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Total restrained domination in graphs of diameter 2 or 3

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Abstract

For a given connected graph $G = (V, E)$, a set $D_{tr} \subseteq V(G)$ is a total restrained dominating set if it is a dominating set and both $\langle D_{tr}\rangle$ and $\langle V(G) - D_{tr}\rangle$ do not contain isolated vertices. The cardinality of the minimum total restrained dominating set in G is the total restrained domination number and is denoted by $\gamma_{tr}(G)$. In this paper, we continue the study of total restrained domination number of graphs. We first give some results on total restrained domination number of graphs. And then, we characterize all graphs G of order n for which (1) $\gamma_{tr}(G) = n$, (2) $\gamma(G) = 1$ and $\gamma_{tr}(G) = 3$, and (3) $\gamma_{tr}(G) = 2$. Furthermore, we give some bounds on total restrained domination number of graphs with diameter 3. Finally, we present some bounds for total restrained domination number of some planar graphs with diameter 2 and γ-set of cardinality 2.

Keywords: Domination number, Total domination number, Restrained domination number, Total restrained domination number

MSC (2000): 05C69

Introduction

In the whole paper, G is a simple graph with vertex set $V(G)$ and edge set $E(G)$ (briefly V and E). For every vertex $v \in V$, the open neighborhood $N(v)$ is the set $\{u \in V \mid uv \in E\}$, and its *closed neighborhood* is the set $N[v] = N(v) \cup \{v\}$. The *open neighborhood* of a set $S \subseteq V$ is the set $N(S) = \cup_{v\in S} N(v)$, and the *closed neighborhood* of S is the set $N[S] = N(S) \cup S$. In a graph, a stem is a vertex adjacent to at least one end vertex. The sets of all end vertices and all stems are denoted by $\Omega(G)$ and $\Omega_1(G)$, respectively. In K_2, a vertex is both an end vertex and a stem. For two vertices u and v in a connected graph G, the distance $d_G(u, v)$ between u and v is the length of a shortest $u - v$ path in G. For a set $S \subseteq V$ and a vertex $v \in V$, the distance $d_G(v, S)$ between v and S is the minimum distance between v and a vertex of S. If a vertex u is adjacent to a vertex v, we write $u \sim v$, while if u and v are nonadjacent, we write $u \nsim v$. If v is adjacent to no vertex in a set $A \subseteq V(G)$, then we write $v \nsim A$, and if v is adjacent to every vertex in A, then we write $v \sim A$. A plane graph is a planar graph together with an embedding in the plane. From the Jordan closed curve theorem, we know that a cycle C in a plane graph separates the plane into two regions, the interior of C and the exterior of C. We use [1] for the terminology and notation which are not defined in this study.

A set $D \subseteq V(G)$ is a dominating set of G if for every vertex $v \in V(G) - D$, there exists a vertex $u \in D$ such that v and u are adjacent. The minimum cardinality of a dominating set in G is the domination number denoted by $\gamma(G)$. The literature on domination has been surveyed in the two books by Haynes et al. [2,3].

A set $D \subseteq V(G)$ is a total dominating set (TDS) of a graph G if each vertex of G has a neighbor in D. Equivalently, a set $D \subseteq V(G)$ is a TDS of a graph G if D is a dominating set of G and $\langle D\rangle$ does not contain an isolate vertex. The cardinality of a minimum TDS in G is the total domination number and is denoted by $\gamma_t(G)$. A minimum TDS of a graph G is called a $\gamma_t(G)$-set. The notion of total domination in graphs was introduced by Cockayne et al. [4] in 1980, and for a survey of total domination in graphs, see [5] and for more detail, see [6].

A subset S of vertices of G is a *restrained dominating set* if $N[S] = V$ and the subgraph induced by $V - S$ has no isolated vertex. The *restrained domination number* $\gamma_r(G)$

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is the minimum cardinality of a restrained dominating set of G. The restrained domination number was introduced by Domke et al. [7], and it has been studied by several authors, for example, see [8,9].

