Chin and Sahebi *Mathematical Sciences* 2013, **7**:23
http://www.iaumath.com/content/7/1/23

Mathematical Sciences
a SpringerOpen Journal

ORIGINAL RESEARCH Open Access

A note on abelian stongly k-Engel π-regular rings

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Abstract

Let R be an associative ring with identity and let $k \geq 1$ be a fixed integer. An element $(x, y) \in R \times R$ is said to be left (right) k-Engel π-regular if there exists a positive integer n and an element $z \in R$ such that $[x, y]_k^n = z[x, y]_k^{n+1}$ ($[x, y]_k^n = [x, y]_k^{n+1} z$). If every element of $R \times R$ is left (right) k-Engel π-regular, then R is said to be left (right) k-Engel π-regular. An element $(x, y) \in R \times R$ is strongly k-Engel π-regular if it is both left and right k-Engel π-regular. The ring R is strongly k-Engel π-regular if every element of $R \times R$ is strongly k-Engel π-regular. In this paper, we investigate properties of abelian strongly k-Engel π-regular ring.

Keywords: Strongly π-regular, k-Engel, Abelian ring

MSC: 16E50, 16D70, 16U99

Introduction

Let R be an associative ring with identity. An element $x \in R$ is said to be right π-regular if there exists a positive integer n and an element $y \in R$ such that $x^n = x^{n+1}y$. If every element of R is right π-regular, then R is said to be right π-regular. By [1], this definition is left-right symmetric. An element of R is strongly π-regular if it is both left and right π-regular. R is strongly π-regular if every element of R is strongly π-regular. In [2], it was shown that if an element x in the ring R is strongly π-regular, then there exists a positive integer n and an element $y \in R$ such that $x^n = x^{n+1}y$ and $xy = yx$. In the case where $n = 1$, the element x is said to be strongly regular.

If $(x_i)_{i \in \mathbb{N}}$ is a sequence of elements of R and k is a positive integer, we define $[x_1, \ldots, x_{k+1}]$ inductively as follows:

$$[x_1, x_2] = x_1x_2 - x_2x_1,$$
$$[x_1, \ldots, x_k, x_{k+1}] = [[x_1, \ldots, x_k], x_{k+1}].$$

If $x_1 = x$ and $x_2 = \cdots = x_{k+1} = y$, the notation $[x, y]_k$ is used to denote $[x_1, \ldots, x_{k+1}]$, and $[x, y]_k$ is called a k-Engel element. For $k = 1$, $[x, y]_k = [x, y]_1$ is usually just denoted by $[x, y]$. An element $(x, y) \in R \times R$ is said to be left (right) k-Engel π-regular if there exists a positive integer n and an element $z \in R$ such that $[x, y]_k^n = z[x, y]_k^{n+1}$ ($[x, y]_k^n = [x, y]_k^{n+1} z$). If every element of $R \times R$ is left (right) k-Engel π-regular, then R is said to be left (right) k-Engel π-regular. An element $(x, y) \in R \times R$ is strongly k-Engel π-regular if it is both left and right k-Engel π-regular. The ring R is strongly k-Engel π-regular if every element of $R \times R$ is strongly k-Engel π-regular. Clearly, if (x, y) is strongly k-Engel π-regular, then $[x, y]_k$ is strongly π-regular. Therefore, there exists a positive integer n and an element $z \in R$ such that $[x, y]_k^n = [x, y]_k^{n+1} z$ and $[x, y]_k z = z[x, y]_k$ (by [2]).

Division rings are examples of strongly k-Engel π-regular rings. Other examples include full matrix rings over division rings and triangular matrix rings over fields. It is clear that rings which satisfy the k-Engel condition are strongly k-Engel π-regular. In [3], we studied the conditions for strongly k-Engel π-regular rings to be commutative (hence, has k-Engel condition). Now in this paper, we investigate the properties of abelian strongly k-Engel π-regular rings and obtain some characterisations of these rings. All rings in this paper are assumed to have

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identity. A ring is said to be abelian if all of its idempotents are central. For a ring R, we use the notation $N(R)$ and $Id(R)$ to denote the set of all nilpotent elements of R and the set of all idempotents of R, respectively.

Main results

Proposition 2.1. *Let R be an abelian strongly k-Engel π-regular ring. Suppose that $N(R)$ is an ideal of R. Then for each $x, y \in R$, $[x, y]_k + N(R)$ is strongly regular (hence regular).*

Proof. Let $x, y \in R$. Then there exist $z \in R$ and a positive integer n such that $[x, y]_k^n = [x, y]_k^{n+1} z$ and $[x, y]_k z = z[x, y]_k$. Thus, $e = [x, y]_k^n z^n \in Id(R)$, and hence, $1 - e \in Id(R)$. Then since $[x, y]_k^n = [x, y]_k^{2n} z^n = [x, y]_k^n e$, it follows that $(1 - e)[x, y]_k^n = 0$, and hence, $(1 - e)[x, y]_k \in N(R)$. Therefore,

$$\begin{aligned}[x, y]_k + N(R) &= e[x, y]_k + N(R) \\ &= [x, y]_k^{n+1} z^n + N(R) \\ &= ([x, y]_k + N(R))^2([x, y]_k^{n-1} z^n + N(R)).\end{aligned}$$

It follows that $[x, y]_k + N(R)$ is strongly regular (hence regular).

