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Freeness conditions for quasi 3-crossed modules and complexes of using simplicial algebras with $CW-$bases

Ali Mutlu[1*] and Berrin Mutlu[2]

Abstract
Using free simplicial algebras with given $CW-$basis, it is shown how to construct a free or totally free quasi 3-crossed module on suitable construction data. Quasi 3-crossed complexes are introduced and similar freeness results are given for these are discussed.

Keywords: Free crossed module; 2-crossed module; Quasi 3-crossed module and complex

MSC: 18D35; 18G30; 18G50; 18G55; 55Q05; 55Q20

Introduction

André [1] uses simplicial methods to investigate homological properties of (commutative) algebras. Other techniques that can give linked results include those using the Koszul complex. Any simplicial algebra yields a crossed module derived from the Moore complex [2] and any finitely generated free crossed complex $C \rightarrow R$ of commutative algebras was shown in [3] to have $C \cong R^n/d(\bigwedge^2 R^n)$, i.e. the 2$^{\text{nd}}$ Koszul complex term modulo the 2-boundaries.

Higher dimensional analogues of crossed modules of commutative algebras have been defined: 2-crossed modules by Grandjean and Vale [4] and crossed n-cubes of algebras by Ellis [5]. It would not be sensible to expect a strong link between free 2-crossed modules or free crossed squares and Koszul-like constructions since the former record quadratic information which is less evidently there in the Koszul complex. Nevertheless it seems to be useful to try to define what freeness of such 'gadgets' should mean - for instance, to ask 'free on what?'. Arvasi and Porter [6] solution goes via free simplicial algebras as used by André but they did not use free simplicial algebras with given $CW-$basis. A free simplicial algebra with given $CW-$basis is introduced in [7] and freeness conditoins of 2-crossed modules and complexes in [8]. Also quasi 3- crossed modules of commutative algebras are described [9]. So we can use a free simpicial algebras with given $CW-$basis by André methods step by step.

In this paper, it is a logical step to introduce an intermediate concept, namely a free quasi 3-crossed complexes. These use a quasi 3-crossed module plus a chain complex of modules and we will show how to derive such a thing from a simplicial algebras in via [7-9]. We have therefore included a purely algebraic treatment of quasi 3-crossed complexes giving explicit formulae for the structure involved in the passage from simplicial algebras to quasi 3-crossed complexes and an explicit direct proof of a freeness result due to Baues in [10].

Preliminaries

All algebras will be commutative and will be over the same fixed but unspecified ground ring.

Truncated simplicial algebras

Denoting the usual category of finite ordinals by Δ, we obtain for each $k \geq 0$ a subcategory $\Delta_{\leq k}$ determined by the objects $[j]$ of Δ with $j \leq k$. A simplicial algebra is a functor from the opposite category Δ^{op} to **Alg**; a k-truncated simplicial algebra is a functor from $(\Delta^{op}_{\leq k})$ to **Alg**. We denote the category of k-truncated simplicial algebras by $\mathbf{Tr_k SimpAlg}$. Recall from [11] some facts about the skeleton functor. There is a truncation functor $\mathbf{tr_k}$ from the category of simplicial algebras, denoted by

*Correspondence: abgamutlu@gmail.com
[1]Celal Bayar University, Faculty of Science and Arts, Department of Mathematics Muradiye Campus, 45047, Manisa, Turkey
Full list of author information is available at the end of the article

SimpAlg, to that of k-truncated simplicial algebras, given by restriction. This admits a right adjoint

$$\mathbf{cosk_k} : \mathbf{Tr_kSimpAlg} \longrightarrow \mathbf{SimpAlg}$$

called the *k-coskeleton functor*, and a left adjoint

$$\mathbf{sk_k} : \mathbf{Tr_kSimpAlg} \longrightarrow \mathbf{SimpAlg},$$

called the *k-skeleton functor.*

These can be given neat descriptions as follows:

Assume given that $\mathbf{tr_k}(\mathbf{A}) = \{A_0, A_1, \ldots, A_k\}$ is a k-truncated simplicial algebra as above. A family of homomorphisms

$$(\delta_0, \ldots, \delta_{k+1}) : X_{k+1} \; \underset{\delta_0}{\overset{\delta_{k+1}}{\vdots}} \; A_k \; \underset{d_0}{\overset{d_k}{\vdots}} \; A_{k-1}$$

satisfying $d_i\delta_j = d_{j-1}\delta_i$ for $0 \leq i < j \leq k+1$, is said to be the *simplicial kernel* of the family of face homomorphisms $(d_0, \ldots, d_k)$ if it has the following universal property: given any family $(x_0, \ldots, x_{k+1})$ of $k+2$ homomorphisms

$$Y \; \underset{x_0}{\overset{x_{k+1}}{\vdots}} \; A_k$$

which satisfies the equalities $d_ix_j = d_{j-1}x_i (0 \leq i < j \leq k+1)$ with the top face maps of the truncated simplicial algebra, there exists a unique homomorphism $x =< x_0, \ldots, x_{k+1} >: Y \to X_{k+1}$ such that $\delta_i x = x_i$.

Simplicial kernels exist and given the simplicial kernel X_{k+1}, the family of homomorphisms

$$(\alpha_{k+1,j}, \ldots, \alpha_{1j}, \alpha_{0j})$$

defined by

$$\alpha_{ij} = \begin{cases} s_{j-1}d_i & \text{if } i < j \\ id & \text{if } i = j \text{ or } i = j+1 \\ s_jd_{i-1} & \text{if } i > j+1 \end{cases}$$

satisfies the simplicial identities with the face maps of the truncated simplicial algebra; hence there exists a unique $s_j : A_k \to X_{k+1}$ such that $\delta_is_j = \alpha_{ij}$. The defined $(s_j)_{0 \leq j \leq k}$ form a system of degeneracies and we have now defined a $(k+1)$-truncated simplicial algebra

$$\{A_0, A_1, \ldots, A_k, X_{k+1}\}.$$

By iterating this process we obtain the simplicial algebra

$$\mathbf{cosk_k}(\mathbf{tr_k}(A)) = \{A_0, A_1, \ldots, A_k, X_{k+1}, X_{k+2}, \ldots\}$$

called the *k-coskeleton* of the truncated simplicial algebra. If $\mathbf{F}$ is an simplicial algebra, then any truncated simplicial algebra map $x : \mathbf{tr_k}(\mathbf{A}) \to \mathbf{tr_k}(\mathbf{F})$ extends uniquely to a simplicial map $x : \mathbf{A} \to \mathbf{cosk_k}(\mathbf{tr_k}(\mathbf{F}))$.

The k-skeleton functor can be constructed by a dual process involving *simplicial cokernels*

$$(s_0, \ldots, s_k) : A_k \; \underset{s_0}{\overset{s_k}{\vdots}} \; X_{k+1}.$$

That is, universal systems of $(k+1)$ maps verifying $s_is_j = s_{j+1}s_i$ for $0 \leq i \leq j \leq k-1$; see [11] for details.

We recall that ideal chain complex from [6]. By an *ideal chain complex* of algebras, (X, d) we mean one in which each $\mathrm{Im}d_{i+1}$ is an ideal of X_i. Given any ideal chain complex (X, d) of algebras and an integer n the truncation, $t_{n]}X$, of X at level n will be defined by

$$(t_{n]}X)_i = \begin{cases} X_i & \text{if } i < n \\ X_i/\mathrm{Im}d_{n+1} & \text{if } i = n \\ 0 & \text{if } i > n. \end{cases}$$

The differential d of $t_{n]}X$ is that of X for $i < n$, d_n is induced from the n^{th} differential of X and all other are zero.

Recall that given a simplicial algebra $\mathbf{A}$, *the Moore complex* $(\mathbf{NA}, \partial)$ *of* $\mathbf{A}$ is the chain complex defined by

$$(\mathbf{NA})_n = \bigcap_{i=0}^{n-1} \mathrm{Ker} d_i^n$$

with $\partial_n : NA_n \to NA_{n-1}$ induced from d_n^n by restriction.

