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Dynamics and behaviors of a third-order system of difference equation

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Abstract

Motivated by an Conjecture in the literature "Dynamics of Second Order Rational Difference Equations with Open Problems and Conjecture", we introduce a difference equation system:

$$x_{n+1} = \frac{y_n + y_{n-2}}{x_{n-1}}, y_{n+1} = \frac{x_n + x_{n-2}}{y_{n-1}}$$

where $x_i, y_i \in (0, \infty), i \in \{-2, -1, 0\}$. If the initial value satisfy $x_i = y_i, i = \{-2, -1, 0\}$, then the system reduces into the Conjecture. In this paper, we investigate the asymptotic behaviors, periodicity and oscillatory of the system.

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Introduction

Recently, there has been great interest in studying difference equation systems. One of the reasons for this is the necessity for some techniques that can be used in investigating equations arising in mathematical models describing real life situations in population biology, economics, probability theory, etc. There are many papers related to the difference equations system for example, such as [1-10].

Cinar [1] studied the solutions of the system of difference equations:

$$x_{n+1} = \frac{1}{y_n}, \quad y_{n+1} = \frac{y_n}{x_{n-1}y_{n-1}}, \quad n = 0, 1, \cdots, \tag{1}$$

Camouzis and Papaschinnopoulos [2] studied the global asymptotic behavior of positive solution of the system of rational difference equations:

$$x_{n+1} = 1 + \frac{x_n}{y_{n-m}}, \quad y_{n+1} = 1 + \frac{y_n}{x_{n-m}}, \quad n = 0, 1, \cdots, \tag{2}$$

Ozban [3] studied the system of rational difference equations:

$$x_n = \frac{a}{y_{n-3}}, \quad y_n = \frac{by_{n-3}}{x_{n-q}y_{n-q}}, \quad n = 0, 1, \cdots, \tag{3}$$

Kurbanli et al. [4] studied the behavior of positive solutions of the system of rational difference equations:

$$x_n = \frac{x_{n-1}}{y_n x_{n-1} + 1}, \quad y_n = \frac{y_{n-1}}{x_n y_{n-1} + 1}, \quad n = 0, 1, \cdots. \tag{4}$$

In the monograph of *Dynamics of second order rational difference equations with open problems and conjecture* [11], Kulenović and Ladas give a conjecture (see [11] p196) as following:

Conjecture 11.4.10. Show that every positive solution of the equation

$$x_{n+1} = \frac{x_n + x_{n-2}}{x_{n-1}}, \quad n = 0, 1, \cdots, \tag{5}$$

converges to a period-four solution.

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Stević [12] has solved the Conjecture.

In this paper, we introduce the difference equation system

$$x_{n+1} = \frac{y_n + y_{n-2}}{x_{n-1}}, y_{n+1} = \frac{x_n + x_{n-2}}{y_{n-1}}, \quad n = 0, 1, \cdots \tag{6}$$

where $x_i, y_i \in (0, \infty), i \in \{-2, -1, 0\}$.

Obviously, we can see that if the initial value satisfy $x_i = y_i$ for $i \in \{-2, -1, 0\}$, then the system (6) reduces into the Eq. (5). Hence, we can also solve the Conjecture 11.4.10. However, we study the system (6).

In this paper, we can obtain the solution of system (6) converges to periodic solution. At the same time, we can get the oscillatory of the system (6).

Before giving some results of the system (6), we need some definitions as follows [5]:

Definition 1. A pair of sequences of positive real numbers $\{x_n, y_n\}_{n=-3}^{\infty}$ that satisfies system (6) is a positive solution of system (6). If a positive solution of system (6) is a pair of positive constants $(\bar{x}, \bar{y})$, that solution is the equilibrium solution.

Definition 2. A 'string' of consecutive terms $\{x_s, \cdots, x_m\}$(resp. $\{y_s, \cdots, y_m\}$),($s \geq -3$, $m \leq \infty$) is said to be a positive semicycle if $x_i \geq \bar{x}$(resp. $y_i \geq \bar{y}$), $\{i \in s, \cdots, m\}$, $x_{s-1} < \bar{x}$(resp. $y_{s-1} < \bar{y}$), and $x_{m+1} < \bar{x}$(resp. $y_{m+1} < \bar{y}$). Otherwise, that is said to be a negative semicycle.

