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On the existence of bi-invariant Finsler metrics on Lie groups

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Abstract

In this paper, we study the geometry of Lie groups with bi-invariant Finsler metrics. We first show that every compact Lie group admits a bi-invariant Finsler metric. Then, we prove that every compact connected Lie group is a symmetric Finsler space with respect to the bi-invariant absolute homogeneous Finsler metric. Finally, we show that if *G* is a Lie group endowed with a bi-invariant Finsler metric, then, there exists a bi-invariant Riemanninan metric on *G* such that its Levi-Civita connection coincides the connection of *F*.

Keywords: Invariant Finsler metric; Bi-invariant metric; Lie group; Flag curvature; Homogeneous geodesic

MSC: 53C60, 53C30

Introduction

Lie groups are the most beautiful and most important manifolds. On the one hand, these spaces contain many prominent examples which are of great importance for various branches of mathematics, like homogeneous spaces, symmetric spaces, and Grassmannians. On the other hand, these spaces have much in common, and there exists a rich theory. The study of invariant structures on Lie groups and homogeneous spaces is an important problem in differential geometry. Milnor's research on the properties of invariant Riemannian metrics on a Lie group obtained many interesting and significant results. He computed the connections of these metrics and obtained the formula for geodesics and curvatures. Lie groups are, in a sense, the nicest examples of manifolds and are good spaces on which to test conjectures [1]. Therefore, it is important to study invariant Finsler metrics. In [2], the authors studied invariant Finsler metrics on homogeneous spaces and gave some descriptions of these metrics. Also in [3] and [4], we have studied the homogeneous Finsler spaces and the homogeneous geodesics in homogeneous Finsler spaces.

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Among the invariant metrics, the bi-invariant ones are the simplest kind. They have nice and simple geometric properties but still form a large enough class to be of interest. In [5], we have studied the geometry of Lie groups with bi-invariant Randers metrics, and in [6], we have studied the naturally reductive Randers metrics on homogeneous manifolds.

In this paper, we study the geometry of Lie groups with bi-invariant Finsler metrics. We first show that every compact Lie group admits a non-Riemannian bi-invariant Finsler metric. Then, we prove that every compact connected Lie group is a symmetric Finsler space with respect to the bi-invariant absolutely homogeneous Finsler metric. Finally, we show that If G is a Lie group endowed with a bi-invariant Finsler metric, then there exists a bi-invariant Riemanninan metric on G such that its Levi-Civita connection coincides the connection of F.

Bi-invariant Finsler metrics on Lie groups

A Finsler metric on a manifold M is a continuous function, $F : TM \longrightarrow [0,\infty)$ differentiable on $TM\backslash\{0\}$ and satisfying three conditions [7]:

(a) $F(y) = 0$ if and only if $y = 0$.
(b) $F(\lambda y) = \lambda F(y)$ for any $y \in T_xM$ and $\lambda > 0$.
(c) For any non-zero $y \in T_xM$, the symmetric bilinear form $g_y : T_xM \times T_xM \longrightarrow R$ given by:

$$g_y(u,v) = \frac{1}{2}\frac{\partial^2}{\partial s \partial t}[F^2(y+su+tv)]\,|_{s=t=0}$$

is positive definite.

For each $y \in T_xM - \{0\}$, define:

$$C_y(u,v,w) = \frac{1}{4}\frac{\partial^3}{\partial s \partial t \partial r}[F^2(y+su+tv+rw)]\,|_{s=t=r=0}\,.$$

C is called the Cartan torsion.

Let G be a connected Lie group with Lie algebra $\mathfrak{g} = T_eG$. We may identify the tangent bundle TG with $G \times \mathfrak{g}$ by means of the diffeomorphism that sends (g, X) to $(L_g)_*X \in T_gG$.

Definition 1. *A Finsler function $F : TG \longrightarrow R_+$ will be called G-invariant (left-invariant) if F is constant on all G-orbits in $TG = G \times \mathfrak{g}$; that is, $F(g, X) = F(e, X)$ for all $g \in G$ and $X \in \mathfrak{g}$.*

The G-invariant Finsler functions on TG may be identified with the Minkowski norms on $\mathfrak{g}$. If $F : TG \longrightarrow R_+$ is an G-invariant Finsler function, then, we may define $\widetilde{F} : \mathfrak{g} \longrightarrow R_+$ by $\widetilde{F}(X) = F(e, X)$, where e denotes the identity in G. Conversely, if we are given a Minkowski norm $\widetilde{F} : \mathfrak{g} \longrightarrow R_+$, then $\widetilde{F}$ arises from an G-invariant Finsler function $F : TG \longrightarrow R_+$ given by $F(g, X) = \widetilde{F}(X)$ for all $(g, X) \in G \times \mathfrak{g}$.