The n^{th} *homotopy module* $\pi_n(\mathbf{A})$ of $\mathbf{A}$ is the nth homology of the Moore complex of $\mathbf{A}$, i.e.,

$$\begin{aligned} \pi_n(\mathbf{A}) &\cong H_n(\mathbf{NA}, \partial) \\ &= \bigcap_{i=0}^{n} \mathrm{Ker} d_i^n / d_{n+1}^{n+1}\Big(\bigcap_{i=0}^{n} \mathrm{Ker} d_i^{n+1}\Big). \end{aligned}$$

We say that the Moore complex $\mathbf{NA}$ of a simplicial algebra is of *length k* if $NA_n = 0$ for all $n \geq k+1$ so that a Moore complex is of length k also of length r for $r \geq k$.

The following lemma is in the case of simplicial algebras We give an obvious analogue for the commutative algebra version.

Lemma 1. *Let* $\mathbf{tr_k}(\mathbf{A})$ *be a k-truncated simplicial algebra, and* $\mathbf{cosk_k}(\mathbf{tr_k}(\mathbf{A}))$*, the algebra-theoretic k-coskeleton of* $\mathbf{tr_k}(\mathbf{A})$ *(i.e. calculated within* **Alg***). Then there is a natural epimorphism from* $\mathbf{N}(\mathbf{cosk_k}(\mathbf{tr_k}(\mathbf{A})))$ *to* $t_{k]}\mathbf{NA}$ *with acyclic kernel. Thus* $\mathbf{cosk_k}(\mathbf{tr_k}(\mathbf{A}))$ *and* $t_{k]}(\mathbf{NA})$ *have the same weak homotopy type.*

Proof. Following Conduché [12], the Moore complex of $\mathbf{cosk_k}(\mathbf{tr_k}(\mathbf{A}))$ is given by:

$$\begin{aligned} N(\mathbf{cosk_k}(\mathbf{tr_k}(\mathbf{A})))_l &= 0 \quad \text{if } l > k+1 \\ N(\mathbf{cosk_k}(\mathbf{tr_k}(\mathbf{A})))_{k+1} &= \mathrm{Ker}(\partial_k : NA_k \longrightarrow NA_{k-1}) \\ N(\mathbf{cosk_k}(\mathbf{tr_k}(\mathbf{A})))_l &= NA_l \quad \text{if } l \leq k. \end{aligned}$$

The natural epimorphism gives on Moore complexes

$$\begin{tikzcd}
\mathbf{N}(\mathbf{cosk_k}(\mathbf{tr_k}(\mathbf{A}))) \arrow[r] \arrow[d] & NA_{k+1} \arrow[r] \arrow[d] & NA_k \arrow[r, "\partial_k"] \arrow[d] & NA_{k-1} \arrow[r] & \cdots \\
t_{k]}(\mathbf{NA}): 0 \arrow[r] & 0 \arrow[r] & NA_k/\partial NA_{k+1} \arrow[r] & NA_{k-1} \arrow[r] & \cdots
\end{tikzcd}$$

and it is immediate that the kernel is acyclic as required. □

2-Crossed modules of algebras

Crossed modules of groups were initially defined by Whitehead as models for (homotopy) 2-types. Conduché, [12], in 1984 described the notion of 2-crossed module as a model for 3-types. Both crossed modules and 2-crossed modules have been adapted for use in the context of commutative algebras (cf. [2-4]). Throughout this paper we denote an action of $r \in R$ on $c \in C$ by $r \cdot c$.

A *crossed module* is an algebra morphism $\partial : C \to R$ with an action of R on C satisfying $\partial(r \cdot c) = r\partial c$ and $\partial(c) \cdot c' = cc'$ for all $c, c' \in C, r \in R$.

In this section, we give description of a 2-crossed module (cf. [4]) and a free 2-crossed module of algebras by using the second order Peiffer elements of [2].

We recall from Grandjeán and Vale [4] the definition of 2-crossed module:

A 2-crossed module of **k**-*algebras* consists of a complex of C_0-algebras

$$C_2 \xrightarrow{\partial_2} C_1 \xrightarrow{\partial_1} C_0$$

with ∂_2, ∂_1 morphisms of C_0-algebras, where the algebra C_0 acts on itself by multiplication, such that

$$C_2 \xrightarrow{\partial_2} C_1$$

is a crossed module in which C_1 acts on C_2 via C_0, (we require thus that for all $x \in C_2$, $y \in C_1$ and $z \in C_0$ that $(xy)z = x(yz)$), further, there is a C_0-bilinear function giving

$$\{\ \ \otimes\ \ \}: C_1 \otimes_{C_0} C_1 \longrightarrow C_2,$$

called a Peiffer lifting, which satisfies the following axioms:

$$\begin{array}{llrcl}
PL1: & & \partial_2\{y_0 \otimes y_1\} & = & y_0y_1 - y_0 \cdot \partial_1(y_1), \\
PL2: & & \{\partial_2(x_1) \otimes \partial_2(x_2)\} & = & x_1x_2, \\
PL3 & & \{y_0 \otimes y_1y_2\} & = & \{y_0y_1 \otimes y_2\} + \partial_1 y_2 \cdot \{y_0 \otimes y_1\}, \\
PL4: & a) & \{\partial_2(x) \otimes y\} & = & y \cdot x - \partial_1(y) \cdot x, \\
 & b) & \{y \otimes \partial_2(x)\} & = & y \cdot x, \\
PL5: & & \{y_0 \otimes y_1\} \cdot z & = & \{y_0 \cdot z \otimes y_1\} = \{y_0 \otimes y_1 \cdot z\},
\end{array}$$

for all $x, x_1, x_2 \in C_2$, y, y_0, y_1, $y_2 \in C_1$ and $z \in C_0$.

A morphism of 2-crossed modules of algebras may be pictured by the diagram

$$\begin{tikzcd}
C_2 \arrow[r, "\partial_2"] \arrow[d, "f_2"'] & C_1 \arrow[r, "\partial_1"] \arrow[d, "f_1"'] & C_0 \arrow[d, "f_0"'] \\
C_2' \arrow[r, "\partial_2'"'] & C_1' \arrow[r, "\partial_1'"'] & C_0'
\end{tikzcd}$$

such that $f_0\partial_1 = \partial_1' f_1$, $f_1\partial_2 = \partial_2' f_2$ and such that

$$f_1(c_0 \cdot c_1) = f_0(c_0) \cdot f_1(c_1),\ f_2(c_0 \cdot c_2) = f_0(c_0) \cdot f_2(c_2),$$

and

$$\{\ \ \otimes\ \ \}f_1 \otimes f_1 = f_2\{\ \ \otimes\ \ \},$$

for all $c_2 \in C_2$, $c_1 \in C_1$, $c_0 \in C_0$.

We thus define the category of 2-crossed module denoting it by $\mathbf{X_2Mod}$.

We denote the category of simplicial algebras with Moore complexes of length n by $\mathbf{SimpAlg}_{\leq n}$ in the following.

Some indication of the argument has been given above as it will be adapted dimension 2 later on. First we recall from [6] the following results.

Proposition 2. *Let* **A** *be a simplicial commutative algebra with the Moore complex* **NA**. *Then the complex of algebras*

$$NA_2/\partial_3(NA_3 \cap D_3) \xrightarrow{\partial_2} NA_1 \xrightarrow{\partial_1} NA_0$$

is a 2-crossed module of algebras, where the Peiffer lifting map is defined as follows:

$$\begin{array}{rcl}
\{\ \ \otimes\ \ \}: NA_1 \otimes NA_1 & \longrightarrow & NA_2/\partial_3(NA_3 \cap D_3) \\
(y_0 \otimes y_1) & \longmapsto & s_1y_0(s_1y_1 - s_0y_1).
\end{array}$$

(Here the right hand side denotes a coset in $NA_2/\partial_3(NA_3 \cap D_3)$ *represented by the corresponding element in* NA_2*).* □

Free 2-crossed modules

The definition of a free 2-crossed module is similar in some ways to the corresponding definition of a free crossed module. However, the construction of a free 2-crossed module is naturally a bit more complicated.

It will be helpful to have the notion of a *pre-crossed module:* this is just a homomorphism $\partial : C \to R$ with an action satisfying $\partial(r \cdot c) = r\partial c$ for $c \in C$, $r \in R$.