A 'string' of consecutive terms $\{(x_s, y_s), \cdots, (x_m, y_m)\}$ is said to be a positive(resp.negative) semicycle if $\{x_s, \cdots, x_m\}$, $\{y_s, \cdots, y_m\}$ are positive (resp.negative) semicycle.

A solution $\{x_n\}$(resp.$\{y_n\}$) oscillates about $\bar{x}$(*resp.*$\bar{y}$) if for every $i \in N$, there exist $s, m \in N, s \leq i, m \geq i$, such that $(x_s - \bar{x})(x_m - \bar{x}) < 0$(resp. $(y_s - c)(y_m - c) < 0$). We say that a solution $\{x_n, y_n\}_{n=-3}^{\infty}$ of system oscillates about$(\bar{x}, \bar{y})$ if $\{x_n\}$ oscillates about $\bar{x}$ or $\{y_n\}$ oscillates about $\bar{y}$.

Some Lemmas

We now present some Lemmas which will be usefully in the proof of the following Theorems.

Lemma 1. *The system of* (6) *has a unique positive equilibrium* $\bar{x} = \bar{y} = 2$.

The proof of Lemma 1 is very easy , thus we omit it.

Lemma 2. *Assume that* $p > 0$, $q > 0$, $pm > 1$, $r > 0$, $m > 0$, $rq > 1$. *Then* $\{p, q\}$, $\{r, m\}$, $\left\{\frac{m+p^2}{pm-1}, \frac{q+r^2}{rq-1}\right\}$, $\left\{\frac{r+q^2}{rq-1}, \frac{p+m^2}{pm-1}\right\}, \cdots$, $\{p, q\}$,$\{r, m\}$,$\left\{\frac{m+p^2}{pm-1}, \frac{q+r^2}{rq-1}\right\}$, $\left\{\frac{r+q^2}{rq-1}, \frac{p+m^2}{pm-1}\right\}$ *is a period-four solution of the system (6).*

Proof. Let $\{p_1, p_2\}$,$\{p_3, p_4\}$,$\{p_5, p_6\}$,$\{p_7, p_8\}$,$\cdots$,$\{p_1, p_2\}$, $\{p_3, p_4\}$,$\{p_5, p_6\}$,$\{p_7, p_8\}$ $\cdots$ be a period-four solution of system (6).

Then by Eq. (6), we obtain

$$\begin{cases} p_1 = \frac{p_4+p_8}{p_5} \\ p_4 = \frac{p_1+p_5}{p_8} \\ p_5 = \frac{p_4+p_8}{p_1} \\ p_8 = \frac{p_1+p_5}{p_4} \end{cases} \begin{cases} p_2 = \frac{p_3+p_7}{p_6} \\ p_3 = \frac{p_2+p_6}{p_7} \\ p_6 = \frac{p_3+p_7}{p_2} \\ p_7 = \frac{p_2+p_6}{p_3} \end{cases} \tag{7}$$

Noting that Eq. (7) can be changed into

$$\begin{cases} p_1p_5 = p_4 + p_8 \\ p_4p_8 = p_1 + p_5 \\ p_2p_6 = p_3 + p_7 \\ p_3p_7 = p_2 + p_6 \end{cases} \tag{8}$$

We suppose that $p_1 = p, p_2 = q, p_3 = r, p_4 = m.$

Then by Eq. (8) we can get

$$p_5 = \frac{m + p^2}{pm - 1}, p_6 = \frac{q + r^2}{rq - 1}, p_7 = \frac{r + q^2}{rq - 1}, p_8 = \frac{p + m^2}{pm - 1}.$$

Hence, we complete the proof. □

Lemma 3. *Assume that the initial points* $x_i, y_i \in (0, \infty), i \in \{-2, -1, 0\}$, $\{x_n, y_n\}_{n=-2}^{\infty}$ *is a positive solution of system* (6). *Then the following statements are true:*

(a) *If* $y_{-1}^2 + x_0 + x_{-2} < x_{-2}y_{-1}x_0$, $x_{-1}^2 + y_0 + y_{-2} < y_{-2}x_{-1}y_0$, *then* $\{x_{4n+i}, y_{4n+i}\}_{n=0}^{\infty}$ *is decreasing, where* $i \in \{-2, -1, 0, 1\}$.