A set $D_{tr} \subseteq V(G)$ is a total restrained dominating set (TRDS) of a graph G if it is a dominating set and the induced subgraphs by D_{tr} and $V(G) - D_{tr}$ do not contain an isolated vertex. The cardinality of a minimum TRDS in G is the total restrained domination number and is denoted by $\gamma_{tr}(G)$. A minimum TRDS of a graph G is called a $\gamma_{tr}(G)$-set. Thus, the total restrained dominating set of a graph combines the properties of both a total dominating set and a restrained dominating set. We assume that every graph without an isolated vertex has a TRDS and $D_{tr} = V(G)$ is such a set. Moreover, the above definitions imply that for any graph G without an isolated vertex, every TRDS is a TDS, so $\gamma_t(G) \leq \gamma_{tr}(G)$. The total restrained domination number of a graph was defined by Ma et al. [10] in 2005, for more, see [11]y. We state the following result which derived from Goddard and Henning [12]:

Theorem 1. *[12] If G is a planar graph with diam*$(G) =$ *2, then* $\gamma(G) \leq 2$ *or* $G = G_9$*, where* G_9 *is the graph of Figure 1.*

Our aim was to continue the study of total restrained domination number of graphs. In this article, we first give some results on total restrained domination number of graphs. And then, we characterize all graphs G of order n for which

(1) $\gamma_{tr}(G) = n$;
(2) $\gamma(G) = 1$ and $\gamma_{tr}(G) = 3$;
(3) $\gamma_{tr}(G) = 2$.

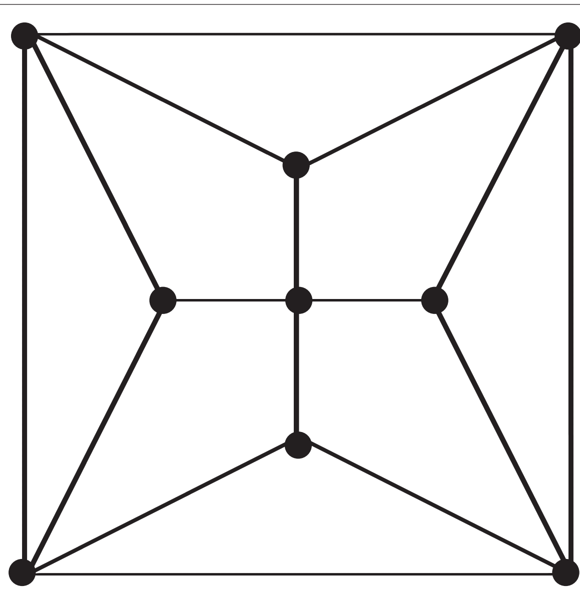

Figure 1 The graph G_9.

Furthermore, we give some bounds on total restrained domination number of graphs with diameter 3. Finally, we present some bounds for total restrained domination number of some planar graphs with diameter 2 and γ-set of cardinality 2.

Results

We begin with the following result that has a straightforward proof.

Lemma 2. *Let G be a nontrivial connected graph of order n. Then,*

(1) $\gamma(G) \leq \gamma_t(G) \leq \gamma_{tr}(G)$ *and* $\gamma_r(G) \leq \gamma_{tr}(G)$*. Furthermore,* $\gamma_{tr}(G) \geq max\{\gamma_r(G), \gamma_t(G)\}$*;*
(2) $\Omega(G) \cup \Omega_1(G) \subseteq D_{tr}$*;*
(3) $2 \leq \gamma_{tr}(G) \neq n - 1$*.*

Lemma 3. $\gamma_{tr}(K_n) = 2$*, where* $n \neq 3$*, and for* $n = 3$*,* $\gamma_{tr}(K_n) = 3$*.*

Proof. It is easy to see that any 2-subset is a total restrained dominating set of K_n, where $n \neq 3$. Hence, the result follows. □

Lemma 4. *Let T be a tree of diameter 2 or 3 and of order* $n \geq 3$*. Then,* $\gamma_{tr}(T) = n$*.*

Proof. Clearly, $\mathrm{diam}(T) = 2$ if and only if $T = K_{1,n-1}$. Also, $\mathrm{diam}(T) = 3$ if and only if T is a double star graph, say $S_{p,q}$. By these facts and by Observation 2, it follows that $\gamma_{tr}(K_{1,n-1}) = n$ and $\gamma_{tr}(S_{p,q}) = |V(S_{p,q})|$. □

In the following theorem, we show that for any $n \geq 4$, there exists a connected graph G of order n and $\mathrm{diam}(G) = \gamma_{tr}(G) = 2$.