Let (C, R, ∂) be a pre-crossed module, let Y be a set and let $\nu : Y \to C$ be a function, then (C, R, ∂) is said to be a *free pre-crossed R-module with basis* ν or, alternatively, on the function $\partial\nu : Y \to R$ if for any pre-crossed R-module (C', R, ∂') and function $\nu' : Y \to C'$ such that $\partial'\nu' = \partial\nu$, there is a unique morphism

$$\phi : (C, R, \partial) \to (C', R, \partial')$$

such that $\phi\nu = \nu'$.

The pre-crossed module (C, R, ∂) is *totally free* if R is a free algebra.

We recall that [6] the following construction.

Let $\{C_2, C_1, C_0, \partial_2, \partial_1\}$ be a 2-crossed module, let Y be a set and let $\vartheta : Y \to C_2$, then $\{C_2, C_1, C_0, \partial_2, \partial_1\}$ is said to be a *free 2-crossed module* with basis ϑ or, alternatively, on the function $\partial_2\vartheta : Y \to C_1$, if for any 2-crossed module $\{C_2', C_1, C_0, \partial_2', \partial_1\}$ and function $\vartheta' : Y \to C_2'$ such that $\partial_2\vartheta = \partial_2'\vartheta'$, there is a unique morphism $\Phi : C_2 \to C_2'$ such that $\partial_2'\Phi = \partial_2$.

Remark. 'Freeness' in any setting corresponds to a left adjoint to a forgetful, so what are the categories involved here? □

Let $2CM/PCM$ be the category whose object consists of a pre-crossed module (C, D, ∂) and (Y, ν) where $\nu : Y \to C$ is simply a function to the underlying set of the algebra C such that $\partial\nu = 0$. Morphisms of such object consist of a pair ϕ, ϕ', where $\phi' : Y \to Y'$, is a function such that $\nu'\phi' = \nu\phi$. Forgetting the algebra structure of the top algebra, C_2, of a 2-crossed module provides one with a functor from 2-crossed module to this category. The object of $2CM/PCM$ are thought of as 2-(dimensional) construction data on given pre-crossed module. 2-crossed will be showed that always exits on such 'gadgets' and hence that the forgetful functor describe above has left adjoint.

The describe of this category $2CM/PCM$ may seem a bit artificial, but given an algebra present as a quotient of say, a polynomial algebra by a ideal, the free crossed module of that presentation has been found to contain valuable information on the algebra. Given a crossed module, for example, an arbitrary algebra R together with an ideal I in R, what should one mean by presentation of (R, I) or more generally of a crossed on R? The 'yoga' of crossed algebraic methods suggest several possible replies. For the sake of exposition, let (R, I) be ideal-pair case will be described only. Picking a set generators X for I gives a function $\nu_1 : X \to R$ and hence a free pre-crossed module $(R^+[X], R, \nu_1^\sharp)$ on ν_1. This gives a morphism of pre-crossed modules from $(R^+[X], R, \nu_1^\sharp)$ to (I, R, inc) measuring in part the freeness of I on X. Taking the kernel of this morphism $K \to R$ we pick a set of generators of $K, \nu_2 : Y \to K$ as a pre-crossed module and we have an object of $2CM/PCM$. Thus to analysis an ideal pair homological, one natural method to use is to compare it via a free 2-crossed module, with a free pre-crossed module. This process is based to some extent on the intuition from related CW-complex constructions from topology. André use [1] of simplicial resolutions provides the bridge between the two settings. The sort of 2-construction data one obtains from a simplicial resolution corresponds to a special type of 2-crossed module:

A free 2-crossed module $\{C_2, C_1, C_0, \partial_2, \partial_1\}$ is *totally free* if $\partial_1 : C_1 \to C_0$ is a totally free pre-crossed module.

At the moment, totally free 2-crossed modules seem to be of more immediate use and interest than those that are merely free. We will therefore concentrate on their constructions, but later on will indicate how to adapt that constructions to the more general case.

The following notation and terminology is due to [6]. We give an explicit description of the construction of a totally free 2-crossed module. For this, we need to recall the 2-skeleton of the free simplicial algebra which is

$$\mathbf{A}^{(2)} : \quad \ldots \; (R[s_0(X), s_1(X)])[Y] \underset{s_0, s_1}{\overset{d_0, d_1, d_2}{\rightrightarrows}} R[X] \underset{s_0}{\overset{d_0, d_1}{\rightrightarrows}} R,$$

with the simplicial structure defined as in section 2.2 of [6]. Analysis of this 2-*dimensional construction data,* (cf. [2]), shows that it consists of some 1-dimensional data: namely the function $\varphi : X \to R$ that is used to induce $d_1 : R[X] \to R$, together with strictly 2-dimensional data consisting of the function $\psi : Y \to R^+[X]$ where $R^+[X]$ is the positively graded part of $R[X]$ and which is used to induce d_2 from $R[s_0(X), s_1(X)][Y]$ to $R[X]$. We will denote this 2-dimensional construction data by $(Y, X; \psi, \varphi, R)$.

Theorem 3. *(See [6].) A totally free 2-crossed module* $\{L, A, R, \psi', \partial_1\}$ *exists on the 2-dimensional construction data* $(Y, X; \psi, \varphi, R)$.

Proof. (See [6].) Suppose given the 2-dimensional construction data as above, i.e., given a function $\varphi : X \to R$ and $\psi : Y \to R^+[X]$. Set $A = R^+[X]$. With the obvious action of R on A, the function φ gives a free pre-crossed module

$$\partial_1 : A \longrightarrow R.$$

Now take $D = R[s_0(X)]^+ [s_1(X), Y] \cap (s_0(X) - s_1(X))$, so that A acts on D by multiplication via s_1. The function ψ induces a morphism of A-algebras,

$$\theta : R[s_0(X)]^+ [s_1(X), Y] \cap (s_0(X) - s_1(X)) \longrightarrow R^+[X]$$

given by $\theta(y) = \psi(y)$, (of course $D = NA_2^{(2)}$, the Moore complex of the 2-skeleton of the free simplicial algebra on the data.)

Let $\{B, A, R, \delta, \eta\}$ be any 2-crossed module and let $\vartheta' : Y \to A$. Recall from [2] the second order Peiffer ideal P_2 in D. It is easily checked that $\psi\ (P_2) = 0$ as all generator elements of P_2 are in $\mathrm{Ker}d_2$. By taking the factor algebra $L = D/P_2$, there exists a morphism $\psi' : L \to A$ such that the diagram,

$$\begin{tikzcd} D \arrow[rr, "\theta"] \arrow[dr, "q"'] & & E \\ & L \arrow[ur, "\psi'"'] & \end{tikzcd}$$

commutes, where q is the quotient morphism of algebras. Also ψ' is a crossed module. Indeed, given the elements $y + P_2, y' + P_2 \in L$,

$$\begin{aligned} \psi'(y+P_2)\cdot(y'+P_2) &= \theta(y)\cdot y' + P_2 \\ &= s_1d_2(y)y' + P_2 \\ &\equiv yy' + P_2 \\ &= (y+P_2)(y'+P_2). \end{aligned}$$

Hence there exists a unique morphism $\Phi : L \to A$ given by $\Phi(y+P_2) = \vartheta'(y)$ such that $\delta\Phi = \psi'$. Therefore $\{L, A, R, \psi', \varphi\}$, or the complex

$$\frac{R[s_0(X)]^+\,[s_1(X), Y] \cap (s_0(X) - s_1(X))}{P_2} \xrightarrow{\psi'} R^+[X] \xrightarrow{\varphi} R$$

is the required free 2-crossed module on $(Y, X; \psi, \varphi, R)$. □

Note: In the algebra case, a closely related structure to that of 2-crossed module is that of a *quadratic module,* defined by Baues [13]. Although it seems intuitively clear that the results above should extend to an algebra version of quadratic modules, we have not checked all the details and so have omitted a study of this idea from this paper.

Having given the construction above, we will briefly turn on the more general case of a free 2-crossed module generated by 2-construction data over a given pre-crossed module, i.e. by an object $((C, R, \partial), (Y, \nu))$ of $2CM/PCM$. The pre-crossed module (C, R, ∂) gives us a simplicial algebra $\mathbf{A}$ with $A_0 = R$, $A_1 = C \rtimes R$, $A_2 = C \rtimes C \rtimes R$ and so on. The 2-construction data for gluing in the new 2-generators $\nu : Y \to C$ allows one to form a new simplicial algebra $\mathbf{F}$ with $F_0 = A_0$, $F_1 = A_1$, $F_2 = A_2[Y]$, etc., as the step-by-step constructions of a simplicial resolution, cf. André [1]. The 2-crossed modules associated to $\mathbf{F}$ will be the desired free 2-crossed module on the construction data. The proof is essentially the same as that of Theorem 3 above.

