{\mu}_{n,2}^{q}).$$

In view of (7), we have

$$\left|\mathcal{L}{*}_{n,q}^{(\alpha,\beta)}(f,x)-f(x)\right| \leq C\omega_2\left(f,\sqrt{\tilde{\mu}_{n,2}^{q}}\right),$$

which proves the theorem. □

Theorem 6. *Assume that $q_n \in (0,1), q_n \to 1$ as $n \to \infty$. Then for any $f \in C^*_{x^2}(I_n)$ such as $f', f'' \in C^*_{x^2}(I_n)$, we have*

$$\lim_{n\to\infty}[n]_{q_n}\left[\mathcal{L}{*}_{n,q_n}^{(\alpha,\beta)}(f,x)-f(x)\right] = x(1+x)f''(x), \quad \text{for every } x \in I_n.$$

Remark 4. One can discuss rate of approximation in weighted spaces for the operators $\mathcal{L}_{n,q}^{(\alpha,\beta)}$. we are omitting the details as it is similar to Theorem 3 and 4 in [20].

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
VNM and PP computed the moments of modified operators and established the asymptotic formula. VNM conceived of the study and participated in its design and coordination. VNM and PP contributed equally and significantly in writing this paper. All the authors drafted the manuscript, read and approved the final manuscript.

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