(b) *If* $y_{-1}^2 + x_0 + x_{-2} = x_{-2}y_{-1}x_0$, $x_{-1}^2 + y_0 + y_{-2} = y_{-2}x_{-1}y_0$, *then* $x_{4n-2} = x_{-2}$, $x_{4n-1} = x_{-1}, x_{4n} = x_0$ *and* $x_{4n+1} = x_1; y_{4n-2} = y_{-2}$, $y_{4n-1} = y_{-1}, y_{4n} = y_0$ *and* $y_{4n+1} = y_1, \quad n = 0, 1, \cdots$.

(c) *If* $y_{-1}^2 + x_0 + x_{-2} > x_{-2}y_{-1}x_0$, $x_{-1}^2 + y_0 + y_{-2} > y_{-2}x_{-1}y_0$, *then* $\{x_{4n+i}, y_{4n+i}\}_{n=0}^{\infty}$ *is increasing, where* $i \in \{-2, -1, 0, 1\}$.

(d) *If* $y_{-1}^2 + x_0 + x_{-2} \leq x_{-2}y_{-1}x_0$, $x_{-1}^2 + y_0 + y_{-2} \geq y_{-2}x_{-1}y_0$, *then* $\{x_{4n-2}\}_{n=0}^{\infty}$, $\{x_{4n}\}_{n=0}^{\infty}$, $\{y_{4n-1}\}_{n=0}^{\infty}$, $\{y_{4n+1}\}_{n=0}^{\infty}$ *is decreasing, and* $\{y_{4n-2}\}_{n=0}^{\infty}$, $\{y_{4n}\}_{n=0}^{\infty}$, $\{x_{4n-1}\}_{n=0}^{\infty}$, $\{x_{4n+1}\}_{n=0}^{\infty}$ *is increasing.*

(e) *If* $y_{-1}^2 + x_0 + x_{-2} \geq x_{-2}y_{-1}x_0$, $x_{-1}^2 + y_0 + y_{-2} \leq y_{-2}x_{-1}y_0$, *then* $\{x_{4n-2}\}_{n=0}^{\infty}$, $\{x_{4n}\}_{n=0}^{\infty}$, $\{y_{4n-1}\}_{n=0}^{\infty}$, $\{y_{4n+1}\}_{n=0}^{\infty}$ *is increasing, and* $\{y_{4n-2}\}_{n=0}^{\infty}$, $\{y_{4n}\}_{n=0}^{\infty}$, $\{x_{4n-1}\}_{n=0}^{\infty}$, $\{x_{4n+1}\}_{n=0}^{\infty}$ *is decreasing.*

Proof. Part(a):

By system (6), we can get

$$x_{n+1} = \frac{y_n + y_{n-2}}{x_{n-1}} = \frac{y_n + y_{n-2}}{\frac{y_{n-2}+y_{n-4}}{x_{n-3}}},$$

$$y_{n+1} = \frac{x_n + x_{n-2}}{y_{n-1}} = \frac{x_n + x_{n-2}}{\frac{x_{n-2}+x_{n-4}}{y_{n-3}}}$$

i.e.

$$\frac{x_{n+3}}{x_{n-1}} = \frac{y_n + y_n}{y_n + y_{n-2}}, \quad \frac{y_{n+3}}{y_{n-1}} = \frac{x_n + x_n}{x_n + x_{n-2}}, \tag{9}$$

for $n \geq 0$.

By the condition $y_{-1}^2 + x_0 + x_{-2} < x_{-2}y_{-1}x_0$, $x_{-1}^2 + y_0 + y_{-2} < y_{-2}x_{-1}y_0$, we can obtain

$$x_2 = \frac{y_{-1} + y_1}{x_0} = \frac{y_{-1} + \frac{x_{-2}+x_0}{y_{-1}}}{x_0} = \frac{{y_{-1}}^2 + x_{-2} + x_0}{y_-1x_0} < x_{-2}$$

$$y_2 = \frac{x_{-1} + x_1}{y_0} = \frac{x_{-1} + \frac{y_{-2}+y_0}{x_{-1}}}{y_0} = \frac{{x_{-1}}^2 + y_{-2} + y_0}{x_-1x_0} < y_{-2}$$