Free 3-crossed modules

The definition of a free quasi 3-crossed module is similar in some ways to the corresponding definition of a free 2-crossed module. However, the construction of a free 3-crossed module is a bit more complicated and will be given by means of the 3-skeleton of a free simplicial algebra with given $CW-$basis.

It will be helpful to have the notion of a *pre-crossed module*: this is just a homomorphism $\partial : C \longrightarrow R$ with an action satisfying $\partial(r \cdot c) = r\partial(c)$ for $c \in C,\ r \in R$.

Let (C, R, ∂) be a pre-crossed module, let S be a set and let $\nu : S \longrightarrow C$ be a function, then (C, R, ∂) is said to be a *free pre-crossed R-module* on ν or alternatively, *on the function* $\partial\nu : S \longrightarrow R$ if for any pre-crossed R-module (C', R, ∂') and function $\nu' : S \longrightarrow C'$ such that $\partial'\nu' = \partial\nu$, there is a unique morphism

$$\Phi : (C, R, \partial) \longrightarrow (C', R, \partial')$$

such that $\phi\nu = \nu'$.

The pre-crossed module (C, R, ∂) is *totally free* if R is a free algebra.

We recall from A. and B. Mutlu [9] that definition of a pre 2-crossed module and quasi 3-crossed modules of algebras.

A *pre-2-crossed modules* of k-algebras consists of complex of C_0-algebras

$$C_2 \xrightarrow{\partial_2} C_1 \xrightarrow{\partial} C_0$$

with ∂_2, ∂_1 morphism of C_0-algebras, where the algebra C_0 acts on itself by multiplication such that

$$C_2 \xrightarrow{\partial_2} C_1$$

is pre crossed module in which C_1 acts on C_2. (we require that for all $x \in C_2$, $y \in C_1$ and $z \in C_0$ that $(xy)z = x(yz)$), further, there is a C_0-bilinear function giving

$$\{\otimes\} : C_1 \otimes C_1 \to C_2$$

called Peiffer lifting, which satisfying the following axioms:

$$\begin{aligned} &2CM1_p \quad \partial_2\{y_0 \otimes y_1\} &&= y_0y_1 - y_0\partial_1(y_1) \\ &2CM2_p \quad \{y_0 \otimes y_1y_2\} &&= \{y_0y_1 \otimes y_2\} + \partial_1(y_2)\,\{y_0 \otimes y_1\} \\ &2CM3_p \quad \{y_0 \otimes y_1\}\cdot z &&= \{y_0 \otimes y_1 \cdot z\} \end{aligned}$$

for all $x, x_1, x_2 \in C_2$, $y, y_0, y_1, y_2 \in C_1$ and, $z \in C_0$.

Let $\{C_2, C_1, C_0, \partial_2, \partial_1\}$ be a free pre 2-crossed module, let Y_2 be a set and let $\vartheta : Y_2 \to C_2$, be a function then $\{C_2, C_1, C_0, \partial_2, \partial_1\}$ is said to be a *free pre 2-crossed module* with basis ϑ or, alternatively, on the function $\partial_2\vartheta : Y_2 \to M$ if for any free pre 2-crossed module $\{C_2', C_1, C_0, \partial_2', \partial_1\}$ and function, $\vartheta' : Y_2 \to L'$ such that $\partial_2\vartheta = \partial_2'\vartheta'$, there is a unique morphism

$$\Phi : C_2 \longrightarrow C_2'$$

such that $\partial_2'\Phi = \partial_2$.

The free pre 2-crossed module $\{C_2, C_1, C_0, \partial_2, \partial_1\}$ is *totally free* if $\partial_1 : C_1 \to C_0$ is a totally free pre-crossed module.

Definition 4. *A quasi 3-crossed module of k-algebras consists of a complex C_0-algebras*

$$C_3 \xrightarrow{\partial_3} C_2 \xrightarrow{\partial_2} C_1 \xrightarrow{\partial_1} C_0$$

with $\partial_3, \partial_2, \partial_1$ morphism of C_0-algebras, where the algebra C_0 acts on itself by multiplication, such that

$$C_3 \xrightarrow{\partial_3} C_2$$

is a crossed module and

$$C_2 \xrightarrow{\partial_2} C_1 \xrightarrow{\partial_1} C_0$$

is a pre 2-crossed module. Thus C_2 acts on C_3 and we require that for all $w \in C_3,\ x \in C_2,\ y \in C_1$ and $z \in C_0$ that

$$(wx)(yz) = \Big(w(x(yz))\Big).$$

Furthermore there is a C_0-equivalent function

$$\{\ \otimes\ \} : C_2 \otimes C_2 \to C_3$$

Mutlu mapping is defined as follows

$$\{\otimes\} = H(c_2 \otimes c_2') = s_1(c_2)s_0(c_2') - s_1(c_2)s_1(c_2') + s_2(c_2)s_2(c_2')$$

if the following conditions are verified.

$3CM1_q$ ∂_2, ∂_1 *are pre-crossed module,* ∂_3 *is a crossed module*

$3CM2_q$ $C_2 \xrightarrow{\partial_2} C_1 \xrightarrow{\partial_1} C_0$ *is a pre 2-crossed module*

$3CM3_q$ $\partial_3 H(c_2 \otimes c_2') = s_1d_2(c_2)s_0d_2(c_2') - s_1d_2(c_2)s_1d_2(c_2') + c_2c_2'$

$3CM4_q$ (a) $H(c_2 \otimes \partial_3(c_3)) = s_2(c_2)c_3$

(b) $H(\partial_3(c_3) \otimes c_2) = s_2(c_2)c_3$

$3CM5_q$ $H(c_2 \otimes \partial_3(c_3))H(\partial_3(c_3) \otimes c_2) = 0$

$3CM6_q$ $H(\partial_3(c_3) \otimes \partial_3(c_3')) = c_3c_3'$

where $c_2, c_2' \in C_2$ and $c_3, c_3' \in C_3$.

We denote such a quasi 3-crossed module of algebras by $\{C_3, C_2, C_1, C_0, \partial_3, \partial_2, \partial_1\}$. A morphism of a quasi 3-crossed modules of algebra may be pictured as a diagram

$$\begin{tikzcd} C_3 \arrow[r, "\partial_3"] \arrow[d, "f_3"'] & C_2 \arrow[r, "\partial_2"] \arrow[d, "f_2"'] & C_1 \arrow[r, "\partial_1"] \arrow[d, "f_1"'] & C_0 \arrow[d, "f_0"'] \\ C_3' \arrow[r, "\partial_3'"'] & C_2' \arrow[r, "\partial_2'"'] & C_1' \arrow[r, "\partial_1'"'] & C_0' \end{tikzcd}$$

of algebras and homomorphisms such that $f_0\partial_1 = \partial_1' f_1$, $f_1\partial_2 = \partial_2' f_2$, $f_2\partial_3 = \partial_3' f_3$ and such that

$$f_1(c_0 \cdot c_1) = {}^{(f_0(c_0))}f_1(c_1), \quad f_2(c_0 \cdot c_2) = {}^{(f_0(c_0))}f_2(c_2), \quad f_2(c_0 \cdot c_3) = {}^{(f_0(c_0))}f_3(c_3)$$

and

$$\{\ \otimes\ \}f_2 \otimes f_2 = f_3\{\ \otimes\ \},$$

for all $c_3 \in C_3$, $c_2 \in C_2$, $c_1 \in C_1$, $c_0 \in C_0$. We thus define the category of quasi 3-crossed modules denoting it by$\mathbf{X_3Mod}$.

