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Coupled fixed point theorems for contractions involving altering distances in ordered metric spaces

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Abstract

In this paper, we establish a coupled fixed point result of $F : X \times X \to X$ having the mixed monotone property involving generalized altering distance functions in five variables on ordered metric spaces. An example is given to support the usability of our results. We also give a coupled fixed point result involving a contraction of integral type.

Keywords: Coupled fixed point, Complete metric space, Generalized altering distance function, Weakly contractive condition, Mixed monotone property, Partially ordered set

MSC: Primary 54H25; Secondary 47H10.

Introduction and preliminaries

The very famous Banach contraction principle [1] can be stated as follows.

Theorem 1. [1]. *Let* (X, d) *be a complete metric space and* T *be a mapping of* X *into itself satisfying:*

$$d(Tx, Ty) \le kd(x, y), \forall\, x, y \in X, \tag{1.1}$$

where k *is a constant in* $(0, 1)$*. Then,* T *has a unique fixed point* $x^* \in X$*.*

In the literature, there is a great number of generalizations of the Banach contraction principle. Khan et al. [2] introduced the notion of an altering distance function, which is a control function that alters distance between two points in a metric space.

Definition 2. [2]. A function $\varphi : [0, +\infty) \to [0, +\infty)$ is called an *altering distance function*if and only if

(i) φ Is continuous,
(ii) φ Is non-decreasing, and
(iii) $\varphi(t) = 0 \Leftrightarrow t = 0$.

Afterwards, a number of works have appeared in which altering distances have been used. Altering distances have been generalized to a two-variable function by Choudhury and Dutta [3] and to a three-variable function by Choudhury [4] and was applied for obtaining fixed point results in metric spaces.

Definition 3. [4]. Let Ψ_3 denote the set of all functions $\varphi : [0, +\infty) \times [0, +\infty) \times [0, +\infty) \to [0, +\infty)$. Then φ is said to be a *generalized altering distance function* if and only if

(i) φ is continuous,
(ii) φ is nondecreasing in all the three variables, and
(iii) $\varphi(x, y, z) = 0 \Leftrightarrow x = y = z = 0$.

In [5], Rao et al. introduced the generalized altering distance function in five variables as a generalization of three variables.

Definition 4. [5]. Let Ψ_5 denote the set of all functions $\varphi : [0, +\infty) \times [0, +\infty) \times [0, +\infty) \times [0, +\infty) \times [0, +\infty) \to [0, +\infty)$. Then φ is said to be a generalized altering distance function if and only if

(i) φ is continuous,
(ii) φ is non-decreasing in all five variables, and
(iii) $\varphi(t_1, t_2, t_3, t_4, t_5) = 0 \Leftrightarrow t_1 = t_2 = t_3 = t_4 = t_5 = 0$.

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On the other hand, the first result on existence of fixed points in partially ordered sets was given by Turinici [6], where he extended the Banach contraction principle in partially ordered sets. Ran and Reurings [7] presented some applications of Turinici's theorem to matrix equations. The obtained result by Turinici was further extended and refined in [8-33]. Subsequently, Harjani and Sadarangani [17] generalized their own results [18] by considering a pair of altering functions (ψ, φ). Nashine and Altun [20] and Nashine and Bessem [22] generalized the results of Harjani and Sadarangani [17,18]. Nashine, Samet, and Vetro [23] also had fixed point theorems for T-weakly isotone-increasing mappings which satisfy a generalized nonlinear contractive condition in complete ordered metric spaces and gave an application to an existence theorem for a solution of some integral equations. Jachymski[30] established a geometric lemma (Lemma 1 in [30]) giving a list of equivalent conditions for some subsets of the plane. Using this lemma, he proved that some very recent fixed point theorems for generalized contractions on ordered metric spaces obtained by Harjani and Sadarangani [17,18], and Amini-Harandi and Emami [10] follow from an earlier result of O'Regan and Petruşel (Theorem 3.6 in [28]).

Now, we introduce some known notations and definitions that will be used later.

Definition 5. Let X be a nonempty set. Then $(X, d, \preceq)$ is called an ordered metric space iff

(i) (X, d) is a metric space, and
(ii) $(X, \preceq)$ is partially ordered.

Definition 6. Let $(X, \preceq)$ be a partial ordered set. Then $x, y \in X$ are called comparable if $x \preceq y$ or $y \preceq x$ holds.

Let $(X, \preceq)$ be a partially ordered set. The concept of a mixed monotone property of the mapping $F : X \times X \to X$ has been introduced by Bhaskar and Lakshmikantham [13].