Next, by Eq. (9), we can get

$$y_3 < y_{-1}, \quad x_4 < x_0, \quad y_5 < y_1, \quad x_6 < x_2, \quad y_7 < y_3,$$
$$x_8 < x_4, \quad y_9 < y_5, \quad x_{10} < x_6, \quad \cdots$$

$$x_3 < x_{-1}, \quad y_4 < y_0, \quad x_5 < x_1, \quad y_6 < y_2, \quad x_7 < x_3,$$
$$y_8 < y_4, \quad x_9 < x_5, \quad y_{10} < y_6, \quad \cdots$$

Therefore by induction, we can get $\left\{x_{4n+i}, y_{4n+i}\right\}_{n=0}^{\infty}$ is decreasing for $i \in \{-2, -1, 0, 1\}$.

Part(b): By system (6), we can easily prove that part(b) holds. Hence, we omit the proof of part(b).

Using the same method in the proof of part(a), we can prove part(c) also holds.

Part(d): By the condition $y_{-1}^2 + x_0 + x_{-2} \leq x_{-2}y_{-1}x_0$, $x_{-1}^2 + y_0 + y_{-2} \geq y_{-2}x_{-1}y_0$, we obtain

$$x_2 = \frac{y_{-1} + y_1}{x_0} = \frac{y_{-1} + \frac{x_{-2}+x_0}{y_{-1}}}{x_0} = \frac{{y_{-1}}^2 + x_{-2} + x_0}{y_{-1}x_0} \leq x_{-2}$$

$$y_2 = \frac{x_{-1} + x_1}{y_0} = \frac{x_{-1} + \frac{y_{-2}+y_0}{x_{-1}}}{y_0} = \frac{{x_{-1}}^2 + y_{-2} + y_0}{x_{-1}x_0} \geq y_{-2}$$

Then, by Eq. (9), we can get

$$y_3 \leq y_{-1}, \quad x_4 \leq x_0, \quad y_5 \leq y_1, \quad x_6 \leq x_2, \quad y_7 \leq y_3,$$
$$x_8 \leq x_4, \quad y_9 \leq y_5, \quad x_{10} \leq x_6, \quad \cdots$$
$$x_3 \geq x_{-1}, \quad y_4 \geq y_0, \quad x_5 \geq x_1, \quad y_6 \geq y_2, \quad x_7 \geq x_3,$$
$$y_8 \geq y_4, \quad x_9 \geq x_5, \quad y_{10} \geq y_6, \quad \cdots$$

Therefore by induction, we can get part(d) also holds.

Part(e): The proof is similar with the part(d), so we omit it.

Hence, we complete the proof of Lemma 3. □

Lemma 4. *Assume that $x_i, y_i \in (2, \infty), i \in \{-2, -1, 0\}$. Then there does not exist a positive solution $\left\{x_n, y_n\right\}_{n=-2}^{\infty}$ of the system (6) such that $\left\{x_{4n+i}, y_{4n+i}\right\}_{n=0}^{\infty}$ is increasing, where $i \in \{-2, -1, 0, 1\}$.*

Proof. Assume, for the sake of contradiction, that there exists a positive solution $\left\{x_n, y_n\right\}_{n=-2}^{\infty}$, such that $\left\{x_{4n+i}, y_{4n+i}\right\}_{n=0}^{\infty}$ is increasing, where $i \in \{-2, -1, 0, 1\}$, $x_i, y_i \in (2, \infty), i \in \{-2, -1, 0\}$.

By system (6), we can get

$$\begin{cases} x_{n+1} = \frac{y_n+y_{n-2}}{x_{n-1}} \leq \frac{y_n+y_{n-2}}{2} \leq \max\left\{y_n, y_{n-2}\right\} \\ y_{n+1} = \frac{x_n+x_{n-2}}{y_{n-1}} \leq \frac{x_n+x_{n-2}}{2} \leq \max\left\{x_n, x_{n-2}\right\} \end{cases} \tag{10}$$

Then, we can get

$$\begin{cases} x_{n+1} \leq \max\left\{x_{n-1}, x_{n-3}, x_{n-5}\right\} \\ y_{n+1} \leq \max\left\{y_{n-1}, y_{n-3}, y_{n-5}\right\}. \end{cases} \tag{11}$$

Because of $x_{n-5} < x_{n-1}$, we can get $x_{n+1} \leq \max\left\{x_{n-1}, x_{n-3}\right\}$.