Proposition 5. *Let G be a planar complete bipartite graph of order n. Then,* $\gamma_{tr}(G) = 2$ *if and only if either* $G = K_2$ *or* $G = K_{2,n-2}$*.*

Proof. It is well known that a complete bipartite graph G is planar if and only if $G = K_{m,n-m}$, where $m = 1, 2$. On the other hand, $\gamma_{tr}(K_{1,n-1}) = n$. Therefore, the desired result follows by combining these assumptions. □

Observation 6. $\gamma_{tr}(K_{m,n-m}) = 2$*, where* $n - m \geq m \geq 2$*.*

As a consequence of Proposition 5 and Observation 6, we have the following corollary:

Corollary 7. *Let G be a graph of order* $n \geq 4$ *which contains* $K_{m,n-m}$ *as subgraph, where* $n - m \geq m \geq 2$*. Then,* $\gamma_{tr}(G) = 2$*.*

Proposition 8. *Let G be a nontrivial connected graph of order n with diam$(G) = 2$ and $\Omega(G) \neq \emptyset$. Then, $\gamma_{tr}(G) = |\Omega(G)| + 1$.*

Proof. Since G is connected and diam$(G) = 2$ and $\Omega(G) \neq \emptyset$, then G contains a $K_{1,n-1}$ as spanning graph. Let x be the unique stem vertex of G. Clearly, all of the non-end vertices must be adjacent to x. It follows that $\gamma_{tr} = |\Omega(G)| + 1$. □

Characterizations

In the following result, we determine all graphs of order n with $\gamma_{tr}(G) = n$.

Proposition 9. *Let G be a connected graph of order $n \geq 2$. Then, $\gamma_{tr}(G) = n$ if and only if either $G = K_3$ or $\Omega(G) \cup \Omega_1(G) \cup S = V(G)$, where $S = \{v | \ v \notin \Omega(G) \cup \Omega_1(G)$ and $N(v) \subseteq \Omega_1(G)$ for all $v \notin \Omega(G) \cup \Omega_1(G)\}$.*

Proof. Necessity: Observation 2(2) asserts that $\Omega(G) \cup \Omega_1(G) \subseteq D_{tr}$. On the other hand, since $N(v) \subseteq \Omega_1(G)$ for all $v \notin \Omega(G) \cup \Omega_1(G)$ and $G - (\Omega(G) \cup \Omega_1(G))$ is union of isolated vertices, too. These imply that $\gamma_{tr}(G) = n$.

Sufficiency: It is trivial to check that $\gamma_{tr}(K_3) = 3$. Now, we assume that $G \neq K_3$. Let $\gamma_{tr}(G) = n$. Assume, to the contrary $N(v) \nsubseteq \Omega(G) \cup \Omega_1(G)$, then there is $u \in V(G) - (\Omega_1(G) \cup \Omega(G))$ such that $uv \in E(G)$. There are $w, r \in V(G)$ such that $uw \in E(G)$ and $vr \in E(G)$, where w and r are not vertices of degree 1. Therefore, $S = V - \{u, v\}$ is a $\gamma_{tr}(G)$-set, a contradiction. □

Proposition 10. *Let G be a connected graph of order n and $\gamma(G) = 1$. Then, (1) $\gamma_{tr}(G) = \Omega(G) \cup \Omega_1(G)$, while G has a pendant edge. (2) $\gamma_{tr}(G) \leq 3$, while G has no pendant edge, with equality if and only if G is union of $k = \frac{n-1}{2}$ copies of complete graph K_3 such that all of them have a common vertex and n is an odd integer.*

Proof. It is well known that $\gamma(G) = 1$ if and only if G has a vertex of degree $n - 1$. Now, if G has a pendant edge, then $\Omega(G) \cup \Omega_1(G)$ is a TRDS of G; this completes the part (1). Finally, if G has no pendant edge, then clearly, we have $\gamma_{tr}(G) \leq 3$.

