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Approximation properties for generalized Baskakov-type operators

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Abstract

In this paper, we give a generalization of the Baskakov-type operators introduced by Baskakov (Doklady Akademii Nauk SSSR 113:249–251, 1957 (in Russian)) and obtain some direct and inverse results for these new operators.

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Introduction

For a continuous function f on $[0,\infty)$ with exponential growth, the Szász operators are given by

$$S_n(f;x)=\sum_{k=0}^{\infty} f(\frac{k}{n})P_{k,n}(x),$$

where $P_{k,n}(x)=e^{-nx}\dfrac{(nx)^k}{k!}$. In [1], Ditzian proved some important inverse theorems for these operators by using the modulus of continuity defined by

$$\begin{aligned}\omega_2(f;\delta,A)&=\sup_{h\le\delta,x\in[0,\infty)}\left|f(x)-2f(x+h)+f(x+2h)\right|e^{-Ax}\\&=\sup_{h\le\delta,x\in[0,\infty)}\left|\Delta_h^2 f(x)\right|e^{-Ax},\end{aligned}$$

where $\sup_{x\in[0,\infty)}\left|f(x)e^{-Ax}\right|<M$.

In 1992, Guo and Zhou [2] gave similar theorems for the following modified Szász operators defined by Mazhar and Totik in [3]:

$$L_n(f;x)=\sum_{k=0}^{\infty}\left(n\int_0^{\infty}f(t)P_{k,n}(t)dt\right)P_{k,n}(x). \tag{1}$$

The authors obtained the following results for these operators:

(1) Let $f\in C[0,\infty)$ be a bounded function. Then, for $0<\alpha<1$,

$$\left|L_n(f;x)-f(x)\right|\le M(\frac{x}{n}+\frac{1}{n^2})^{\alpha/2}$$

holds if and only if

$$\omega_1(f;\delta)=O(\delta^\alpha),\quad(\delta>0),$$

where

$$\omega_1(f;\delta)=\sup_{0\le h\le\delta}\sup_{x\ge h/2}\left|f(x+h/2)-f(x-h/2)\right|. \tag{2}$$

(2) For $f\in C[0,\infty)\cap L_\infty[0,\infty)$ and $0<\alpha<1$,

$$\omega_1(f;\delta)=O(\delta^\alpha)\Longleftrightarrow\left|L_n'(f;x)\right|\le M\left(\min\left\{n^2,n/x\right\}\right)^{(1-\alpha)/2}$$

$$\omega_2(f;\delta)=O(\delta^\alpha)\Longleftrightarrow\left|L_n''(f;x)\right|\le M\left(\min\left\{n^2,n/x\right\}\right)^{(2-\alpha)/2}$$

holds, where $\omega_1(f;\delta)$ is defined by (2) and

$$\omega_2(f;\delta)=\sup_{0<h\le\delta,x\in[0,\infty)}\left|f(x)-2f(x+h)+f(x+2h)\right|. \tag{3}$$

(3) For $f\in C_B[0,\infty)$,

$$\left\|L_nf-f\right\|_\infty\le C\left(\omega_\varphi^2(f;\frac{1}{\sqrt{n}})_\infty+\omega_1(f;\frac{1}{n})+\frac{1}{n}\left\|f\right\|_\infty\right)$$

holds, where C is a constant independent of n, and $\omega_\varphi^2(f;.)_\infty$ is the Ditzian-Totik modulus of smoothness [4] defined by

$$\omega_\varphi^2(f;\delta)_\infty=\sup_{0<h\le\delta,x\in[0,\infty)}\left\|f(x-h\varphi(x))-2f(x)+f(x+h\varphi(x))\right\|_\infty,$$

$$x\ge h,\varphi(x)=\sqrt{x}.$$

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In [5], Baskakov introduced the following sequence of linear operators $\{\mathcal{L}_n\}$ which are generalizations of Bernstein polynomials, Szász operators, and Lupas operators:

$$\mathcal{L}_n(f;x) = \sum_{k=0}^{\infty} \frac{(-x)^k}{k!} \varphi_n^{(k)}(x) f(\frac{k}{n}). \tag{4}$$

Here, $x \in [0,b) \subset \mathbb{R}$ ($b > 0$, b can be ∞), $n \in \mathbb{N}$, and $\{\varphi_n\}_{n\in\mathbb{N}}$ is a sequence of functions defined on $[0,b]$ that have the following properties for all $k, n \in \mathbb{N}$:

(a) φ_n is analytic on the interval $[0,b]$ including the end points,
(b) $\varphi_n(0) = 1$,
(c) φ_n is completely monotone, i.e. $(-1)^k \varphi_n^{(k)}(x) \geq 0$,
(d) there exists a positive integer $m_0 = m_0(n)$, such that
$$-\varphi_n^{(k)}(x) = n\varphi_{n+m_0}^{(k-1)}(x) \ \ (k = 1, 2, ...),$$
(e) $\lim\limits_{n\longrightarrow\infty} \frac{n}{m_0+n} = 1$.

For the operators $\mathcal{L}_n(f;x)$ given by (4), we have (see [5]):

$$\mathcal{L}_n(1;x) = 1, \tag{5}$$

$$\mathcal{L}_n(t;x) = x, \tag{6}$$

$$\mathcal{L}_n(t^2;x) = \frac{n(m_0+n)}{n^2}x^2 + \frac{x}{n}, \tag{7}$$

and

$$\mathcal{L}_n((t-x)^2;x) = \frac{m_0}{n}x^2 + \frac{1}{n}x.$$

In the present paper, inspired by the operators (1) and (4), we introduce a generalization of the Baskakov operators as follows:

$$\mathcal{L}_n^*(f;x) = \sum_{k=0}^{\infty} \left(\frac{1}{\gamma_n} \int_0^{\infty} f(t) \frac{(-t)^k}{k!} \varphi_n^{(k)}(t)\, dt \right) \frac{(-x)^k}{k!} \varphi_n^{(k)}(x), \tag{8}$$

where $\gamma_n = \int\limits_0^{\infty} \varphi_n(t)\, dt < \infty$ for all $n \in \mathbb{N}$ and φ_n also satisfies the following condition:

$$\lim_{x\to\infty} x^k \varphi_n^{(k-1)}(x) = 0, \ \ k, n \in \mathbb{N}$$

and

$$\lim_{n\to\infty} \frac{\gamma_{n-\upsilon m_0}}{\gamma_n} = 1, \ \ \upsilon = 1, 2, 3.$$

In this study, we shall give some direct and inverse results for the new operators defined by (8).

Note that if $\varphi_n(x) = e^{-nx}$ in (8), we get the operators $L_n(f;x)$ defined by (1). Also, very important results were obtained by Mazhar and Totik in [3]. Recently, integral-type modification of some operators based on q-integers have been studied by Gupta et al. [6], Gupta and Kim [7], and Kim [8].

Main results

Now, we give the following lemma which will be used for the proof of theorems:

Lemma 1. *The following equalities hold:*

$$\begin{aligned}
&\mathcal{L}_n^*(1,x) = 1,\\
&\mathcal{L}_n^*(t,x) = \frac{\gamma_{n-m_0}}{\gamma_n}\left(\frac{nx}{n-m_0} + \frac{1}{n-m_0}\right),\\
&\mathcal{L}_n^*(t^2,x) = \frac{\gamma_{n-2m_0}}{\gamma_n}\left[\frac{n(m_0+n)}{(n-2m_0)(n-m_0)}x^2 + \frac{4nx}{(n-2m_0)(n-m_0)} + \frac{2}{(n-2m_0)(n-m_0)}\right].
\end{aligned}$$

From the definition of operators $\mathcal{L}_n^*$ and Lemma 1, we have

$$\begin{aligned}
\mathcal{L}_n^*((t-x)^2,x) = &\left(\frac{\gamma_{n-2m_0}}{\gamma_n}\frac{n(m_0+n)}{(n-2m_0)(n-m_0)} - \frac{2n}{n-m_0}\frac{\gamma_{n-m_0}}{\gamma_n} + 1\right)x^2\\
&+\left(\frac{4n}{(n-2m_0)(n-m_0)}\frac{\gamma_{n-2m_0}}{\gamma_n} - \frac{2}{n-m_0}\frac{\gamma_{n-m_0}}{\gamma_n}\right)x\\
&+\frac{2}{(n-2m_0)(n-m_0)}\frac{\gamma_{n-2m_0}}{\gamma_n}.
\end{aligned} \tag{9}$$