Definition 5. *Let $\{C_3, C_2, C_1, C_0, \partial_2, \partial_1\}$ be a quasi 3-crossed module, let Y_3 be a set and let $\vartheta : Y_3 \to K$, be a function then $\{C_3, C_2, C_1, C_0, \partial_3, \partial_2, \partial_1\}$ is said to be a free quasi 3-crossed module on the function $\partial_3\vartheta : Y_3 \to L$ if for a quasi 3-crossed module $\{C_3', C_2, C_1, C_0, \partial', \partial_2, \partial_1\}$ and function, $\vartheta' : Y_3 \to C_3'$ such that $\partial_3\vartheta = \partial'\vartheta'$, there is a unique morphism*

$$\Phi : C_3 \longrightarrow C_3'$$

such that $\partial'\Phi = \partial_3$.

The free quasi 3-crossed module $\{C_3, C_2, C_1, C_0, \partial_3, \partial_2, \partial_1\}$ is totally free if $\{C_2, C_1, C_0, \partial_2, \partial_1\}$ is a totally free pre 2-crossed module.

We shall give an explicit description of the construction of a totally free quasi 3-crossed module. We use step by step construction of André methods together free simplicial algebras with given CW basis of algebras, but Arvasi and Porter did not use free simplicial algebras with given $CW-$basis in [2]. So we use it. For this, we will need to recall the 3-skeleton of a free simplicial algebra which is

$$\mathbf{F}^{(3)} : \ldots F_3^{(3)} \underset{s_1,s_0,s_2}{\overset{d_0,d_1,d_2,d_3}{\rightleftarrows}} F_2^{(3)} \underset{s_1,s_0}{\overset{d_1,d_0,d_2}{\rightleftarrows}} F_1^{(3)} \underset{s_0}{\overset{d_1,d_0}{\rightleftarrows}} F_0^{(3)}$$

where $F_3^{(3)} = F(s_2s_1s_0(X_0) \cup s_2s_0(Y_1)s_2s_1(Y_1) \cup s_2(Y_2) \cup s_1(Y_2) \cup s_0(Y_2) \cup Y_3)$, $F_2^{(3)} = F(s_1s_0(X_0) \cup s_0(Y_1) \cup s_1(Y_1) \cup Y_2)$, $F_1^{(3)} = F(s_0(X_0) \cup (Y_1))$ and $F_0^{(3)} = F(X_0)$ with the simplicial structure defined in dimension 2 as in [7], but we extend to simplicial structure to dimensional 3. Thus we will assume that X_0 and Y_1 are parts of a CW-basis, $\mathfrak{F}$, with $X_0 = \mathfrak{F}_0$, $Y_1 = \mathfrak{F}_1 \setminus s_0(X_0)$, $Y_2 = \mathfrak{F}_2 \setminus (s_0(\mathfrak{F}_1) \cup s_1(\mathfrak{F}_1))$ and $Y_3 = \mathfrak{F}_3 \setminus (s_0(\mathfrak{F}_2) \cup s_1(\mathfrak{F}_2) \cup s_2(\mathfrak{F}_2))$. Analysis of this *2-dimensional construction data*, shows that it consists of some *1-dimensional data:* namely the function $\varphi : Y_1 \longrightarrow F(X_0)$ that is used to induce $d_1 : F(s_0(X_0) \cup Y_1) \longrightarrow F(X_0)$, together with strictly 2-dimensional data consisting of the function $\psi : Y_2 \longrightarrow [\,Y_1]$ where $[\,Y_1]$ is the ideal of Y_1 in $F(s_0(X_0) \cup Y_1)$. This function is used to induce $d_2 : F(s_1s_0(X_0) \cup s_0(Y_1) \cup s_1(Y_1) \cup Y_2) \longrightarrow F(s_0(X_0) \cup Y_1)$ together with strictly 3-dimensional data consisting of the function $\phi : Y_3 \longrightarrow [(s_1(Y_1) \cup Y_2)] \cap [Z \cup Y_2]$ is the ideal of Y_2. We will denote this 3-dimensional construction data by $(Y_3, Y_2, Y_1, \phi, \psi, \varphi, F(X_0))$ where $Z = \{s_1(y_1) - s_0(y_1) :\ y_1 \in Y_1\}$.

Theorem 6. *Let $(Y_3, Y_2, Y_1, F(X_0), \phi, \psi, \varphi)$ be 3-dimension construction data as defined above, then there is a totally free quasi 3-crossed module $\{C_3, C_2, C_1, F(X_0), \partial_3, \partial_2, \partial_1\}$ defined by the data.*

Proof. Given the construction data we construct a 3-truncated simplicial algebra as above. Set $C_1 = [Y_1]$ and $C_2 = [(s_1(Y_1) \cup Y_2)] \cap [Z \cup Y_2]$. With the obvious action of $F(X_0)$ on M, the function φ induces a free pre-crossed module $\mathcal{C} = (C_1, F(X_0), \partial_1)$.

Now set $Z = \{s_1(y_1) - s_0(y_1) : y_1 \in Y_1\}$, $Z_0 = \{s_2s_1(y_1) - s_2s_0(y_1) : y_1 \in Y_1\}$ $Z_1 = \{s_2s_0(y_1) - s_1s_0(y_1) :$

$y_1 \in Y_1\}$ and $Z_2 = \{s_2(y_2) - s_1(y_2) : y_2 \in Y_2\}$ take $D = [s_2(Y_2) \cup s_1(Y_1) \cup Y_3] \cap [s_2(Y_2) \cup Y_3 \cup Z \cup Z_0] \cap [s_0(Y_2) \cup Y_3 \cup Z_1 \cup Z_2]$ the ideal in $F_3^{(3)}$ so that C_2 acts on D by multiplication via s_2. The function θ induces a morphism of

$$\theta : D \longrightarrow C_2,$$

given by $\theta(y) = \psi(y)$, for $y \in Y_3$. (of course $D = NF_3^{(3)}$, part of the Moore complex of the 3-skeleton of the free simplicial algebra on the data).

Recall from [2,14] three dimension Peiffer ideal P_3 in D. This is the ideal of $NF_3^{(3)}$ that is of D, generated by the elements of the form:

$$\partial_4(NF_3^{(3)}) = \sum_{I,J} [K_I, K_J]$$

where $I \cup J = [3]$. For detail see [2,14]. It is easily checked that $\theta(P_3) = 1$ as all generator elements of P_3 are in $\mathrm{Ker} d_3$ Taking the factor element $C_3 = D/P_3$ we get a morphism $\partial_3 : C_3 \to C_2$ such that the diagram,

$$\begin{tikzcd} D \arrow[rr, "\theta"] \arrow[dr, "q"'] & & C_2 \\ & C_3 \arrow[ur, "\psi'"'] & \end{tikzcd}$$

commutes, where q is the quotient morphism.

We will show that $\{C_3, C_2, C_1, F(X_0), \partial_3, \partial_2, \partial_1\}$ i.e. the complex

$$D/P_3 \xrightarrow{\partial_3} C_2 \xrightarrow{\partial_2} C_1 \xrightarrow{\partial_1} F(X_0)$$

is the required free quasi 3-crossed complex module on $(Y_3, Y_2, Y_1, F(X_0), \phi, \psi, \partial_1)$. The Mutlu mapping

$$\{\ \otimes\ \} : C_2 \times C_2 \longrightarrow D/P_3$$

is induced by the map

$$\omega : C_2 \otimes C_2 \longrightarrow D$$

given by

$$\omega(c_2 \otimes c_2') = s_1(c_2)s_0(c_2') - s_1(c_2)s_1(c_2') + s_2(c_2)s_2(c_2').$$

That $\{C_3, C_2, C_1, F(X_0), \partial_3, \partial_2, \partial_1\}$ is a quasi 3-crossed module is now easy to check.