If G is union of $k = \frac{n-1}{2}$ copies of complete graph K_3 such that all of them have a common vertex and n is an odd integer, then $\gamma_{tr}(G) = 3$.

Conversely, since $\gamma(G) = 1$, then G contains a $K_{1,n-1}$ as spanning graph. We claim that G is the union of $k = \frac{n-1}{2}$ copies of complete graph K_3 such that all of them have a common vertex. Assume, to the contrary, that a copy, say $G_i \neq K_3$, and so $|V(G_i)| \geq 4$. Then, there exists a vertex $w \in V(G_i)$ such that $\{u, w\}$ is a γ_{tr}-set of G, where u is the common vertex of all copies and $G_i - \{u, w\}$ has no isolated vertex, a contradiction with $\gamma_{tr}(G) = 3$. This completes the proof. □

In the following result, we characterize all graphs with $\gamma_{tr}(G) = 2$.

Proposition 11. *Let G be a connected graph of order n. Then, $\gamma_{tr}(G) = 2$ if and only if either (1) G has a vertex v of degree $n - 1$ such that $|\Omega(G)| = 1$ or G has no pendant edge and a component of $G[N(v)]$ is of order at least 3, or (2) G has $S_{p,q}$ as spanning subgraph such that G has no pendant edge and each component of $G[V(G) - \{u, v\}]$ is of order at least 2 and $p + 1 \leq deg_G(u) \leq n - 2$, $q + 1 \leq deg_G(v) \leq n - 2$ for some u and v.*

Proof. Necessity: It is obvious. *Sufficiency:* Since $\gamma_{tr}(G) = 2$, then $\gamma(G) \leq 2$. We consider the following two cases:

> *Case 1:* If $\gamma(G) = 1$, then G contains a $K_{1,n-1}$ as spanning graph. By Observation 2(2), we imply that $|\Omega(G)| \leq 1$, otherwise, a contradiction with $\gamma_{tr}(G) = 2$. If $|\Omega(G)| = 1$ and so $\Omega(G) \cup \Omega_1(G)$ is the unique TRDS of G, hence the desired result follows. Finally, $|\Omega(G)| = 0$. Since G contains a $K_{1,n-1}$ as spanning graph, therefore there exists a vertex $v \in V(G)$ such that $\deg(v) = n - 1$. It is easy to say that $G[N(v)]$ is union of some connected graph such that each vertex of these graphs are adjacent to v in G. We deduce that one of them must be of order at least 3, otherwise, a contradiction with $\gamma_{tr}(G) = 2$.
> *Case 2:* If $\gamma(G) = 2$, then G must be have a $S_{p,q}$ as spanning subgraph. Observation 2(2) implies that G has no pendant edge, and there exist two vertices u and v such that $G[V(G) - \{u, v\}]$ is the union of some connected graph of order at least 2 and $p + 1 \leq \deg_G(u) \leq n - 2$, $q + 1 \leq \deg_G(v) \leq n - 2$, otherwise, a contradiction with $\gamma_{tr}(G) = 2$. □

Bounds on total restrained domination number

Theorem 12. *Let G be a nontrivial connected graph with diam$(G) = 3$ and $|\Omega_1(G)| = 2$. Let $T = \{u | d(u, w) = 3, w \in \Omega(G)\}$ and $|T| = m$. Let v be a vertex in $N(\Omega_1(G))$ such that $|N_T(v)| = k$ and k be the maximum number between vertices such as v's. Then, $|\Omega(G) \cup S| + 2 \leq \gamma_{tr}(G) \leq |\Omega(G) \cup S| + m - k + 3$, where $S = \{v \mid N(v) \subseteq \Omega_1(G)\}$. These bounds are sharp.*