The following lemma is well known and can be found for example in p. 72 of [4]. □

Lemma 2.2. *Let R be a ring and I a nil ideal of R. Then idempotents of R/I can be lifted to R.*

Proposition 2.3. *Let R be an abelian ring. If $N(R)$ is an ideal of R and for each $x, y \in R$, $[x, y]_k + N(R)$ is regular, then R is strongly k-Engel π-regular.*

Proof. Let $x, y \in R$. Since $[x, y]_k + N(R)$ is regular, there exist some $z \in R$ such that $[x, y]_k z [x, y]_k + N(R) = [x, y]_k + N(R)$. Clearly, $(z[x, y]_k)^2 + N(R) = z[x, y]_k + N(R)$. By Lemma 2.2, there is an idempotent $e \in R$ such that $e + N(R) = z[x, y]_k + N(R)$, that is, $e - z[x, y]_k \in N(R)$. Thus, there exists an integer $m \geq 1$ such that $(e - z[x, y]_k)^m = 0$. Since e is central, $e = t[x, y]_k$ for some $t \in R$.

Now $[x, y]_k + N(R) = ([x, y]_k z [x, y]_k) + N(R) = [x, y]_k e + N(R)$ gives us $[x, y]_k - [x, y]_k e \in N(R)$. Hence, there exist some integer $n \geq 1$ with $0 = ([x, y]_k - [x, y]_k e)^n = [x, y]_k^n - [x, y]_k^n e$. Therefore, $[x, y]_k^n = [x, y]_k^n e = e[x, y]_k^n = t[x, y]_k^{n+1}$. Thus, R is strongly k-Engel π-regular.

By Propositions 2.1 and 2.3, we readily have the following: □

Theorem 2.4. *Let R be an abelian ring such that $N(R)$ is an ideal of R. Then R is strongly k-Engel π-regular if and only if for each $x, y \in R$, $[x, y]_k + N(R)$ is regular.*

Proposition 2.5. *Let R be an abelian strongly k-Engel π-regular ring and let P be a prime ideal of R. Then for each $x, y \in R$, $[x, y]_k + P$ is nilpotent or a unit.*

Proof. Let $x, y \in R$. Since R is strongly k-Engel π-regular, by the proof of Theorem 2.1 in [3], we may write $[x, y]_k = fu = uf$ for some near idempotent f and some unit $u \in R$. By near idempotent we mean that there exists a positive integer n such that $e = f^n$ is an idempotent. Then $[x, y]_k^n = eu^n = u^n e$. Since $(1 - e)Re = \{0\} \subseteq P$ and P is a prime ideal, it follows that $e \in P$ or $1 - e \in P$. If $e \in P$, then $[x, y]_k^n = eu^n \in P$; hence, $[x, y]_k + P$ is nilpotent. If $1 - e \in P$, then $[x, y]_k^n + P = eu^n + P = (e + P)(u^n + P) = u^n + P$ is a unit in R/P. It follows that $[x, y]_k + P$ is a unit in R/P. □

Proposition 2.6. *Let R be a strongly k-Engel π-regular ring and I an ideal of R. Then I is strongly k-Engel π-regular as a ring.*

Proof. Let $x, y \in I$. Since R is strongly k-Engel π-regular, then there exist $z \in R$ and a positive integer n such that $[x, y]_k^n = [x, y]_k^{n+1} z$ and $[x, y]_k z = z[x, y]_k$. If $n = 1$, let $t = [x, y]_k z^2$. Then $t \in I$, $[x, y]_k t = t[x, y]_k$ and $[x, y]_k^2 t = [x, y]_k^2 [x, y]_k z^2 = [x, y]_k ([x, y]_k^2 z)z = [x, y]_k^2 z = [x, y]_k$. If $n \geq 2$, let $t = [x, y]_k^{n-1} z^n \in I$. Then $[x, y]_k t = t[x, y]_k$. Therefore, $[x, y]_k^n = [x, y]_k^{n+1} z = \ldots = [x, y]_k^{n+1} ([x, y]_k^{n-1} z^n) = [x, y]_k^{n+1} t$. Thus, I is strongly k-Engel π-regular. □

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
SS conceived of the idea for the study. AYMC participated in the investigation. Both authors read and approved the final manuscript.

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AYMC and SS are lecturers in their respective institutions.

Acknowledgements
The second author thanks the authority of IAUCTB for their support to complete this research.

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Received: 9 January 2013 Accepted: 27 April 2013
Published: 8 May 2013

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doi:10.1186/2251-7456-7-23
Cite this article as: Chin and Sahebi: **A note on abelian stongly k-Engel π-regular rings.** *Mathematical Sciences* 2013 **7**:23.