Theorem 2. *Let $f \in C[0,\infty)$ be a bounded function, and $0 < \alpha \leq 1$. If the usual modulus of smoothness $\omega_1(f,t)$ defined by (2) satisfies*

$$\omega_1(f,t) = O(t^\alpha) \ \ (t > 0), \tag{10}$$

then

$$\begin{aligned}
|\mathcal{L}_n^*(f,x) - f(x)| \leq K\Bigg(&\left(\frac{\gamma_{n-2m_0}}{\gamma_n}\frac{n(m_0+n)}{(n-2m_0)(n-m_0)} - \frac{2n}{n-m_0}\frac{\gamma_{n-m_0}}{\gamma_n} + 1\right)x^2\\
&+\left(\frac{4n}{(n-2m_0)(n-m_0)}\frac{\gamma_{n-2m_0}}{\gamma_n} - \frac{2}{n-m_0}\frac{\gamma_{n-m_0}}{\gamma_n}\right)x\\
&+\frac{2}{(n-2m_0)(n-m_0)}\frac{\gamma_{n-2m_0}}{\gamma_n}\Bigg)^{\alpha/2}
\end{aligned}$$

holds.

Proof. Using the definition of the operators $\mathcal{L}_n^*$ and the equality (9), we obtain the following inequality:

$$\begin{aligned}\left|\mathcal{L}_n^*\left(f,x\right)-f(x)\right| &\le \sum_{k=0}^{\infty}\left(\frac{1}{\int_0^\infty \varphi_n(t)dt}\int_0^\infty \left|f(t)-f(x)\right|\frac{(-t)^k}{k!}\varphi_n^{(k)}(t)dt\right)\frac{(-x)^k}{k!}\varphi_n^{(k)}(x)\\
&\le \sum_{k=0}^{\infty}\left(\frac{1}{\int_0^\infty \varphi_n(t)\,dt}\int_0^\infty \omega_1\left(|t-x|\right)\frac{(-t)^k}{k!}\varphi_n^{(k)}(t)dt\right)\frac{(-x)^k}{k!}\varphi_n^{(k)}(x)\\
&\le \omega_1\left(f,\delta_n\right)\left\{1+\frac{1}{\delta_n}\left(\sum_{k=0}^{\infty}\frac{1}{\int_0^\infty \varphi_n(t)dt}\int_0^\infty |t-x|\,p_{n,k}(t)dt\right)p_{n,k}(x)\right\}\\
&\le \omega_1\left(f\delta_n\right)\left\{1+\frac{1}{\delta_n}\left(\mathcal{L}_n^*\left((t-x)^2,x\right)\right)^{1/2}\right\}\\
&= \omega_1\left(f,\delta_n\right)\left\{1+\frac{1}{\delta_n}\left(\left(\frac{\gamma_{n-2m_0}}{\gamma_n}\frac{n\left(m_0+n\right)}{\left(n-2m_0\right)\left(n-m_0\right)}-\frac{2n}{n-m_0}\frac{\gamma_{n-m_0}}{\gamma_n}+1\right)x^2\right.\right.\\
&\quad+\left(\frac{4n}{\left(n-2m_0\right)\left(n-m_0\right)}\frac{\gamma_{n-2m_0}}{\gamma_n}-\frac{2}{n-m_0}\frac{\gamma_{n-m_0}}{\gamma_n}\right)x\\
&\quad\left.\left.+\frac{2}{n-m_0}\frac{\gamma_{n-2m_0}}{\gamma_n}\right)^{1/2}\right\}.\end{aligned}$$

If we choose δ_n as

$$\begin{aligned}\delta_n=&\left[\left(\frac{\gamma_{n-2m_0}}{\gamma_n}\frac{n\left(m_0+n\right)}{\left(n-2m_0\right)\left(n-m_0\right)}-\frac{2n}{n-m_0}\frac{\gamma_{n-m_0}}{\gamma_n}+1\right)x^2\right.\\
&+\left(\frac{4n}{\left(n-2m_0\right)\left(n-m_0\right)}\frac{\gamma_{n-2m_0}}{\gamma_n}-\frac{2}{n-m_0}\frac{\gamma_{n-m_0}}{\gamma_n}\right)x\\
&\left.+\frac{2}{\left(n-2m_0\right)\left(n-m_0\right)}\frac{\gamma_{n-2m_0}}{\gamma_n}\right]^{1/2},\end{aligned}$$