More, we can get

$$\begin{cases} x_{n+1} \leq \max\left\{x_{n-1}, x_{n-3}\right\} \\ x_{n+2} \leq \max\left\{x_n, x_{n-2}\right\} \\ x_{n+3} \leq \max\left\{x_{n+1}, x_{n-1}\right\} \\ x_{n+4} \leq \max\left\{x_{n+2}, x_n\right\}, \end{cases}$$

which can be written as

$$\begin{cases} x_{n+1} \leq \max\left\{x_{n-1}, x_{n-3}\right\} \\ x_{n+2} \leq \max\left\{x_n, x_{n-2}\right\} \\ x_{n+3} \leq \max\left\{x_{n-1}, x_{n-3}\right\} \\ x_{n+4} \leq \max\left\{x_n, x_{n-2}\right\}. \end{cases}$$

If $x_{n-1} \geq x_{n-3}$, then $x_{n+3} \leq x_{n-1}$, which is contradiction;

If $x_{n-3} \geq x_{n-1}$, then $x_{n+1} \leq x_{n-3}$, which is contradiction;

If $x_n \geq x_{n-2}$, then $x_{n+4} \leq x_n$, which is contradiction;

If $x_{n-2} \geq x_n$, then $x_{n+2} \leq x_{n-2}$, which is contradiction.

They are contradictions and we complete the proof. □

Lemma 5. *Assume that $x_i, y_i \in (0, 2), i = \{-2, -1, 0\}$. Then there does not exist a positive solution $\left\{x_n, y_n\right\}_{n=-2}^{\infty}$ of the system (6) such that $\left\{x_{4n+i}, y_{4n+i}\right\}_{n=0}^{\infty}$ is decreasing, where $i \in \{-2, -1, 0, 1\}$.*

The proof of Lemma 5 is similar with the proof of Lemma 4, so we omit it.

Results and discussion

In this section, we study five cases of the solution of the system (6). We get the solution of system (6) eventually converges to period-four solution.

Theorem 1. *Assume that the initial points $x_i, y_i \in (0, \infty), i \in \{-2, -1, 0\}$, $\left\{x_n, y_n\right\}_{n=-2}^{\infty}$ is a positive solution of the system (6). Then the following statement are true:*

If $y_{-1}^2 + x_0 + x_{-2} < x_{-2}y_{-1}x_0$, $x_{-1}^2 + y_0 + y_{-2} < y_{-2}x_{-1}y_0$, then $\left\{x_{4n-i}, y_{4n-i}\right\}_{n=0}^{\infty}$ is decreasing, where $i \in \{-2,-1,0,1\}$, and $\left\{x_n, y_n\right\}_{n=-2}^{\infty}$ converges to a period-four solution as following

$$\left\{p_1,p_5\right\}\left\{p_2,p_6\right\}\left\{p_3,p_7\right\}\left\{p_4,p_8\right\},\cdots,\quad \left\{p_1,p_5\right\}\left\{p_2,p_6\right\}\left\{p_3,p_7\right\}\left\{p_4,p_8\right\},\cdots.$$

Proof. By Lemma 3(a), we can get

$\left\{x_{4n-i}, y_{4n-i}\right\}_{n=0}^{\infty}$ is decreasing for $i \in \{-2,-1,0,1\}$

Because 'Monotone bounded sequence must have limit,' we can set

$$\lim_{n\to\infty}\left\{x_{4n-2},y_{4n-2}\right\} = (p_1,p_2), \quad \lim_{n\to\infty}\left\{x_{4n-1},y_{4n-1}\right\} = (p_3,p_4).$$
$$\lim_{n\to\infty}\left\{x_{4n},y_{4n}\right\} = (p_5,p_6), \quad \lim_{n\to\infty}\left\{x_{4n+1},y_{4n+1}\right\} = (p_7,p_8)$$

With no loss of generality, we consider these three cases as followings:

(a) $p_1 = p_2 = p_3 = p_4 = p_5 = p_6 = p_7 = p_8 = 0$
(b) $\sum_{i=1}^{8} p_i^2 > 0, \quad \prod_{i=1}^{8} p_i = 0$
(c) $\sum_{i=1}^{8} p_i^2 > 0, \quad \prod_{i=1}^{8} p_i > 0$