Let $\{C_3', C_2, C_1, F(X_0), \partial'_3, \partial_2, \partial_1\}$ be any quasi 3-crossed module on the pre 2-crossed module $\mathcal{C}$, and let $\varphi' : Y_3 \longrightarrow C_3'$ be such that $\partial'\theta' = \psi$. This function $\theta' : Y_3 \longrightarrow K$ uniquely extends to an C_2-equivalent homomorphism $D \longrightarrow C_3'$ sending conjugation via s_2 or s_1 to the corresponding actions. This extension sends elements of P_3 to trivial elements of C_3', so induces a morphism $\phi : C_3 \longrightarrow C_3'$ satisfying the conditions to make (ϕ, Id, Id) a morphism of quasi 3-crossed modules. Uniqueness is easily verified. □

Remarks.
1) A slight modification of the above will allow the construction of a free quasi 3-crossed module on a function $f : Y_3 \longrightarrow L$ where $(C_2, C_1, F(X_0), \partial_2, \partial_1)$ is a given pre 2-crossed module and $\partial_2 f = 1$. One forms a truncated simplicial algebra from $(C_2, C_1, F(X_0), \partial_2, \partial_1)$ as above, then taking its skeleton, one attaches new basis elements corresponding to the elements of Y_3 using the given function to get d_3.
2) In [13,15], Baues introduces a notion of quadratic module and a related quadratic complex. He gives a construction of a quadratic module from a simplicial algebra in Appendix B of [13] and discusses free and totally free quadratic modules and complexes. The notion can extend quasi 3-quadratic modules and complexes but we omit in this paper.

The data needed by his construction can also be used to give a 3-crossed module, which is not surprising given the close relationship between the two concepts. It thus can also be used to produce a simplicial algebra having that (totally) free quasi 3-crossed module associated to it. This can be to show directly that the Baues construction does produce all (totally) free quadratic modules.

Using simple techniques from combinatorial algebra theory one can obtained more explicit expressions for the intersection

$$\begin{aligned} &[\,s_2(Y_2) \cup s_1(Y_1) \cup Y_3] \cap [\,s_2(Y_2) \cup Y_3 \cup Z \cup Z_0] \\ &\qquad \cap [\,s_0(Y_2) \cup Y_3 \cup Z_1 \cup Z_2] \end{aligned}$$

that allow for the manipulation and interpretation of the resulting quotient by P_3. We believe this could be of particular significance when A is a simplicial resolution of a algebra, say, constructed by a step-by-step method as we outlined in [7] (based on ideas of M. André, see references in [2,7]). This method not only provides a resolution but, if needed, comes with a CW-basis already chosen. Here we will not explore this in depth as it would take us too far away from our main aims, however some related ideas are explored briefly in the next section. □

For $k = 3$ and $n = 0, 1, 2, 3$ the n-Type of k-Skeleton

The n-type of k-skeleton $(n, k = 0, 1, 2)$ examined group case in [16,20] together with free simplicial group with given $CW-$bases. Now we iterated from 2-type of 3-skeleton together with free simplicial algebra with given $CW-$bases.

Assume now given by the 3-skeleton $\mathbf{F}^{(3)}$ of a free simplicial algebra approximating $\mathbf{A}$

$$\mathbf{F}^{(3)} : \ldots\ F_3^{(3)} \underset{s_1,s_0,s_2}{\overset{d_0,d_1,d_2,d_3}{\rightleftarrows}} F_2^{(3)} \underset{s_1,s_0}{\overset{d_1,d_0,d_2}{\rightleftarrows}} F_1^{(3)} \underset{s_0}{\overset{d_1,d_0}{\rightleftarrows}} F_0^{(3)}.$$

As [7] be, one gets the same π for this $\mathbf{F}^{(3)}$

$$\pi_0(\mathbf{F}^{(3)}) \cong F_0^{(3)}/d_1^1(\mathrm{Ker}d_0^1).$$

and considering the morphism $d_1^1 : \mathrm{Ker}d_0^1 \longrightarrow F(X_0)$, one readily obtains $\mathrm{Im}d_1^1 = N$ and $F_0^{(3)} = F(X_0)$. Thus

$$\pi_0(\mathbf{F}^{(3)}) \cong F(X_0)/N \cong F(X_0).$$

But

$$\pi_1(\mathbf{F}^{(3)}) \cong \mathrm{Ker}([s_1(Y_1)]/\partial_2 NF_2^{(3)} \longrightarrow F(X_0)),$$

where $\partial_2 NF_2^{(3)}$ is generated by P_1 and the image $d_2(Y_2)$, thus $\pi_1(\mathbf{F}^{(3)}) \cong \pi_1(\mathbf{A})$. There is an isomorphism

$$\pi_2(\mathbf{F}^{(3)}) \cong \mathrm{Ker}(NF_2^{(3)}/\partial_3(NF_3^{(3)}) \longrightarrow F_1^{(2)}).$$

Since $NF_2^{(3)} = [s_1(Y_1) \cup Y_2] \cap [Z \cup Y_2]$, the second homotopy module of the 3-skeleton looks like

$$\pi_2(\mathbf{F}^{(3)}) \cong \mathrm{Ker}([s_1(Y_1) \cup Y_2] \cap [Z \cup Y_2]/P_2 \longrightarrow F(s_0(X_0) \cup Y_1))$$

where P_2 is the second dimension Peiffer ideal, see [2,14]. This homotopy module will be a module over $\pi_0(\mathbf{A})$ and is a measure of the identities among the level 1 elements of the construction data for the homotopy type of $\mathbf{A}$.

Note that the free 2-crossed modules correspond to the 2-type of the 3-skeleton of free simplicial algebras, and conversely.

$$\pi_3(\mathbf{F}^{(3)}) \cong \mathrm{Ker}(NF_3^{(3)}/\partial_4 NF_4^{(3)} \longrightarrow NF_2^{(3)})$$

where $\partial_4 NF_4^{(3)}$ is generated by P_3 and image $d_3(Y_3)$, thus $\pi_3(\mathbf{F}^{(3)}) \cong \pi_3(\mathbf{A})$ There is an isomorphism

$$\pi_3(\mathbf{F}^{(3)}) \cong \mathrm{Ker}(NF_3^{(3)}/\partial_4 NF_4^{(3)} \longrightarrow NF_2^{(3)})$$

since $NF_3^{(3)} = [s_2(Y_2) \cup s_1(Y_1) \cup Y_3] \cap [s_2(Y_2) \cup Y_3 \cup Z \cup Z_0] \cap [s_0(Y_2) \cup Y_3 \cup Z_1 \cup Z_2]$, the third homotopy module of the 3-skeleton looks like

$$\pi_3(\mathbf{F}^{(3)}) \cong \mathrm{Ker}(D/P_3 \longrightarrow F_2^{(3)})$$

where P_3 is the third dimension Peiffer ideal.

Note that the free quasi 3-crossed modules correspond to the 3-type of the 3-skeleton of free simplicial algebras, and conversely.

Quasi 3-crossed complexes and simplicial algebras

Any simplicial algebra, $\mathbf{A}$., yields a normal chain complex of algebras, namely its Moore complex, $(\mathbf{NA}, \partial)$. Arvasi and Porter, [2], examined the extra structure inherent in a Moore complex that allows the reconstruction of $\mathbf{A}$. from $\mathbf{NA}$. They gave the term hyper-crossed complex to the resulting structure. Crossed complexes themselves, (cf. Brown and Higgins, [17]) correspond to a class of hypercrossed complexes in which nearly all of the extra structure is trivial, so the only non-abelian algebras occur in dimensions 0 and 1 and are linked by a crossed module structure. The other terms are all modules over $NA_0/\partial NA_1$. Thus a crossed complex looks like a crossed modules with a tail that is a chain complex of $\pi_0(A)$-modules. If the original simplicial group is the Kan loop group of a reduced simplicial set, K, then it is well known that the corresponding complex has the 'chains on the universal cover' in dimensions greater than 1 and a free crossed module in the bottom two dimensions. (This is implicit in much of the work of Baues on crossed (chain) complexes, [10,13,15], and was explicitly proved by Ehlers and Porter [18].)

Crossed modules model algebraic 1-types (and hence topological 2-types) and we have recalled from Conduché's work, [12], that 2-crossed modules model algebraic 2-types (and hence topological 3-types). It is thus natural to give these latter models also a 'tail' and to consider '2-crossed complexes' is defined in Mutlu and Porter [16]. Such gadgets are related to the quadratic complexes of Baues, [13,15], in an obvious way. Furthermore A. Mutlu and B. Mutlu defined a quasi 3-crossed modules model 3-types (and hence topological 4-types) in [9]. If we take to based on a definition of quasi 3-crossed modules, we can extend to definition of a quasi 3-crossed module is called as quasi 3-crossed complex, which as follows.