Similarly, a Finsler metric is right-invariant if each $R_a : G \longrightarrow G$ is an isometry.

Definition 2. *A Finsler metric on G that is both left-invariant and right-invariant is called bi-invariant.*

Let G be a compact Lie group. Fix a base $\omega_1, \omega_2, \ldots, \omega_n$ in T_eG and put $\omega = \omega_1 \wedge \ldots \wedge \omega_n$ where $n = dimG$. Extend ω to a left-invariant differential form Ω on G by putting $\Omega_g = (L_g)^*\omega$. The form Ω never vanishes. The form determines an orientation of G. Recall that a chart with coordinates $x^1, \ldots, x^n$ is called positively if $dx^1 \wedge \ldots \wedge dx^n = f\Omega$ where f is a positive function defined on the coordinate neighborhood. Clearly, the atlas consisting of all positively oriented charts determines an orientation on G. Indeed, if $dy^1 \wedge \ldots \wedge dy^n = h\Omega$ and $h > 0$, then:

$$dy^1 \wedge \ldots \wedge dy^n = \frac{h}{f} dx^1 \wedge \ldots \wedge dx^n.$$

On the other hand,

$$dy^1 \wedge ... \wedge dy^n = \frac{\partial(y^1, \ldots, y^n)}{\partial(x^1, \ldots, x^n)} dx^1 \wedge \ldots \wedge dx^n,$$

so

$$\frac{\partial(y^1, \ldots, y^n)}{\partial(x^1, \ldots, x^n)} = \frac{h}{f} > 0.$$

Now, for any $a \in G$, we can easily see that $R_a^*\Omega$ is left-invariant. It follows that $R_a^*\Omega = f(a)\Omega$. We can easily see that $f(ab) = f(a)f(b)$ that is $f : G \longrightarrow R - \{0\}$ is a continuous homomorphism of G into the multiplicative group of real numbers. Since $f(G)$ is compact connected subgroup, the conclusion $f(G) = 1$ holds. Therefore, $R_a^*\Omega = \Omega$. So Ω is bi-invariant volume element on G.

Theorem 1. *Every compact Lie group admits a bi-invariant Finsler metric.*

Proof. Let Ω be the bi-invariant volume element. Let F_e be a Minkowski norm on $T_eG = \mathfrak{g}$. Then, define the function $\widetilde{F}_e$ on T_eG by:

$$\widetilde{F}_e^2(X) = \int_G F_e^2(Ad_gX)\Omega.$$

Let $K_1, \ldots, K_N$ be a covering of G by cubes, and that let $\phi_1, ..., \phi_N$ be a corresponding partition of unity, and $\Omega = \Omega_{12\ldots n}dx^1 \wedge \ldots \wedge dx^n$. Now we can write:

$$\widetilde{F}_e^2(X) = \int_G F_e^2(Ad_gX)\Omega = \sum_{i=1}^{N} n! \int_{K_i} F_e^2(\phi_i Ad_gX)\Omega_{12\ldots n}dx^1 \wedge \ldots \wedge dx^n.$$

By definition of the orientation, $\Omega_{12\ldots n} > 0$, and $\phi_i(x)$ is positive in K_i; furthermore, $F_e^2(Ad_gX) > 0$ if $X \neq 0$. Since all summands in the above expression for $\int_G F_e^2(Ad_gX)\Omega$ are positive, we come to $\int_G F_e^2(Ad_gX)\Omega > 0$. So, $\widetilde{F}_e$ is well-defined function $\widetilde{F}_e : T_eG \longrightarrow [0, \infty)$, $\widetilde{F}_e(X) > 0 if X \neq 0$. We can easily see that $\widetilde{F}_e(X) = 0$ if and only if $X = 0$. Clearly $\widetilde{F}_e(\lambda X) = \lambda \widetilde{F}_e(X)$ for any $X \in T_eG$, $\lambda > 0$.