then we get

$$\begin{aligned}\left|\mathcal{L}_n^*\left(f,x\right)-f(x)\right| \le\ & 2\omega_1\left(f,\left[\left(\frac{\gamma_{n-2m_0}}{\gamma_n}\frac{n\left(m_0+n\right)}{\left(n-2m_0\right)\left(n-m_0\right)}-\frac{2n}{n-m_0}\frac{\gamma_{n-m_0}}{\gamma_n}+1\right)x^2\right.\right.\\
&+\left(\frac{4n}{\left(n-2m_0\right)\left(n-m_0\right)}\frac{\gamma_{n-2m_0}}{\gamma_n}-\frac{2}{n-m_0}\frac{\gamma_{n-m_0}}{\gamma_n}\right)x\\
&\left.\left.+\frac{2}{\left(n-2m_0\right)\left(n-m_0\right)}\frac{\gamma_{n-2m_0}}{\gamma_n}\right]^{1/2}\right).\end{aligned}$$

Consequently, using (10) in the above inequality, we finally get

$$\begin{aligned}\left|\mathcal{L}_n^*\left(f,x\right)-f(x)\right| \le 2K&\left(\left(\frac{\gamma_{n-2m_0}}{\gamma_n}\frac{n\left(m_0+n\right)}{\left(n-2m_0\right)\left(n-m_0\right)}-\frac{2n}{n-m_0}\frac{\gamma_{n-m_0}}{\gamma_n}+1\right)x^2\right.\\
&+\left(\frac{4n}{\left(n-2m_0\right)\left(n-m_0\right)}\frac{\gamma_{n-2m_0}}{\gamma_n}-\frac{2}{n-m_0}\frac{\gamma_{n-m_0}}{\gamma_n}\right)x\\
&\left.+\frac{2}{\left(n-2m_0\right)\left(n-m_0\right)}\frac{\gamma_{n-2m_0}}{\gamma_n}\right)^{\alpha/2}.\end{aligned}$$

□

Theorem 3. *Let $f\in C[0,\infty)$ be a bounded function, and $0<\alpha<1$. If*

$$\left|\mathcal{L}_n^*\left(f,x\right)-f(x)\right| \le K\left(\frac{1}{n^2}+\frac{1}{n^2(1+m_0x)}\right)^{\alpha/2}$$

for some positive constant K, then

$$\omega_1\left(f,t\right)=O\left(t^{\alpha}\right),\qquad (t>0),$$

where $\omega_1\left(f,t\right)$ is the usual modulus of smoothness of f defined by (2).

Proof. For $\delta>0$, let

$$f_\delta(x)=\frac{1}{\delta}\int_0^\infty f\left(x+s\right)ds.$$

For the function $f \in C[0,\infty) \cap L_\infty[0,\infty)$, the following inequalities hold (see [9]).

$$\|f_\delta - f\|_\infty \le \omega_1(f,\delta), \tag{11}$$

$$\|f'_\delta\|_\infty \le \frac{1}{\delta}\omega_1(f,\delta). \tag{12}$$

Now, we find the derivative of $\mathcal{L}_n^*(f,x)$ with respect to x. From the definition of the operators $\mathcal{L}_n^*$, we can write

$$\begin{aligned}\frac{d}{dx}\mathcal{L}_n^*(f,x) &= \frac{n}{x}\sum_{k=0}^{\infty}\left(\frac{1}{\int_0^\infty \varphi_n(t)dt}\int_0^\infty f(t)\frac{(-t)^k}{k!}\varphi_n^{(k)}(t)\,dt\right)\\ &\quad \frac{(-x)^k}{k!}\varphi_n^{(k)}(x)\left\{\left(\frac{k}{n}-x\right)+x\left(1-\frac{\varphi_{n+m_0}^{(k)}(x)}{\varphi_n^{(k)}(x)}\right)\right\}\\ &= n\sum_{k=0}^{\infty}\left(\frac{1}{\int_0^\infty \varphi_n(t)dt}\int_0^\infty f(t)\left(p_{n,k+1}(t)\right.\right.\\ &\quad \left.\left. -p_{n,k}(t)\right)dt\right)\frac{(-x)^k}{k!}\varphi_{n+m_0}^{(k)}(x).\end{aligned}$$