Definition 7. (see Bhaskar and Lakshmikantham [13]). Let $(X, \preceq)$ be a partially ordered set and $F : X \times X \to X$. Then the map F is said to have mixed monotone property if $F(x, y)$ is monotone nondecreasing in x and is monotone non-increasing in y; that is, for any $x, y \in X$,

$$x_1 \preceq x_2 \text{ implies } F(x_1, y) \preceq F(x_2, y) \text{ for all } y \in X$$

and

$$y_1 \preceq y_2 \text{ implies } F(x, y_2) \preceq F(x, y_1) \text{ for all } x \in X.$$

Definition 8. (see Bhaskar and Lakshmikantham [13]). An element $(x, y) \in X \times X$ is called a *coupled fixed point of a mapping* $F : X \times X \to X$ if

$$F(x, y) = x \text{ and } F(y, x) = y.$$

Bhaskar and Lakshmikantham [13] proved the following coupled fixed point theorem. For other coupled fixed point results, see [34-46].

Theorem 9. *(see Bhaskar and Lakshmikantham [13]). Let $(X, \preceq)$ be a partially ordered set and suppose there is a metric d on X such that (X, d) is a complete metric space. Let $F : X \times X \to X$ be a mapping having the mixed monotone property on X. Assume that there exists a $k \in [0, 1)$ with*

$$d(F(x, y), F(u, v)) \preceq \frac{k}{2} [\, d(x, u) + d(y, v)] \tag{1.2}$$

for all $x, y, u, v \in X$ with $x \preceq u$ and $v \preceq y$. Suppose either F is continuous or X has the following properties:

1. *If a non-decreasing sequence $x_n \to x$, then $x_n \preceq x$ for all n.*
2. *If a non-increasing sequence $y_n \to y$, then $y \preceq y_n$ for all n.*

If there exist $x_0, y_0 \in X$ such that $x_0 \preceq F(x_0, y_0)$ and $F(y_0, x_0) \preceq y_0$, then there exist $x, y \in X$ such that $x = F(x, y)$ and $y = F(y, x)$, that is, F has a coupled fixed point.

In this paper, we first obtain a coupled fixed point result for $F : X \times X \to X$ having the mixed monotone property satisfying a contractive condition which involves generalized altering distance functions in five variables in complete ordered metric spaces. An example is also been given to the validity of our results. In particular, in this example, we will show that the result of Bhaskar and Lakshmikantham [13] cannot be applied. Finally, we establish a coupled fixed point result involving a contraction of integral type.

Main results

Before stating our main theorem, the following lemma is needed:

Lemma 10. Let (X, d) be a metric space and let $\{y_n\}$ be a sequence in X such that

$$\lim_{n \to \infty} d(y_{n+1}, y_n) = 0.$$

If $\{y_n\}$ is not a Cauchy sequence, then there exist $\epsilon > 0$ and two sequences $\{m_k\}$ and $\{n_k\}$ of positive integers such that the following four sequences tend to ε when $k \to \infty$:

$$d(y_{m_k}, y_{n_k}), \quad d(y_{m_k}, y_{n_k+1}), \quad d(y_{m_k-1}, y_{n_k}), \quad d(y_{m_k-1}, y_{n_k+1}). \tag{2.1}$$

Our main theorem is the following.

Theorem 11. *Let $(X, \preceq)$ be a partially ordered set and suppose that there exists a metric d on X such that (X, d) is a complete metric space. Let $F : X \times X \longrightarrow X$ be a mapping having the mixed monotone property on X such that*

$$\begin{aligned}\Phi_1(d(F(x,y),F(u,v))) \leq\ & \psi_1(d(x,u),d(y,v),d(x,F(x,y)),\\ & d(u,F(u,v)),\frac{1}{2}[\,d(x,F(u,v))+d(u,F(x,y))]\,)\\ & -\psi_2((d(x,u),d(y,v),d(x,F(x,y)),d(u,F(u,v)),\\ & \frac{1}{2}[\,d(x,F(u,v))+d(u,F(x,y))]\,),\end{aligned} \tag{2.2}$$

for $x, y, u, v \in X$ with $x \succeq u$ and $y \preceq v$, where ψ_1 and ψ_2 are generalized altering distance functions and $\Phi_1(x) = \psi_1(x,x,x,x,x)$. Assume either

1. *F is continuous, or*
2. *X has the following properties:*
    (a) *if a non-decreasing sequence $x_n \to x$, then $x_n \preceq x$ for all n,*
    (b) *if a non-increasing sequence $y_n \to y$, then $y \preceq y_n$ for all n.*

If there exist $x_0, y_0 \in X$ such that $x_0 \preceq F(x_0, y_0)$ and $y_0 \succeq F(y_0, x_0)$, then F has a coupled fixed point, that is, there exists $(x, y) \in X$ such that $F(x, y) = x$ and $F(y, x) = y$.