Since F_e is C^∞ on $T_eG - \{0\}$, we see that $\widetilde{F}_e$ is C^∞ on $T_eG - \{0\}$. Now for any $y \neq 0, u, v \in T_eG$ by a direct computation, we have:

$$\widetilde{g}_y(u,v) = \int_G g_{Ad_g(Y)}(Ad_g(u), Ad_g(v))\Omega.$$

Since $g_{Ad_g(Y)}$ is positively definite, hence $\widetilde{g}_y$ is positive definite. So $\widetilde{F}(e)$ is a Minkowski norm on T_eG. In the following, we show that $\widetilde{F}_e$ is $Ad(G)-$invariant i.e.:

$$\forall h \in G, \widetilde{F}_e(Ad_{(h)}X) = \widetilde{F}_e(X).$$

Applying the definition of $\widetilde{F}_e$ to the left side of this expression, we find that:

$$\widetilde{F}_e^2(Ad_hX) = \int_G F_e^2(Ad_gAd_hX)\Omega = \int_G F_e^2(Ad_{gh}X)\Omega.$$

The diffeomorphism L_h and R_h preserve the orientation because Ω is bi-invariant. So, for the diffeomorphism $I_h = R_{x^{-1}} \circ L_x : G \longrightarrow G$, we have:

$$\int_{I_h(G)} F_e^2(Ad_gX)\Omega = \int_G F_e^2(Ad_{gh}X)R_h^*\Omega.$$

Since $I_h(G) = G$ and $R_h^*\Omega = \Omega$, we see that:

$$\widetilde{F}_e^2(Ad_hX) = \widetilde{F}_e^2(X).$$

Thus, $\widetilde{F}_e(Ad_h X) = \widetilde{F}_e(X)$.

Extend the Minkowski norm in T_eG, thus defined to a left-invariant Finsler metric on G, by putting:

$$\widetilde{F}(X) = \widetilde{F}_e((L_{a^{-1}})_{*a}X),$$

whenever $X \in T_aG$.

We show that this Finsler metric is bi-invariant. We only need to check the right-invariance, we have:

$$\widetilde{F}((R_a)_*X) = \widetilde{F}((L_{a^{-1}})_*(R_a)_*X).$$

Furthermore, $Ad_{g^{-1}} = (L_{a^{-1}})_*(R_a)_*$. Consequently, by $Ad-$invariance of $\widetilde{F}$:

$$\widetilde{F}((R_a)_*X) = \widetilde{F}(Ad_{a^{-1}}X) = \widetilde{F}(X).$$

□

A connected Riemannian manifold M is said to be symmetric Riemannian space if to each $p \in M$ there is an associated isometry $\sigma_p : M \longrightarrow M$ which is (i) involutive, and (ii) has p as an isolated fixed point. As an example, every compact connected Lie group G is a symmetric Riemannian space with respect to the bi-invariant Riemannian metric [8,9].

The definition of symmetric Finsler space is a natural generalization of the definition of symmetric Riemannian spaces [3].

Definition 3. *A connected Finsler space (M, F) is said to be symmetric, if to each $p \in M$ there is associated an isometry $s_p : M \longrightarrow M$ which is:*

(a) *involutive (s_p^2 is the identity).*
(b) *has p as an isolated fixed point, that is, there is a neighborhood U of p in which p is the only fixed point of s_p.*

As p is an isolated fixed point of σ_p, it follows that $(d\sigma_p)_p = -id$, and therefore, symmetric Finsler spaces have reversible metrics and geodesics.

In the following theorem similar to the Riemmanian case, we show that Lie groups with bi-invariant absolutely homogeneous Finsler metrics are symmetric Finsler space.

Theorem 2. *Every compact connected Lie group G is a symmetric Finsler space with respect to the bi-invariant absolutely homogeneous Finsler metric.*

Proof. Let $\psi : G \longrightarrow G$ denote the inversion map $g \longrightarrow g^{-1}$. Clearly ψ is involutive and is an isometry of G with e as isolated fixed point. Let $X \in T_eG$, since $\psi(\exp(tX)) = \exp(-tX)$ we obtain:

$$\psi_*(X_e) = \frac{d}{dt}|_{t=0}\psi(\exp(tX)) = \frac{d}{dt}|_{t=0}\exp(-tX) = -X.$$

This means that $\psi_{*e} = -Id$, hence ψ is an isometry of G, and e is an isolated fixed point of ψ.