Now, using the properties of the operators (4), we obtain

$$\begin{aligned}\left|\frac{d}{dx}\mathcal{L}_n^*(f_\delta - f,x)\right| &= \frac{n}{x}\sum_{k=0}^{\infty}\left(\frac{1}{\int_0^\infty \varphi_n(t)dt}\int_0^\infty |f_\delta(t)-f(t)|\frac{(-t)^k}{k!}\varphi_n^{(k)}(t)dt\right)\\ &\quad \times\frac{(-x)^k}{k!}\varphi_n^{(k)}(x)\left[\left(\frac{k}{n}-x\right)+x\left(1-\frac{\varphi_{n+m_0}^{(k)}(x)}{\varphi_n^{(k)}(x)}\right)\right]\\ &\le \frac{n}{x}\|f_\delta - f\|_\infty\{\mathcal{L}_n(|t-x|,x)\\ &\quad + x\sum_{k=0}^{\infty}\frac{(-x)^k}{k!}\varphi_n^{(k)}(x)\left(1-\frac{\varphi_{n+m_0}^{(k)}(x)}{\varphi_n^{(k)}(x)}\right)\right\}\\ &= \frac{n}{x}\|f_\delta - f\|_\infty\left(\frac{m_0}{n}x^2+\frac{x}{n}\right)^{1/2}\\ &\le \frac{n}{x}\omega_1(f;\delta)\left(\frac{m_0}{n}x^2+\frac{x}{n}\right)^{1/2},\end{aligned} \tag{13}$$

where we have used the inequality,

$$\mathcal{L}_n(|t-x|,x) \le \left(\mathcal{L}_n(t-x)^2,x\right)^{1/2} = \left(\frac{m_0}{n}x^2+\frac{x}{n}\right)^{1/2}.$$

On the other hand, we also have

$$\begin{aligned}&\left|\frac{d}{dx}\mathcal{L}_n^*(f_\delta - f,x)\right|\\ &\quad = \left|n\sum_{k=0}^{\infty}\left(\frac{1}{\int_0^\infty \varphi_n(t)dt}\int_0^\infty |f_\delta(t)-f(t)|\right.\right.\\ &\qquad \left.\left.\times\left(p_{n,k+1}(t)-p_{n,k}(t)\right)dt\right)\frac{(-x)^k}{k!}\varphi_{n+m_0}^{(k)}(x)\right|\\ &\quad \le 2n\|f_\delta - f\|_\infty\varphi_{n+m_0}(0)\\ &\quad \le 2n\omega_1(f;\delta).\end{aligned}$$

Using the two estimates of $\left|\frac{d}{dx}\mathcal{L}_n^*(f_\delta - f,x)\right|$ obtained above, we get

$$\left|\frac{d}{dx}\mathcal{L}_n^*(f_\delta - f,x)\right| \le 2\omega_1(f;\delta)\min\left\{\left(nm_0+\frac{n}{x}\right)^{1/2},n\right\}.$$

Also, one can easily obtain that

$$\begin{aligned}\left|\frac{d}{dx}\mathcal{L}_n^*(f_\delta,x)\right| &= \left|\frac{n}{m_0-n}\sum_{k=0}^{\infty}\left(\frac{1}{\int_0^\infty \varphi_n(t)dt}\int_0^\infty f_\delta(t)\left(p'_{n-m_0,k+1}(t)\right)dt\right)\right.\\ &\quad \left.\times\frac{(-x)^k}{k!}\varphi_{n+m_0}^{(k)}(x)\right|\\ &\le \frac{1}{\delta}\frac{\gamma_{n-m_0}}{\gamma_n}\frac{n}{n-m_0}\omega_1(f;\delta).\end{aligned}$$