Proof. It is clear to see that $|\Omega(G) \cup S| + 2 \leq \gamma_{tr}(G)$. Now, we show that $\gamma_{tr}(G) \leq |\Omega(G) \cup S| + 2 + (1 + m - k)$. Let v be a vertex such that $|N_T(v)| = k$. Let the other vertices in T be totally dominated by at most $m - k$ vertices in $N(\Omega_1(G)) \cap N(T)$. Let W be at most these $m - k$ vertices. Then, $D = \Omega(G) \cup S \cup \Omega_1(G) \cup W \cup \{v\}$ is a total dominating set, and D and $G - D$ have no isolated vertices. Therefore, $\gamma_{tr} \leq |\Omega(G) \cup S| + 2 + (1 + m - k)$.

The sharpness of the lower bound is trivial. To show the sharpness of the upper bound, we define the graph $G_{2,m,k}$, as shown in Figure 2, where $m = 6$ and $k = 4$.

It is easy to check that $S = \{u_1, u_3\}$, $\Omega(G) = \{x, y\}$, $T = \{w_2, w_3, z_1, z_2, z_3, z_4\}$, $N(\Omega_1(G)) = \{u_1, u_2, u_3, w_1, w_4, a, b, x, y\}$, and $N_T(u_2) = \{z_1, z_2, z_3, z_4\}$ such that $|N_T(u_2)|$ is the maximum number between vertices in $N(\Omega_1(G))$. Then, $\gamma_{tr}(G) = |\Omega(G) \cup S| + m - k + 3$. Furthermore, $\{a, b, x, y, w_1, w_4, u_1, u_2, u_3\}$ is a γ_{tr}-set for G. □

Theorem 13. *Let G be a nontrivial connected graph with $diam(G) = 3$ and $|\Omega_1(G)| = 1$. Then, $|\Omega(G)| + 2 \leq \gamma_{tr}(G) \leq deg(v) + |S| + 1$, where $v \in \Omega_1(G)$ and $S = \{u|\, N(u) \subseteq N(v)\} - \{u|V(G[N[u] - \{v\}]) \subseteq N[v]\}$. These stated bounds are sharp.*

Proof. Let w, v, s, t be a diametral path in G and $w \in \Omega(G)$ and $v \in \Omega_1(G)$. Clearly, every vertex from $V(G) - N[v]$ will be joined to a vertex from $N[v] - \Omega(G)$. Let $V(G) - N[v] = S \cup \overline{S}$, where $S = \{z|\, N(z) \subseteq N[v]\} - \{z|V(G[N[z] - \{v\}]) \subseteq N[v]\}$. It is not so difficult to see that for every vertex $r \in \overline{S}$, there exist a vertex $x \in \overline{S}$ such that $x \sim r$. Therefore, $N[v] \cup S$ is a total restrained dominating set of G. This completes the proof.

The sharpness of the lower bound is trivial. To show the sharpness of the upper bound, we consider the graph G_k as shown in Figure 3, where $k = \deg(v) + |S| + 1$. Furthermore, $S = \{u\}$ and $\{w_1, w_2, \ldots, w_t, v, a, b, u\}$ is a γ_{tr}-set of the graph. □

Total restrained domination number of some planar graphs

If G is a planar graph of diameter 2. Then, by Theorem 1, we have $G = G_9$ or $\gamma(G) \leq 2$. It is straightforward to see that $\gamma_{tr}(G_9) = 3$. Now, suppose $\gamma(G) \leq 2$. If $\gamma(G) = 1$, then we can apply Proposition 10 to obtain total restrained domination number of G. Now, in the following results, we discuss total restrained domination number of planar graph G of diameter 2 and $\gamma(G) = 2$.