Now, if we can prove case(a) and cade(b) do not hold, then we can obtain only case(c) holds.
By Lemma 5, we know that case(a) does not hold.
Next we try to prove the case(b) do not hold.
By the system (6), we can get

$$\begin{cases} x_{n+1}x_{n-1} = y_n + y_{n-2} \\ y_{n+1}y_{n-1} = x_n + x_{n-2}. \end{cases} \tag{12}$$

By limiting both sides of Eq. (12), we can get

$$\begin{cases} \lim_{n\to\infty} x_{n+1}x_{n-1} = \lim_{n\to\infty} y_n + \lim_{n\to\infty} y_{n-2} \\ \lim_{n\to\infty} y_{n+1}y_{n-1} = \lim_{n\to\infty} x_n + \lim_{n\to\infty} x_{n-2}. \end{cases} \tag{13}$$

If $\prod_{i=1}^{8} p_i = 0$, the identical Equation (12) do not hold. Hence, case(b) does not hold.
At last, only case(c) holds. We complete the proof. □

Corollary 1. Suppose that $\left\{x_n, y_n\right\}_{n=-2}^{\infty}$ is a positive solution of system (6). Then the following statement is true:
If $y_{-1}^2+x_0+x_{-2} < x_{-2}y_{-1}x_0$, $x_{-1}^2+y_0+y_{-2} < y_{-2}x_{-1}y_0$, then the solution of system (6) oscillates about equilibrium $(\bar{x},\bar{y})$. Besides the positive semicycle have at most three terms, at least one term; the negative semicycle have at most three terms and at least one term.

Theorem 2. *Assume that the initial points $x_i, y_i \in (0,\infty), i \in \{-2,-1,0\}$, $\left\{x_n, y_n\right\}_{n=-2}^{\infty}$ are a positive solution of the system (6). Then the following statements are true:*
If $y_{-1}^2 + x_0 + x_{-2} = x_{-2}y_{-1}x_0$, $x_{-1}^2 + y_0 + y_{-2} = y_{-2}x_{-1}y_0$, then $\left\{x_{-2},y_{-2}\right\}$, $\left\{x_{-1},y_{-1}\right\}$, $\left\{x_0,y_0\right\}$, $\left\{x_1,y_1\right\}$, $\cdots$, $\left\{x_{-2},y_{-2}\right\}$, $\left\{x_{-1},y_{-1}\right\}$, $\left\{x_0,y_0\right\}$, $\left\{x_1,y_1\right\}$, $\cdots$ is a period-four solution of the system (6), where $x_0 = \frac{x_{-2}+x_{-1}^2}{x_{-2}x_{-1}-1}$, $x_1 = \frac{x_{-2}^2+x_{-1}}{x_{-2}x_{-1}-1}$.

Proof. By the Lemma 8 and Lemma 3(b), we can prove the Theorem 2. Hence, we omit it. □

Theorem 3. *Assume that the initial points $x_i, y_i \in (0,\infty), i \in \{-2,-1,0\}$, $\left\{x_n, y_n\right\}_{n=-2}^{\infty}$ are a positive solution of the system (6). Then the following statements are true:*
If $y_{-1}^2 + x_0 + x_{-2} > x_{-2}y_{-1}x_0$, $x_{-1}^2 + y_0 + y_{-2} > y_{-2}x_{-1}y_0$, then $\left\{x_{4n-i}, y_{4n-i}\right\}_{n=0}^{\infty}$ is decreasing, where $i \in \{-2,-1,0,1\}$ and $\left\{x_n, y_n\right\}_{n=-2}^{\infty}$ converges to a period-four solution as following:

$$\left\{p_1,p_5\right\}\left\{p_2,p_6\right\}\left\{p_3,p_7\right\}\left\{p_4,p_8\right\},\cdots,\quad \left\{p_1,p_5\right\}\left\{p_2,p_6\right\}\left\{p_3,p_7\right\}\left\{p_4,p_8\right\},\cdots.$$

Proof. By Lemma 3(c), we can get:

$\left\{x_{4n-i}, y_{4n-i}\right\}_{n=0}^{\infty}$ is increasing for $i \in \{-2,-1,0,1\cdots\}$.