For any $t>0$ and $0<h\le t$, $t\in(0,\infty)$, we can write

$$\begin{aligned}|f(x+h)-f(x)| &\le |f(x+h)-\mathcal{L}_n^*(f,x+h)| + |f(x)-\mathcal{L}_n^*(f,x)|\\ &\quad + \left|\int_0^h \frac{d}{dx}\mathcal{L}_n^*(f_\delta,x+u)\,du\right| + \left|\int_0^h \frac{d}{dx}\mathcal{L}_n^*(f-f_\delta,x+u)\,du\right|\\ &\le 2K(\delta(n,m_0,x,h))^\alpha + \int_0^h \frac{1}{\delta}\frac{2n}{n-m_0}\frac{\gamma_{n-m_0}}{\gamma_n}\omega_1(f;\delta)\,du\\ &\quad + \int_0^h 2\omega_1(f;\delta)\min\left\{\left(nm_0+\frac{n}{x+u}\right)^{1/2},n\right\}du\\ &= 2K(\delta(n,m_0,x,h))^\alpha + \frac{h}{\delta}\frac{2n}{n-m_0}\frac{\gamma_{n-m_0}}{\gamma_n}\omega_1(f;\delta)\\ &\quad + 2\omega_1(f;\delta)\int_0^h \min\left\{\left(nm_0+\frac{n}{x+u}\right)^{1/2},n\right\}du\\ &\le 2K(\delta(n,m_0,x,h))^\alpha + \frac{h}{\delta}\frac{2n}{n-m_0}\frac{\gamma_{n-m_0}}{\gamma_n}\omega_1(f;\delta)\\ &\quad + 6\omega_1(f;\delta)\frac{h}{\sqrt{\frac{1}{n^2}+\frac{1}{n^2(1+m_0(x+h))}}}\\ &\le 2K(\delta(n,m_0,x,h))^\alpha + 6h\omega_1(f;\delta)\left[\frac{n}{n-m_0}\frac{1}{\delta}+\frac{1}{\delta(n,m_0,x,h)}\right],\end{aligned}$$

where

$$\delta\left(n,m_0,x,h\right)=\left(\frac{1}{n^2}+\frac{1}{n^2(1+m_0(x+h))}\right)^{1/2}.$$

Note that for any $n\in\mathbb{N}$, we have

$$\frac{1}{2}\delta\left(n,m_0,x,h\right)\le\delta\left(n+1,m_0,x,h\right)\le\delta\left(n,m_0,x,h\right).$$

Hence, for $0<\delta<\frac{1}{2}$, we can choose $n\in\mathbb{N}$ such that

$$\delta\left(n,m_0,x,h\right)<\delta\le 2\delta\left(n,m_0,x,h\right).$$

For sufficiently large n, we get

$$\begin{aligned}\left|f\left(x+h\right)-f(x)\right| &\le 2K\left(\delta\left(n,m_0,x,h\right)\right)^{\alpha}+6h\omega_1\left(f;\delta\right)\left[\frac{2}{\delta}+\frac{1}{\delta\left(n,m_0,x,h\right)}\right]\\ &\le 2K\delta^{\alpha}+24\frac{h}{\delta}\omega_1\left(f;\delta\right)\\ &\le \max\left\{2K,24\right\}\left(\delta^{\alpha}+\frac{h}{\delta}\omega_1\left(f;\delta\right)\right).\end{aligned}$$

From last inequality, for $0<h\le t$, we get

$$\omega_1\left(f;t\right)\le K'\left(\delta^{\alpha}+\frac{t}{\delta}\omega_1\left(f;\delta\right)\right)$$

which implies $\omega_1\left(f;t\right)=O(t^{\alpha})$, as desired. □

Theorem 4. *For $f\in C[0,\infty)\cap L_{\infty}[0,\infty)$, $0<\alpha<2$, we have*

$$\begin{aligned}\omega_2\left(f,t\right)=O\left(t^{\alpha}\right)\iff&\left|\frac{d^2}{dx^2}\mathcal{L}_n^*\left(f,x\right)\right|\\ &\le M\min\left\{\frac{2n}{x}+4n^2+6nm_0,4n\left(m_0+n\right)\right\}^{(2-\alpha)/2},(n>2m_0),\end{aligned}$$

where $\omega_2(f,.)$ is the modulus of smoothness of f defined by (3).