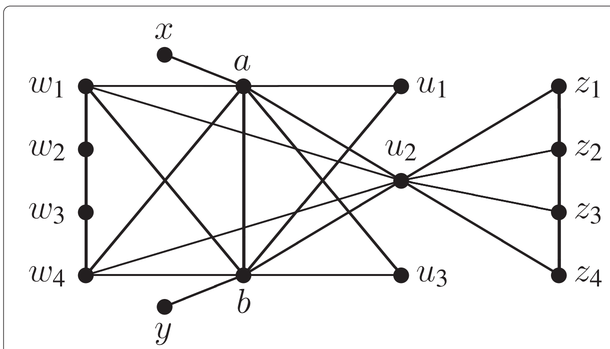

Figure 2 The graph $G_{2,m,k}$.

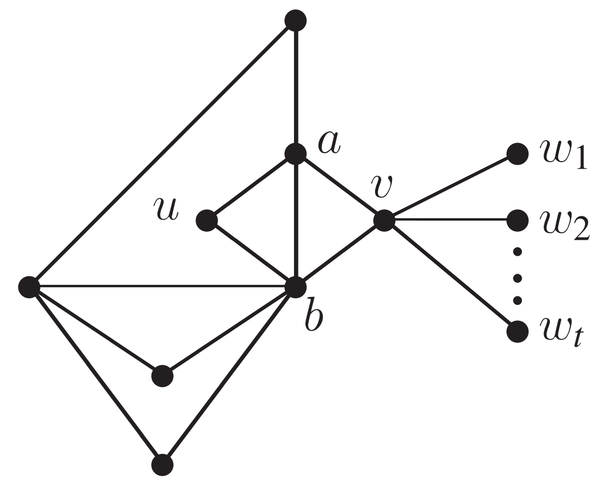

Figure 3 The graph G_k.

Proposition 14. *Let G be a planar graph of diameter 2 and with a $\gamma(G)$-set $\{a, b\} \subseteq V(G)$, $d(a, b) = 1$ and $|N(a) \cap N(b)| \leq 1$. Then, $\gamma_{tr}(G) \leq 3$.*

Proof. Let $\gamma(G) = 2$ and $d(a, b) = 1$; then, G has no pendant edge, otherwise, a contradiction with $\text{diam}(G) = 2$ or $\gamma(G) = 2$. If $|N(a) \cap N(b)| = 0$, then it is easy to see that $\{a, b\}$ is a D_{tr}. If $|N(a) \cap N(b)| = 1$, we may assume that $N(a) \cap N(b) = \{u\}$. If $\deg(u) = 2$, then $\{a, b, u\}$ is a total restrained domination number of G, otherwise, $\{a, b\}$ is a D_{tr}. This completes the proof. □

Theorem 15. *Let G be a planar graph of diameter 2 and with a $\gamma(G)$-set $\{a, b\} \subseteq V(G)$, $d(a, b) = 2$ and $|N(a) \cap N(b)| \leq 2$. Then, $\gamma_{tr}(G) \leq 3$.*

Proof. Let $\gamma(G) = 2$ and $d(a, b) = 2$; then, G have no pendant edge. Assume, to the contrary, let G has a pendant edge. Then, it must be at t, where t lies on $a - t - b$ path, a contradiction by the stated γ-set. Let A, B, and C be three sets such as $N(a) \cap N(b) = C$, $N(a) - C = A$, and $N(b) - C = B$. Since $d(a, b) = 2$, so $|C| \geq 1$. We process the following cases:

Case 1: $|C| = 1$ and so $C = \{c\}$. If $c \sim (A \cup B)$, then contradicting with $\gamma(G) = 2$. Hence, there is a vertex from $A \cup B$ such that is not adjacent to c. Without the loss of generality, we may assume that $x \in A$ and $x \nsim c$. Let B_1 and B_2 be partitions of B, such that B_1 is the set of those vertices which has a neighbor in $A \cup B$ and B_2 is the set of those vertices that are adjacent to c and has no neighbor in $A \cup B$.
Claim 1. $B_2 = \emptyset$.