For every $g \in G$, clearly $\psi = R_{g^{-1}} \circ \psi \circ L_{g^{-1}}$. Therefore $\psi_{*g} : \mathfrak{g} \longrightarrow \mathfrak{g}$ is an isometry for any $g \in G$. Now for $g \in G$, Let $\sigma_g(x) = gx^{-1}g$, $x \in G$. The mapping σ_g is an isometry because:

$$\sigma_g = R_g \circ \psi \circ R_{g^{-1}}.$$

Obviously, σ_g fixing the point g and is involutive $\sigma_g^2(x) = g(gx^{-1}g)^{-1}g = x$. Now, it suffices to show that g is isolated fixed point. If $X \in \mathfrak{g}$, then for an arbitrary $g \in G$, we have:

$$(\sigma_g)_*X = (R_g)_*\psi_*(R_{g^{-1}})_*(X) = (R_g)_*(-(R_{g^{-1}})_*X) = -X,$$

and the proof is complete. □

Theorem 3. *If G is a Lie group endowed with an absolutely homogeneous bi-invariant Finsler metric, then, the geodesics through the identity of G are exactly one-parameter subgroups.*

Proof. Suppose $\gamma(t)$ is a geodesic in G with $e = \gamma(0)$. We show that $\gamma(t)$ is a one-parameter subgroup. With the help of symmetries, we have $\sigma_{\gamma(c)}\sigma_e\gamma(t) = \gamma(t+tc)$. Since $\sigma_{\gamma(c)}\sigma_e(x) = \gamma(c)x\gamma(c)$, so $\gamma(c)\gamma(t)\gamma(c) = \gamma(t+2c)$. In particular, $\gamma(2c) = \gamma(c)^2$, and by an induction, we have $\gamma(nc) = \gamma(c)^n$. Now, if $\frac{c_1}{c_2} \in Q$, $c_1 = \alpha n_1$, $c_2 = \alpha n_2$ where n_1 and n_2 are integers, then:

$$\gamma(c_1+c_2) = \gamma((n_1+n_2)\alpha) = \gamma(\alpha)^{n_1+n_2} = \gamma(\alpha)^{n_1}\gamma(\alpha)^{n_2}$$
$$= \gamma(c_1)\gamma(c_2).$$

Now, by the continuity of $\gamma(t)$ for arbitrary c_1 and c_2, we have:

$$\gamma(c_1 + c_2) = \gamma(c_1)\gamma(c_2).$$

Hence, $\gamma(t)$ is a one-parameter subgroup of G.

Now, we show that one-parameter subgroups are geodesics. Suppose $\gamma(t)$ is a one-parameter subgroup. Let $\xi = \frac{d\gamma(t)}{dt}|_{t=0}$ be the tangent vector of $\gamma(t)$ at e. There is a geodesic $x(t)$ through e determined by ξ. We have already shown that $x(t)$ is a one-parameter subgroup, so $x(t) = \gamma(t)$, and $\gamma(t)$ is a geodesic. □

Definition 4. *Let G be a connected Lie group, $\mathfrak{g} = T_eG$ its Lie algebra identified with the tangent space at the identity element, $\widetilde{F} : \mathfrak{g} \longrightarrow R_+$ a Minkowski norm and F the left-invariant Finsler metric induced by $\widetilde{F}$ on G. A geodesic $\gamma : R \longrightarrow G$ is said to be homogeneous if there is a $Z \in \mathfrak{g}$ such that $\gamma(t) = \exp(tZ)\gamma(0)$, $t \in R$ holds. A tangent vector $X \in T_eG - \{0\}$ is said to be a geodesic vector if the 1-parameter subgroup $t \longrightarrow \exp(tX)$, $t \in R$, is a geodesic of F.*

For results on homogeneous geodesics in homogeneous Finsler manifolds, we refer to [4,10-12]. The basic formula characterizing geodesic vector in the Finslerian case was derived in [4], Theorem 3.1. For Lie groups with left invariant metrics, we have the following theorem.