Proof. (:⟹) We assume that $\omega_2\left(f,t\right)\le Mt^{\alpha}$. For $g\in C_B\left[0,\infty\right)$, we get the second-order derivative of the operator $\mathcal{L}_n^*\left(g,x\right)$ with respect to x as

$$\begin{aligned}\frac{d^2}{dx^2}\mathcal{L}_n^*\left(g,x\right)&=n\left(m_0+n\right)\sum_{k=0}^{\infty}\left(\frac{1}{\int_0^{\infty}\varphi_n(t)dt}\int_0^{\infty}g(t)\left(p_{n,k}(t)\right.\right.\\ &\left.\left.-2p_{n,k+1}\left(t\right)+p_{n,k+2}(t)\right)dt\right)\frac{(-x)^k}{k!}\varphi_{n+m_0}^{(k)}(x)\\ &=n\left(m_0+n\right)x^{-2}\sum_{k=0}^{\infty}\left(\frac{1}{\int_0^{\infty}\varphi_n(t)dt}\int_0^{\infty}f(t)p_{n,k}(t)dt\right).\end{aligned}$$

Hence,

$$\left|\frac{d^2}{dx^2}\mathcal{L}_n^*\left(g,x\right)\right|\le 4n\left(m_0+n\right)\left\|g\right\|_{\infty}$$

and

$$\left|\frac{d^2}{dx^2}\mathcal{L}_n^*\left(g,x\right)\right|\le\left(\frac{2n}{x}+4n^2+6nm_0\right)\left\|g\right\|_{\infty}.$$

Now for $f\in C[0,\infty)\cap L_{\infty}[0,\infty)$, let us define the Steklov function as

$$f_d(x)=\frac{4}{d^2}\int\int_0^{d/2}\left(2f\left(x+u+v\right)-f\left(x+2u+2v\right)\right)dudv.$$

Then,

$$\left\|f-f_d\right\|\le\omega_2\left(f,d\right)$$

and

$$\left\|f_d''\right\|\le\frac{9}{d^2}\omega_2\left(f,d\right)$$

For f_d, one can verify

$$\begin{aligned}\left|\frac{d^2}{dx^2}\mathcal{L}_n^*\left(f_d,x\right)\right|&=\left|\sum_{k=0}^{\infty}\left(\frac{1}{\int_0^{\infty}\varphi_n(t)dt}\int_0^{\infty}f_d''(t)p_{n+2m_0,k+2}\left(t\right)dt\right)\right|\\ &\le\frac{1}{n\left(m_0+n\right)}\frac{9}{d^2}\omega_2\left(f,d\right).\end{aligned}$$

Choosing $d=\min\left\{\frac{2n}{x}+4n^2+6nm_0,4n\left(m_0+n\right)\right\}^{-1/2}$, we get

$$\begin{aligned}\left|\frac{d^2}{dx^2}\mathcal{L}_n^*\left(f,x\right)\right|&\le\left|\frac{d^2}{dx^2}\mathcal{L}_n^*\left(f_d,x\right)\right|+\left|\frac{d^2}{dx^2}\mathcal{L}_n^*\left(f-f_d,x\right)\right|\\ &\le\frac{1}{n\left(m_0+n\right)}\frac{9}{d^2}\omega_2\left(f,d\right)+\min\left\{\frac{2n}{x}\right.\\ &\left.+4n^2+6nm_0,4n\left(m_0+n\right)\right\}\omega_2\left(f,d\right),\end{aligned}$$

which proves the necessity part of the theorem. (⟸:) Now, in order to prove the sufficiency part of the theorem, we define the combination of $\mathcal{L}_{n,1}^*$ as follows

$$\mathcal{L}_{n,1}^*\left(f,x\right)=a_0(n)\mathcal{L}_{n_0}\left(f,x\right)+a_1(n)\mathcal{L}_{n_1}\left(f,x\right),$$

where $|a_0(n)|+|a_1\left(n\right)|\le B$, $n=n_0<n_1\le An$ with A and B as absolute constants having the property

$$\mathcal{L}_{n,1}^*\left(t^i,x\right)=x^i,\quad i=0,1.$$

Using the methods in [9,10] for $f\in C[0,\infty)\cap L_{\infty}[0,\infty)$, we have

$$\left|\mathcal{L}_{n,1}^*\left(f,x\right)-f(x)\right|\le M\omega_2\left(f,\sqrt{\delta_n(x)}\right),\delta_n(x)=\frac{2\left(m_0x+1\right)\left(\left(n+m_0\right)x+1\right)}{\left(n-2m_0\right)\left(n-m_0\right)}.$$

For $m, n \in N, x \in (0, \infty), 0 < h \le t$, we have

$$\left|\mathcal{L}_m^*(f,x) - 2\mathcal{L}_m^*(f,x+h) + \mathcal{L}_m^*(f,x+2h)\right| \le 4M\omega_2\left(\mathcal{L}_m^* f, \sqrt{\delta_n(x+2h)}\right) + \int\int_0^h \left|\frac{d^2}{dx^2}\mathcal{L}_n^*\left(\mathcal{L}_{n,1}^* f, x+u+v\right)\right| dudv. \quad (14)$$