Proof. Assume, to the contrary, that $B_2 \neq \emptyset$ and so $y \in B_2$. It is easy to check that $d(x, y) \geq 3$, a contradiction with $\operatorname{diam}(G) = 2$. Hence, the desired result follows. □

Thus, each vertex of B must be adjacent to $A \cup B$. Now, we consider the following two cases:

(1): $\deg(z) = 2$ for some $z \in A$ such that $z \sim \{a, c\}$. It is easy to check that $c \sim B$. Assume, to the contrary, that there exists a vertex $b' \in B$ such that $c \nsim b'$, then $d(z, b') \geq 3$, a contradiction with $\operatorname{diam}(G) = 2$. Certainly, the planarity of G and $\operatorname{diam}(G) = 2$ shows that $|B| \leq 2$, otherwise, a contradiction with $\operatorname{diam}(G) = 2$. On the other side, all vertices of A, except those vertices such as z's, are adjacent to c or only a vertex of B, say b_1, otherwise, a contradiction with $\operatorname{diam}(G) = 2$. It implies that $\{b_1, c, b\}$ is a TRDS of G, where $|B| = 1$, and $\{b_1, c\}$ is a TRDS of G, where $|B| = 2$. Hence, the result follows.

(2): $\deg(z) \geq 3$ for all $z \in A$. Hence, $\{a, b, c\}$ is a TRDS of G. Hence, the result follows.

Case 2: If $|C| = 2$, and let $C = \{c_1, c_2\}$. We have the following easy claim. **Claim 2.** Every vertex $x \in A \cup B$ has a neighbor in $A \cup B$. Now, we continue to complete of the proof by the following:

(1) If $c_1 \sim y$ or $c_2 \sim y$, where $y \in A \cup B$. Without the loss of generality, we may assume that $c_1 \sim y$. Then, by using Claim 2, $\{a, c_2, b\}$ is a TRDS of G.

(2) If $c_1 \nsim y$ and $c_2 \nsim y$ for every $y \in A \cup B$. Therefore, every vertex of A must be adjacent to a vertex of B and converse. Otherwise, if there exist two vertices $a_1 \in A$ and $b_1 \in B$ such that $a_1 \nsim b_1$, a contradiction with $\operatorname{diam}(G) = 2$ (because $d(a_1, b_1) \geq 3$, $d(a_1, b) \geq 3$ or $d(a, b_1) \geq 3$). Now, we consider the following claim.

Claim 3. If $c_i \nsim A \cup B$ for $i = 1, 2$, then $|A| \ngeq 4$ and $|B| \ngeq 4$.

Proof. Assume, to the contrary, that $|A| \geq 4$ and $|B| \geq 4$. Since, $c_i \nsim A \cup B$, and by assumption $\operatorname{diam}(G) = 2$, it implies that there exists a vertex $x \in A \cup B$, say $x \in A$, such that $d(x, y) \geq 3$ for some $y \in B$, otherwise, a contradiction with $\operatorname{diam}(G) = 2$. Hence, the result follows. □

Claim 3 and our assumptions imply that one of the following holds.

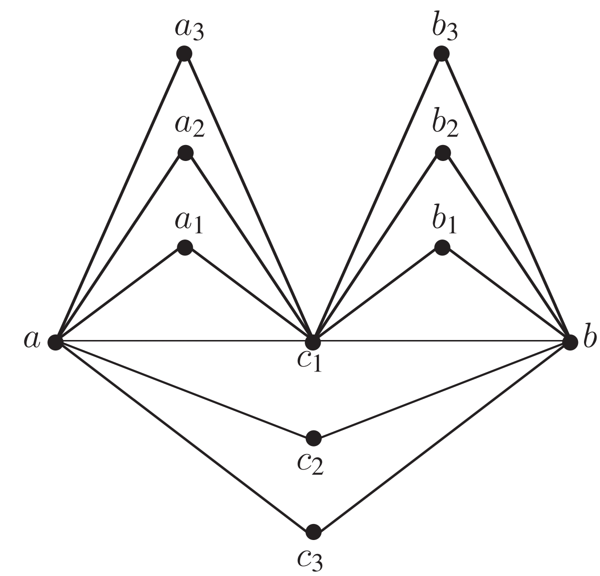

Figure 4 The graph $G_{|C|}$.