Proof. Let $x_0, y_0 \in X$ be arbitrarily chosen and define $x_1 := F(x_0, y_0)$ and $y_1 := F(y_0, x_0)$. Next, we consider $x_2 := F(x_1, y_1)$ and $y_2 := F(y_1, x_1)$. Continuing in this way, we construct two sequences $\{x_n\}$ and $\{y_n\}$ in X such that

$$x_{n+1} = F(x_n, y_n) \text{ and } y_{n+1} = F(y_n, x_n) \ \forall\ n \geq 0. \tag{2.3}$$

Using the fact that F has a mixed monotone property, we have as in [13]

$$x_n \preceq x_{n+1} \quad \text{and} \quad y_{n+1} \preceq y_n \quad \forall\ n \geq 0. \tag{2.4}$$

Since $x_n \succeq x_{n-1}$ and $y_n \preceq y_{n-1}$, then from (2.2)

$$\begin{aligned}\Phi_1(d(x_{n+1},x_n)) &= \Phi_1(d(F(x_n,y_n),F(x_{n-1},y_{n-1})))\\ &\leq \psi_1(d(x_n,x_{n-1}),d(y_n,y_{n-1}),d(x_n,F(x_n,y_n)),d(x_{n-1},\\ &\qquad F(x_{n-1},y_{n-1})),\\ &\qquad \frac{1}{2}[\,d(x_n,F(x_{n-1},y_{n-1}))+d(x_{n-1},F(x_n,y_n))]\,)\\ &\quad -\psi_2(d(x_n,x_{n-1}),d(y_n,y_{n-1}),d(x_n,F(x_n,y_n)),\\ &\qquad d(x_{n-1},F(x_{n-1},y_{n-1})),\\ &\qquad \frac{1}{2}[\,d(x_n,F(x_{n-1},y_{n-1}))+d(x_{n-1},F(x_n,y_n))]\,)\\ &= \psi_1(d(x_n,x_{n-1}),d(y_n,y_{n-1}),d(x_n,x_{n+1}),d(x_{n-1},x_n),\\ &\qquad \frac{1}{2}d(x_{n-1},x_{n+1}))\\ &\quad -\psi_2(d(x_n,x_{n-1}),d(y_n,y_{n-1}),d(x_n,x_{n+1}),d(x_{n-1},x_n),\\ &\qquad \frac{1}{2}d(x_{n-1},x_{n+1}))\\ &\leq \psi_1(d(x_n,x_{n-1}),d(y_n,y_{n-1}),d(x_n,x_{n+1}),d(x_{n-1},x_n),\\ &\qquad \frac{1}{2}d(x_{n-1},x_{n+1})).\end{aligned} \tag{2.5}$$

Since ψ_1 is monotone increasing with respect to the first variable, we have for all $n \geq 1$

$$d(x_{n+1}, x_n) \leq d(x_n, x_{n-1}). \tag{2.6}$$

Since $x_n \succeq x_{n-1}$ and $y_n \preceq y_{n-1}$, then from (2.2)

$$\begin{aligned}\Phi_1(d(y_n,y_{n+1})) &= \Phi_1(d(F(y_{n-1},x_{n-1}),F(y_n,x_n)))\\ &= \Phi_1(d(F(y_n,x_n),F(y_{n-1},x_{n-1})))\\ &\leq \psi_1(d(y_n,y_{n-1}),d(x_n,x_{n-1}),d(y_n,F(y_n,x_n)),d(y_{n-1},\\ &\qquad F(y_{n-1},x_{n-1})),\\ &\qquad \frac{1}{2}[\,d(y_n,F(y_{n-1},x_{n-1}))+d(y_{n-1},F(y_n,x_n))]\,)\\ &\quad -\psi_2(d(y_n,y_{n-1}),d(x_n,x_{n-1}),d(y_n,F(y_n,x_n)),d(y_{n-1},\\ &\qquad F(y_{n-1},x_{n-1})),\\ &\qquad \frac{1}{2}[\,d(y_n,F(y_{n-1},x_{n-1}))+d(y_{n-1},F(y_n,x_n))]\,)\\ &= \psi_1(d(y_n,y_{n-1}),d(x_n,x_{n-1}),d(y_n,y_{n+1}),d(y_{n-1},y_n),\\ &\qquad \frac{1}{2}d(y_{n-1},y_{n+1}))\\ &\quad -\psi_2(d(y_n,y_{n-1}),d(x_n,x_{n-1}),d(y_n,y_{n+1}),d(y_{n-1},y_n),\\ &\qquad \frac{1}{2}d(y_{n-1},y_{n+1}))\\ &\leq \psi_1(d(y_n,y_{n-1}),d(x_n,x_{n-1}),d(y_n,y_{n+1}),d(y_{n-1},y_n),\\ &\qquad \frac{1}{2}d(y_{n-1},y_{n+1})).\end{aligned} \tag{2.7}$$