Definition 7. *A quasi 3-crossed complex of algebras is a sequence of algebras*

$$C: \qquad \ldots \to C_n \xrightarrow{\partial_n} C_{n-1} \to \ldots C_3 \xrightarrow{\partial_3} C_2 \xrightarrow{\partial_2} C_1 \xrightarrow{\partial_1} C_0$$

in which

(i) C_0 *acts on* C_n, $n \geq 1$, *the action of* ∂C_1 *being trivial on* C_n *for* $n \geq 4$;
(ii) *each* ∂_n *is a* C_0*-algebra homomorphism and* $\partial_i \partial_{i+1} = 0$ *for all* $i \geq 1$;

and

(iii) $C_3 \xrightarrow{\partial_2} C_2 \xrightarrow{\partial_2} C_1 \xrightarrow{\partial_1} C_0$ *is a quasi 3-crossed module.*

Note that for any quasi 3-crossed module,

$$C_3 \xrightarrow{\partial_3} C_2 \xrightarrow{\partial_2} C_1 \xrightarrow{\partial_1} C_0,$$

$J = Ker\partial_3$ is commutative, since $C_3 \xrightarrow{\partial_3} C_2$ is a crossed module, but more is true. The action of C_2 on C_3 via C_1 restricts to one on J, but by axiom $3CM4_q$, the action is trivial. This implies that the action of C_1 itself on J factors through one of $C_1/\partial_2 C_2$. Thus in any quasi 3-crossed complex,

$$\ldots \to C_5 \to C_4 \to C_3 \to \ker \partial_2$$

is a chain complex of $C_0/\partial_1 C_1$-modules and a quasi 3-crossed complex is just a quasi 3-crossed module with a chain complex as tail added on.

As usual there is a algebras version with C_0 a algebras and each C_n a family of algebras indexed by the objects of C_0.

Given a simplicial algebras, A, define

$$C_n = \begin{cases} NA_n & \text{for } n = 0,1,2 \\ NA_3/d_4(NA_4 \cap D_4) & \text{for } n = 3 \\ (NA_n/(NA_n \cap D_n)) + d_{n+1}(NG_{n+1} \cap D_{n+1}) & \text{for } n \geq 4 \end{cases}$$

with ∂_n induced by the differential of NA. Note that the bottom three terms ($n = 0,1,2$ and 3) form the quasi 3-crossed module already considered in [9] and that for $n \geq 4$, the algebras are all $\pi_0(A)$-modules, since in these dimensions C_n is the same as the corresponding crossed complex term (cf. Ehlers and Porter, [18], for instance or use the hypercrossed complex theory of Carrasco and Cegarra, [19]).

Proposition 8. *With the above structure (C_n, ∂_n) is a quasi 3-crossed complex.*

Proof. The only thing remaining is to check that $\partial_3\partial_4$ is trivial which is straightforward. □

If the simplicial algebra $\mathbf{A}$ is the loop algebra of a simplicial set, X, then for $n \geq 4$, the corresponding quasi 3-crossed complex term, C_n, will be the $(n+1)^{st}$ module of the chains on the universal cover of X since that is the description of the corresponding (and isomorphic) crossed complex term.

The notion of morphism for quasi 3-crossed complexes should be clear. A morphism will be a morphism of graded algebras restricting to a morphism of quasi 3-crossed modules on the bottom three terms and compatible with the action. This will give us a category, $\mathbf{X_3} - \mathbf{Comp}$ of quasi 3-crossed complexes and morphisms between them. We will similarly denote by $\mathbf{X_2} - \mathbf{Comp}$ the category of 2-crossed complexes together with their morphisms. It is easily seen that the construction C is functorial from the category of simplicial algebras to that of quasi 3-crossed complexes.

A quasi 3-crossed complex C will be said to be *free* if for $n \geq 4$, the $C_0/\partial C_1$-modules, C_n are free and the quasi 3-crossed module at the base is a free quasi 3-crossed module. It will be *totally free* if in addition the base quasi 3-crossed module is totally free.

Before turning to a detailed examination of freeness in quasi 3-crossed complexes, we will consider the relation between crossed complexes and quasi 3-crossed complexes.

Suppose

$$\mathcal{C}: \qquad C_3 \overset{\partial_3}{\to} C_2 \overset{\partial_2}{\to} C_1 \overset{\partial_1}{\to} C_0$$

is a 3-truncated crossed complex, then $(C_2, C_1, C_0, \partial_2, \partial_1)$ is a 2-crossed module, C_2 is a module over C_1 on which $\partial_2 C_2$ acts trivially, and $\partial_1\partial_2\partial_3 = 0$.

Lemma 9. *The 3-truncated crossed complex yields a quasi 3-crossed module by taking $\{c \otimes c'\} = 0 \in C_3$ for all $c, c' \in C_2$ and in which the actions of C_2 on C_3 are both trivial.* □

Proof. The proof is easy. □

As a consequence such truncated crossed complexes form a full subcategory of the category of quasi 3-crossed modules, as the two definitions of morphism clearly coincide on this subclass of quasi 3-crossed modules. Passing to the complex version of this one clearly gets:

Proposition 10. *There is a full embedding, E of the category, $\mathbf{X_2} - \mathbf{Comp}$, of 2-crossed complexes into that $\mathbf{X_3} - \mathbf{Comp}$, of quasi 3-crossed complexes.* □

We will think of $\mathbf{X_2} - \mathbf{Comp}$ as a full subcategory of $\mathbf{X_3} - \mathbf{Comp}$ via this embedding.

Theorem 11. *The full subcategory of 2-crossed complexes is a reflexive subcategory of $\mathbf{X_3} - \mathbf{Comp}$.*

Proof. We have to show that the functor, E, thought of now as an inclusion, has a left adjoint, K. We first look at a slightly simpler situation.

Suppose that D is a quasi 3-truncated crossed complex as above, and

$$C: \quad C_3 \to C_2 \to C_1 \to C_0,$$

with morphisms, $\partial_3, \partial_2, \partial_1$ and Mutlu mapping, $\{\ \otimes\ \}$, is a quasi 3-crossed module. If we are given a morphism, $\underline{f} = (f_3, f_2, f_1, f_0)$ of quasi 3-crossed modules, $\underline{f} : C \to \overline{E(D)}$, then if $m_1, m_2 \in C_2$, $f_3\{m_1, m_2\} = 1$ since within $E(D)$, the Mutlu mapping is trivial. This in turn implies that $f_2 < m_1, m_2 >= 1$, where $< m_1, m_2 >= s_1(m_1)s_0(m_2) - s_1(m_1)s_1(m_2) + s_2(m_1)s_2(m_2)$ is the Peiffer multiplication of m_1 and m_2. Thus any morphism $\underline{f}$ from C to $E(D)$ has a kernel that contains the subalgebra, $\{C_2 \otimes C_2\}$ generated by the Mutlu maps in dimension 3, and the Peiffer ideal, P_2 of the pre 2-crossed module, $C_2 \overset{\partial_2}{\to} C_1 \overset{\partial_1}{\to} C_0$ in dimension 2.

We form $K(C)$ as follows:

$$K(C)_0 = C_0$$
$$K(C)_1 = C_1/P_1$$
$$K(C)_2 = C_2/\{C_1 \otimes C_1\}$$
$$K(C)_2 = C_3/\{C_2 \otimes C_2\},$$

with the induced morphisms and actions. The previous discussion makes it clear that $K(C)$ is a quasi 3-truncated crossed complex, and K is clearly functorial. Of course $\underline{f} : C \to E(D)$ yields $K(\underline{f}) : K(C) \to KE(D) \cong D$, so K is the required reflection, at least on this subcategory of truncated objects.

Extending K to all crossed complexes is then simple as we take $K(C)_n = C_n$ if $n \geq 4$ with $K(\partial)_n = \partial_n$ if $n > 4$ and

$$K(\partial)_4 : C_4 \to C_3/\{C_2 \otimes C_2\} = K(C)_3$$

given by the composite of ∂_4 and the quotient from C_3 to $K(C)_3$. The details are easy so will be omitted. □

We thus have functors from the category of simplicial algebras to both $\mathbf{X_2} - \mathbf{Comp}$ and $\mathbf{X_3} - \mathbf{Comp}$ and a relationship between these two categories given by the last result. The first two functors will, for greater precision, be denoted $C^{(2)}$ and $C^{(3)}$, respectively, so that $C^{(2)}$ is studied, for instance, in Ehlers-Porter, [18], and Mutlu-Porter, [20], whilst $C^{(3)}$ was introduced in this section above. We first note that the three functors have the 'right' sort of interrelationship.