Theorem 4. *[4] Let G be a connected Lie group with Lie algebra $\mathfrak{g}$, and let F be a left-invariant Finsler metric on G. Then $X \in \mathfrak{g} - \{0\}$ is a geodesic vector if and only if:*

$$g_X(X, [X, Z]) = 0$$

holds for every $Z \in \mathfrak{g}$.

Corollary 1. *If G is a Lie group endowed with a bi-invariant Finsler metric, then, the geodesics through the identity of G are homogeneous geodesics.*

Theorem 5. *Let G be a Lie group with a bi-invariant Finsler metric F. Then, there exists a bi-invariant Riemanninan metric on G such that its Levi-Civita connection coincides the connection of F.*

Proof. If F is a bi-invariant Finsler metric on G, as:

$$Ad_g = (R_{a^{-1}})_{*a} \circ (L_a)_{*e},$$

it is clear that Ad_g is an isometry on $\mathfrak{g}$. Now define:

$$I_e = \{x \in \mathfrak{g} | F(x) = 1\}.$$

Then, the adjoint group $Ad(G) \subseteq GL(\mathfrak{g})$ leaves I_e invariant. Let G_1 be the subgroup of the general linear group $GL(\mathfrak{g})$ consisting of the elements which leaves I_e invariant. Then, G_1 is a compact Lie group, and $Ad(G)$ is a subgroup of G_1. So, $Ad(G)$ has compact closure in $GL(\mathfrak{g})$. Therefore, the Lie group G admits a bi-invariant Riemannian metric g.

For (G, F) and (G, g), we have that the geodesics of (G, F) and (G, g) coincide, and hence (G, F) is a Berwald space. Now, we show that the Levi-Civita connection of (G,g) coincides the connection of (G, F).

Let ∇ be the connection of F. For any left invariant vector fields X, Y, Z on G, we have:

$$Yg_X(Z, X) = g_X(\nabla_Y Z, X) + g_X(Z, \nabla_Y X) \quad (1)$$

Similarly,

$$Zg_X(Y, X) = g_X(\nabla_Z Y, X) + g_X(Y, \nabla_Z X) \quad (2)$$

$$Xg_X(Z, X) = g_X(\nabla_X Z, X) + g_X(Z, \nabla_X X) \quad (3)$$

All covariant derivatives have $\widetilde{X}$ as reference vector.

Subtracting (2) from the summation of (1) and (3), we get:

$$g_X(Z, \nabla_{X+Y} X) + g_X(X - Y, \nabla_Z X) = Yg_X(Z, X) - Zg_X(Y, X) + Xg_X(Z, X) - g_X([Y, Z], X) - g_X([X, Z], X),$$

where we have used the symmetry of the connection, i.e., $\nabla_Z X - \nabla_X Z = [Z, X]$. Set $Y = X - Z$ in the above equation, we obtain:

$$2g_X(Z, \nabla_X X) = 2Xg_X(Z, X) - Zg_X(X, X) - 2g_X([X, Z], X). \quad (4)$$

Since F is left-invariant, dL_x is a linear isometry between the spaces $T_eG = \mathfrak{g}$ and T_xG, $\forall x \in G$. Therefore, for any left-invariant vector field X, Z on G, we have:

$$g_X(Z, X) = g_{X_e}(Z_e, X_e)$$

i.e., the functions $g_X(Z, X)$, $g_X(X, X)$ are constant. Therefore, from (4) and Theorem 4, the following is obtained:

$$g_X(Z, \nabla_X X) |_e = -g_X([X, Z], X) |_e = -g_X([X, Z], X) = 0.$$

Consequently for any left invariant X, we have:

$$\nabla_X X = 0.$$

Now using the identity:

$$\nabla_X Y = \frac{1}{2}(\nabla_{X+Y}(X + Y) - \nabla_X X - \nabla_Y Y + [X, Y]),$$

we get:

$$\nabla_X Y = \frac{1}{2}[X, Y].$$

Consequently, the assertion of the theorem follows. □

Corollary 2. *If a connected Lie group G admits a bi-invariant Finsler metric, then, it is isomorphic to the cartesian product of a compact group and an additive vector group.*

Competing interests

The authors declare that they have no competing interests.

Authors' contributions

DL and MT carried out the proof. Both authors read and approved the final manuscript.

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