Then, by Eq. (3), we can get

$$\begin{cases} \dfrac{x_{n+3}}{x_{n-1}} = \dfrac{y_{n+2}+y_n}{y_n+y_{n-2}} < \dfrac{y_{n+2}}{y_{n-2}} \\ \dfrac{y_{n+3}}{y_{n-1}} = \dfrac{x_{n+2}+x_n}{x_n+x_{n-2}} < \dfrac{x_{n+2}}{x_{n-2}} \end{cases} \tag{14}$$

i.e.

$$\frac{x_{n-2}}{x_{n+2}} < \frac{y_{n-1}}{y_{n+3}} < \frac{x_n}{x_{n+4}} < \frac{y_{n+1}}{y_{n+5}} \tag{15}$$

$$\frac{y_{n-2}}{y_{n+2}} < \frac{x_{n-1}}{x_{n+3}} < \frac{y_n}{y_{n+4}} < \frac{x_{n+1}}{x_{n+5}} \tag{16}$$

Next, we can get:

$$\frac{y_{-2}}{y_2} < \frac{x_{-1}}{x_3} < \frac{y_0}{y_4} < \frac{x_1}{x_5} < \frac{y_2}{y_6} < \frac{x_3}{x_7} < \frac{y_4}{y_8} < \frac{x_5}{x_9} < \frac{y_6}{y_{10}} < \frac{x_7}{x_{11}} < \frac{y_8}{y_{12}}\cdots$$

$$\frac{x_{-2}}{x_2} < \frac{y_{-1}}{y_3} < \frac{x_0}{x_4} < \frac{y_1}{y_5} < \frac{x_2}{x_6} < \frac{y_3}{y_7} < \frac{x_4}{x_8} < \frac{y_5}{y_9} < \frac{x_6}{x_{10}} < \frac{y_7}{y_{11}} < \frac{x_8}{x_{12}}\cdots$$

Besides, if $0 < a < b < c < d$, we can get $ac < bc < bd$. Hence by induction and the above inequality, we can get:

$$\frac{y_{-2}}{y_{4n+2}} < \frac{x_{-1}}{x_{4n+3}} < \frac{y_0}{y_{4n+4}} < \frac{x_1}{x_{4n+5}} < \frac{y_2}{y_{4n+6}} < \frac{x_3}{x_{4n+7}} \tag{17}$$

$$\frac{x_{-2}}{x_{4n+2}} < \frac{y_{-1}}{y_{2n+3}} < \frac{x_0}{x_{4n+4}} < \frac{y_1}{y_{4n+5}} < \frac{x_2}{x_{4n+6}} < \frac{y_3}{y_{4n+7}} \tag{18}$$

We can set $\lim_{n\to\infty} x_{4n+1} = p_1, \lim_{n\to\infty} x_{4n+2} = p_2, \lim_{n\to\infty} x_{4n+3} = p_3, \lim_{n\to\infty} x_{4n+4} = p_4, \lim_{n\to\infty} y_{4n+1} = p_5, \lim_{n\to\infty} y_{4n+2} = p_6, \lim_{n\to\infty} y_{4n+3} = p_7, \lim_{n\to\infty} y_{4n+4} = p_8$.

From Lemma 10, we know that there at least one $p_i < 2$. Then by Limiting Theorem, we can get at least one of the limiting of p_i must exist. With no loss ordinary, we set the limit of $\{x_{4n+1}\}$ exists, i.e. $p_1 < 2$.

By limiting the inequation (17), we can get

$$\lim_{n\to\infty} \frac{y_{-2}}{y_{4n+2}} \le \lim_{n\to\infty} \frac{x_{-1}}{x_{4n+3}} \le \lim_{n\to\infty} \frac{y_0}{y_{4n+4}} \le \lim_{n\to\infty} \frac{x_1}{x_{4n+5}} \le \lim_{n\to\infty} \frac{y_2}{y_{4n+6}} \tag{19}$$

Hence, we can get the limits of $\{y_{4n+2}\}$, $\{x_{4n+3}\}$ and $\{y_{4n+4}\}$ are all exists. We can prove $p_3 < \infty, p_6 < \infty$, and $p_8 < \infty$.