(2-1) $|A| = 1$ and $|B| \geq 1$. Let $A = \{a_1\}$. The vertex a_1 must be adjacent to all vertices of B, otherwise, a contradiction with $\operatorname{diam}(G) = 2$. It is easy to check that $\{a_1, a, c_1\}$ is a TRDS of G. Hence, the result follows.

(2-2) $|A| = 2$ and $|B| \geq 2$. We simply imply that there exists a vertex in A or B; without the loss of generality, we may assume that $x \in A$, such that $x \sim A - \{x\}$, otherwise, a contradiction with $\operatorname{diam}(G) = 2$. Thus, there exists a vertex $y \in B$ with $x \sim y$ such that $\{x, b, y\}$ is a TRDS of G.

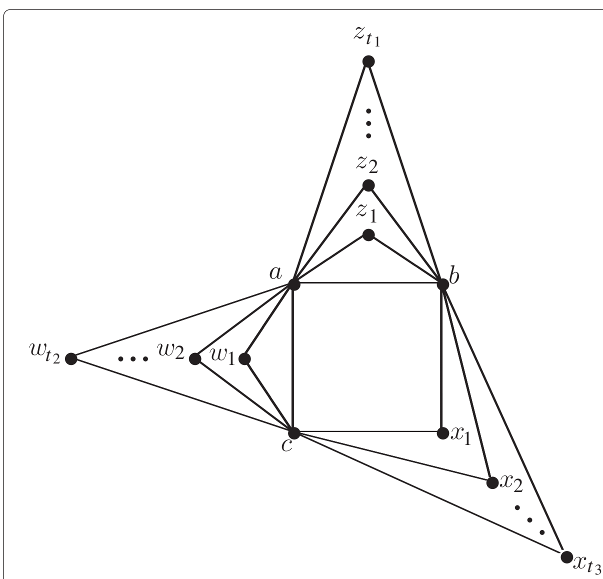

Figure 5 The graph G_{t+2}.

(2-3) $|A| = 3$ and $|B| \geq 3$. An argument similar to that described in the proof of Case (2-2) shows that the result holds.

The following graph shows that Theorem 15 is not true for $|C| = 3$.

Let $C = \{c_1, c_2, c_3\}$, $A = N(a) - C = \{a_1, a_2, a_3\}$, and $B = N(b) - C = \{b_1, b_2, b_3\}$. Let $E(G) = \{ac_i,\ bc_i|\ i = 1, 2, 3\} \cup \{aa_i, c_1a_i, c_1b_i, bb_i|\ i = 1, 2, 3\}$ (see the graph $G_{|C|}$ as shown in Figure 4). Then, G is a planner graph with $\mathrm{diam}(G) = 2$, but $\gamma_{tr}(G) = 4$.

We conclude our results, with the following result. As a consecutive of the Proposition 14 and Theorem 15, we have the following corollary:

Corollary 16. *Let G be a planar graph of diameter 2 and with a $\gamma(G)$-set $\{a, b\} \subseteq V(G)$ and $|N(a) \cap N(b)| \leq t$. Then, $\gamma_{tr}(G) \leq t + 2$. This bound is sharp.*

To show the sharpness of Corollary 16, we construct the graph G_{t+2} as shown in Figure 5.

It is easy to check that $\{a, b\}$, $\{a, c\}$, and $\{b, c\}$ are γ-sets of the constructed graph. Furthermore, $d(a, b) = d(a, c) = 1$ and $d(b, c) = 2$. Also, $|N(a) \cap N(b)| = t_1$, $|N(a) \cap N(c)| = t_2$, and $|N(b) \cap N(c)| = t_3 + 1$, where $t_1 \leq t_2 \leq t_3 - 1$ and $t = min\{t_1, t_2, t_3 - 1\}$.

Competing interests
The authors declare that they have no competing interests.

Authors' contribution
ZT, DSN, HAA, DAM, and YZ contributed equally. All authors read and approved the final manuscript.

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