Now, we shall estimate the second term on the right-hand side of the above inequality. Firstly, we have

$$\left|\frac{d^2}{dx^2}\mathcal{L}_{n,1}^*(f,x)\right| \le 4B(An)^{2-\alpha}.$$

On the other hand, we also have

$$\left|\frac{d^2}{dx^2}\mathcal{L}_{n,1}^*(f,x)\right| \le B\left(\frac{An}{x} + 6(An)^2\right)^{(2-\alpha)/2} \le MBAn^{(2-\alpha)/2}\left(\frac{1}{x}\right)^{(2-\alpha)/2}.$$

From which, we can write

$$\left(\frac{1}{x}\right)^{\alpha/2-1}\left|\frac{d^2}{dx^2}\mathcal{L}_{n,1}^*(f,x)\right| \le MBAn^{(2-\alpha)/2}.$$

Hence, using the above inequality, we get

$$\left|\left(\frac{1}{x}\right)^{\alpha/2-1}\frac{d^2}{dx^2}\mathcal{L}_m^*\left(\mathcal{L}_{n,1}^* f, x\right)\right| = \left|\left(\frac{1}{x}\right)^{\alpha/2-1}\sum_{k=0}^{\infty}\left(\frac{1}{\int_0^\infty \varphi_n(t)dt}\int_0^\infty \frac{d^2}{dx^2}\mathcal{L}_{n,1}^*(f,t)\,p_{n+2m_0,k+2}(t)dt\right) \times \frac{(-x)^k}{k!}\varphi_{n+m_0}^{(k)}(x)\right| \le MBAn^{(2-\alpha)/2}.$$

After making some arrangements and then taking the integral of both sides of the above inequality, we get

$$\int\int_0^h \left|\frac{d^2}{dx^2}\mathcal{L}_n^*\left(\mathcal{L}_{n,1}^* f, x+u+v\right)\right| dudv \le MBAn^{(2-\alpha)/2}\int\int_0^h \left|\frac{1}{x+u+v}\right|^{\frac{\alpha-1}{2}} dudv \le MBAn^{(2-\alpha)/2}h^2\left(\frac{n}{x+2h}\right)^{(2-\alpha)/2}. \quad (15)$$

Now, substituting (15) into (14), we finally obtain

$$\left|\mathcal{L}_m^*(f,x) - 2\mathcal{L}_m^*(f,x+h) + \mathcal{L}_m^*(f,x+2h)\right| \le 4M\omega_2\left(\mathcal{L}_m^* f, \sqrt{\delta_n(x+2h)}\right) + M_1 h^2 \left(M_n(x+2h)\right)^{(2-\alpha)}, \quad (16)$$

where $M_n(x) = \sqrt{\frac{x}{n}}$. Choosing $n \in \mathbb{N}$ such that

$$\frac{t}{2C} \le \max\left\{\sqrt{\delta_n(x+2h)}, M_n(x+2h)\right\} \le \frac{t}{C},$$

we obtain from (16) by induction

$$\omega_2(\mathcal{L}_m^* f, t) \le 4M\omega_2\left(\mathcal{L}_m^* f, \frac{t}{C}\right) + (2C)^{2-\alpha}M_2 t^\alpha$$
$$\vdots$$
$$\le t^2(4M)^k C^{-2k}\left\|\frac{d^2}{dx^2}\mathcal{L}_m^* f\right\| + (2C)^{2-\alpha}M_2 t^\alpha \frac{C^\alpha}{C^\alpha - 4M}. \quad (17)$$

If we take $C = (1+4M)^{1/\alpha}$ and let $k \to \infty$, we obtain

$$\omega_2\left(\mathcal{L}_m^* f, t\right) \le \frac{1}{C^\alpha - 4M}4C^2M_2t^\alpha$$

which implies that $\omega_2(f,t) = O(t^\alpha)$, where $\frac{1}{C^\alpha - 4M}4C^2M_2$ is independent of m. So, the proof is completed. □

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
ÇA, SKS, and İB equally contributed to the making of this paper. All authors read and approved the final manuscript.

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