Proposition 12.

$$KC^{(3)} \cong C^{(2)}$$

Proof. The key is to identify $\{C_2 \otimes C_2\}$, when $C = C^{(3)}(\mathbf{A})$ for $\mathbf{A}$ a simplicial algebra, but by the results of [2,14], this is $NA_3 \cap D_3$ (cf. [21-23]) or rather its image in the quotient, $C^{(3)}(\mathbf{A})_3$. The result follows since

$$C^{(2)}(\mathbf{A})_3 = \frac{NA_3}{(NA_3 \cap D_3) + d_4(NA_4 \cap D_4)}.$$

□

The functor K preserves freeness.

Proposition 13. *If $\mathcal{C}$ is a (totally) free quasi 3-crossed complex, then $K(C)$ is a (totally) free crossed complex.*

Proof. Above dimension 3, K does nothing and as

$$C_0/\partial C_1 \cong K(C)_0/\partial K(C)_1,$$

the freeness of the modules $K(C)_n$, $n \geq 4$ is not in doubt. In the base quasi 3-crossed module, we have merely to check that $K(C)_3$ is a free $K(C)_0/\partial K(C)_1$-module, as the behaviour of K on $(C_2, C_1, C_0, \partial)$ is just that of the quotienting operation that turns a pre 2-crossed module into a 2-crossed module and this preserves freeness.

Suppose therefore that $C_3 \to C_2 \to C_1 \to C_0$ is a free quasi 3-crossed module with basis $\theta : Y_2 \to C_3$. Suppose also given a module L over $G = C_0/\partial C_1$ and a function $\phi : Y_2 \to C_2$. We need to show that ϕ extends over $K(C)_3$. To do this we construct a quasi 3-crossed complex as follows: The base is the pre 2-crossed module $(C_2, C_1, C_0, \partial_2, \partial_1)$. To complete this we put $Ker\partial_2 \times C_2$ in dimension 3 with as ∂_3 the inclusion on $\mathrm{Ker}\partial_2$ and the trivial map on C_2,

$$D := \{D : \mathrm{Ker}\partial_2 \times L \to C_1 \to C_0\}.$$

The Mutlu mapping is just the product of the Peiffer map from $C_2 \otimes C_2$ to $\mathrm{Ker}\partial_2$ and the axioms are easy to check. Now define $\bar{\phi}$ from our given free quasi 3-crossed module to this one, D, by defining $\bar{\phi}(y) = (\partial\theta y, \phi y)$ for $y \in Y_3$. Compose $\bar{\phi}$ with the obvious projection from D to the crossed complex

$$C_2 \to C_1 \to 0 \to A,$$

where as before, $A = C_0/\partial C_1$. The composed map factors through $K(C)$ giving a morphism $K(C)_3 \to L$ extending ϕ. This is the unique extension of ϕ since at each stage uniqueness was a consequence of the conditions. □

The functor $C^{(3)}$ has a right adjoint, just as does $C^{(2)}$. Given a quasi 3-crossed complex, C, one first constructs the simplicial algebra corresponding to the quasi 3-crossed module at the base, using Mutlu's mapping. We also form the simplicial algebra from the chain complex given by all C_i, $i \geq 3$. The fact that C_3 may be non-abelian does not cause problem but does force semidirect products to be used rather than products. The two parts are then put together via a semidirect product much as in Ehlers and Porter, [18], Proposition 2.4.

Remark. An alternative but equivalent approach follows the route via hypercrossed complexes (cf. Carrasco and Cegarra, [19]), and the extension of the Dold-Kan theorem.

We can also adapt the methods use in Ashley, [24]. In this case NA is a crossed complex if and only if $NA_n \cap D_n$ is always trivial. (In fact, $NA \cong C^{(1)}A$ since $C^{(1)}A$ is obtained by dividing NG_n by $(NG_n \cap D_n) + d_{n+1}(NA_{n+1} \cap D_{n+1})$ in dimension n, and of course this is assumed to be trivial for all $n \geq 2$.)

A similar argument applies if $NA_n \cap D_n$ is trivial for $n \geq 3$. Then $C^{(2)} \simeq NA$, so the Moore complex is a 2-crossed complex in [14]. Also a similar argument applies if $NA_n \cap D_n$ is trivial for $n \geq 4$. Then $C^{(3)} \simeq NA$ so the Moore complex is a quasi 3-crossed complex.

Similar structures have been studied by Duskin, Glenn and Nan Tie under the name of 'n-hypergroupoids' (here $n = 3$). The simplicial algebras that give rise to 2-crossed complexes would seem to be 2-hypergroupoids internal to the category of groups, or a slight variant of such things. We have not investigated this connection in any depth. We hope n-hyperalgebras will be defined that we postponed another article.

The category $\mathbf{X_3} - \mathbf{Comp}$ is clearly equivalent to a reflexive subcategory of the category of hypercrossed complexes of Carrasco and Cegarra, [19], and this completes the chain of linked structures, since this implies once again that $\mathbf{X_3} - \mathbf{Comp}$ is equivalent to a reflexive subcategory (in fact a variety) in the category of simplicial algebras. (Each of these statements is the result of direct verification using the constructions and structures outlined above.)

Given the linkages between the various categories above one would the following: □

Theorem 14. *If F is a free simplicial algebras, then $C^{(3)}F$ is a totally free quasi 3-crossed complex. If $\mathfrak{F}$ is a CW-basis for F, then $\mathfrak{F}$ gives construction data for $C^{(3)}F$.*

Proof. We have already seen that the base quasi 3-crossed module of $C^{(3)}F$ is totally free on construction data derived from the CW-basis. It remains to show that the $C_0/\partial C_1$-modules in higher dimension are free on the corresponding data, but here we can use the case of crossed complexes, and that was proved in [20]. □

Remarks. There are various things to note:

(i) The proof given in [2,14] that if $\mathbf{F}$ is a simplicial resolution of A then $C^{(1)}\mathbf{F}$ is a free crossed resolution of A, does not immediately extend to '3-crossed resolutions'. The notion of 3-crossed resolution clearly would make sense and seems to be need for handling certain problems in algebra extension theory, however we have not given a construction of a tensor product of a pair of quasi 3-crossed complexes and the result for crossed resolutions used $\pi(1) \otimes \_$ where $\pi(1)$ is the free crossed complex on one generator in dimension 1 and thus is also $\pi(\Delta(1))$ the crossed complex of the 1-simplex, and $\otimes$ is the tensor products of crossed complexes defined by Brown and Higgins [17]. This construction could be avoided by using enriched tensor $K\otimes\_$, in the simplicially enriched category of quasi 3-crossed complexes and $\Delta\bar{\otimes}\_$, which should give the same result, but as we have not yet investigated colimits of quasi 3-crossed complexes that construction must also be put off for a future date. It should be pointed out the Baues has in [13] defined a tensor product of totally free quadratic complexes using a fairly obvious construction, so it seems unlikely that the conjectured constructions are technically difficult.

(ii) Although $C^{(3)}\mathbf{F}$ is totally free for $\mathbf{F}$ a free simplicial algebra, it seems almost certain that not all totally free quasi 3-crossed complexes arise in this way. The problem is that in a CW-basis, the new generators are used to build $\pi_n\mathbf{F}$ or $\pi_{n-1}\mathbf{F}$ either as generators or relations. In a quasi 3-crossed complex, the generators at each level influence the *relative* homotopy groups, $\pi_n(\mathbf{F}^{(n)}, \mathbf{F}^{(n-1)})$. This is, of course, more or less equivalent to the realisation problem of Whitehead discussed at length by Baues, [13], but here in a purely algebraic context. Clearly this algebraic realisation problem is important for the analysis of the difference in the homotopical information that can be gleaned from crossed or 3-crossed, as against simplicial, methods. □

Competing interests
The authors declare that they have no competing interests.

Author details
[1]Celal Bayar University, Faculty of Science and Arts, Department of Mathematics Muradiye Campus, 45047, Manisa, Turkey. [2]Hasan Türek Anatolian High School, Mathematics Teacher Manisa, 45020, Turkey.

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