From Eq (18), we can get:

$$\frac{x_{-2}}{y_{-1}} < \frac{x_{4n+2}}{y_{4n+3}} < \frac{x_{4n+6}}{y_{4n+3}} < \frac{x_2}{y_{-1}},$$

$$\frac{x_0}{y_1} < \frac{x_{4n+4}}{y_{4n+5}} < \frac{x_{4n+8}}{y_{4n+5}} < \frac{x_4}{y_1}$$

Hence, we can obtain $\lim_{n\to\infty} \frac{x_{4n+2}}{y_{4n+3}} = k, \lim_{n\to\infty} \frac{x_{4n+4}}{y_{4n+5}} = l$, and $0 < k < \infty, 0 < l < \infty$

Assume that $\lim_{n\to\infty} x_{4n+2} = \infty, \lim_{n\to\infty} x_{4n+4} = \infty, \lim_{n\to\infty} y_{4n+1} = \infty, \lim_{n\to\infty} y_{4n+3} = \infty$.

By the system (6), we can get

$$x_{4n+2}x_{4n} = y_{4n+1} + y_{4n-1}$$

which can be changed into

$$1 = \frac{y_{4n+1} + y_{4n-1}}{x_{4n+2}x_{4n}}. \tag{20}$$

By limiting the both side of Eq. (20), we can get the right side of equation is $\lim_{n\to\infty} \frac{y_{4n+1}+y_{4n-1}}{x_{4n+2}x_{4n}} = 0$, which is contradiction.

Hence, use the same method, we can get $\lim_{n\to\infty} x_{4n+2} < \infty, \lim_{n\to\infty} x_{4n+4} < \infty, \lim_{n\to\infty} y_{4n+1} < \infty, \lim_{n\to\infty} y_{4n+3} < \infty$.
i.e.

$$p_i < +\infty, i = \{1, 2, 3, 4, 5, 6, 7, 8\}$$

Therefore, we complete the proof. □

Corollary 2. Suppose that $\{x_n, y_n\}_{n=-2}^{\infty}$ is a positive solution of system (6). Then the following statement is true:

If $y_{-1}^2 + x_0 + x_{-2} > x_{-2}y_{-1}x_0$, $x_{-1}^2 + y_0 + y_{-2} > y_{-2}x_{-1}y_0$, then the solution of system (6) oscillates about equilibrium $(\bar{x}, \bar{y})$. Besides, the positive semicycle have at most three terms, at least one term; the negative semicycle have at most three terms, at least one term.

Theorem 4. *Assume that the initial points $x_i, y_i \in (0, \infty), i \in \{-2, -1, 0\}$, $\{x_n, y_n\}_{n=-2}^{\infty}$ are a positive solution of the system* (6)*. Then, the following statements are true:*

If $y_{-1}^2 + x_0 + x_{-2} \le x_{-2}y_{-1}x_0$, $x_{-1}^2 + y_0 + y_{-2} \ge y_{-2}x_{-1}y_0$, then the solution of system (6) *converges to a period-four solution as following*

$$\{p_1, p_5\}\{p_2, p_6\}\{p_3, p_7\}\{p_4, p_8\}, \cdots,$$

$$\{p_1, p_5\}\{p_2, p_6\}\{p_3, p_7\}\{p_4, p_8\}, \cdots.$$

Proof. By the Theorem 1 and Theorem 3, we can get Theorem 4 holds. Hence, we omit it. □

Theorem 5. *Assume that the initial points $x_i, y_i \in (0, \infty), i \in \{-2, -1, 0\}$, $\{x_n, y_n\}_{n=-2}^{\infty}$ are a positive solution of the system* (6)*. Then the following statements are true:*

If $y_{-1}^2 + x_0 + x_{-2} \ge x_{-2}y_{-1}x_0$, $x_{-1}^2 + y_0 + y_{-2} \le y_{-2}x_{-1}y_0$, then the solution of system (6) *converges to a period-four solution as following:*

$$\{p_1, p_5\}\{p_2, p_6\}\{p_3, p_7\}\{p_4, p_8\}, \cdots,$$

$$\{p_1, p_5\}\{p_2, p_6\}\{p_3, p_7\}\{p_4, p_8\}, \cdots.$$

Proof. By the Theorem 1 and Theorem 3, we can get Theorem 5 holds. Hence, we omit it. □

Competing interests
The authors declare that they have no competing interests.

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