Since ψ_1 is monotone increasing with respect to the first variable, we have for all $n \geq 1$

$$d(y_{n+1}, y_n) \leq d(y_n, y_{n-1}). \tag{2.8}$$

In view of (2.6) and (2.8), the sequences $\{d(x_{n+1}, x_n)\}$ and $\{d(y_{n+1}, y_n)\}$ are nonincreasing, so there exist $r \geq 0$ and $\gamma \geq 0$ such that

$$\lim_{n\to\infty} d(x_{n+1}, x_n) = r \quad \text{and} \quad \lim_{n\to\infty} d(y_{n+1}, y_n) = \gamma.$$

Again, since ψ_1 is monotone increasing with respect to the fifth variable, from (2.5), we have by triangular inequality

$$d(x_{n+1}, x_n) \leq \frac{1}{2} d(x_{n-1}, x_{n+1}) \leq \frac{1}{2} d(x_{n-1}, x_n) + \frac{1}{2} d(x_n, x_{n+1});$$

so in the limit, we have

$$\lim_{n\to\infty} d(x_{n-1}, x_{n+1}) = 2r.$$

Similarly, from (2.7)

$$\lim_{n\to\infty} d(y_{n-1}, y_{n+1}) = 2\gamma.$$

Passing on the limit $n \to \infty$ in (2.5) and (2.7) respectively and using the continuities of ψ_1, ψ_2, we get

$$\Phi_1(r) \leq \psi_1(r, \gamma, r, r, r) - \psi_2(r, \gamma, r, r, r),$$

and

$$\Phi_1(\gamma) \leq \psi_1(\gamma, r, \gamma, \gamma, \gamma) - \psi_2(\gamma, r, \gamma, \gamma, \gamma).$$

Assume that $\gamma \neq r$. Without loss of generality, suppose that $\gamma < r$, so

$$\Phi_1(r) \leq \psi_1(r, \gamma, r, r, r) - \psi_2(r, \gamma, r, r, r) \leq \Phi_1(r) - \psi_2(r, \gamma, r, r, r),$$

which holds unless $\psi_2(r, \gamma, r, r, r) = 0$, that is, $r = \gamma$ (which is equal to 0 using the same idea), a contradiction. We deduce that

$$\lim_{n\to\infty} d(x_{n+1}, x_n) = \lim_{n\to\infty} d(y_{n+1}, y_n) = 0. \tag{2.9}$$

Now we shall show that $\{x_n\}$ and $\{y_n\}$ are Cauchy sequences in (X, d).

Suppose that $\{x_n\}$ and $\{y_n\}$ are not Cauchy sequences. Then, Lemma 10 implies that there exist $\varepsilon > 0$ and two sequences $\{m_k\}$ and $\{n_k\}$ of positive integers (with for all positive integer k, $m(k) > n(k)$) such that the sequences

$$d(x_{m_k}, x_{n_k}), \quad d(x_{m_k}, x_{n_k+1}), \quad d(x_{m_k-1}, x_{n_k}), \quad d(x_{m_k-1}, x_{n_k+1}), \tag{2.10}$$

and

$$d(y_{m_k}, y_{n_k}), \quad d(y_{m_k}, y_{n_k+1}), \quad d(y_{m_k-1}, y_{n_k}), \quad d(y_{m_k-1}, y_{n_k+1}) \tag{2.11}$$

tend to ε (from above) when $k \to \infty$. It follows that

$$\limsup_{k\to\infty} d(x_{n_k-1}, x_{m_k+1}) \leq \varepsilon \quad \text{and} \quad \limsup_{k\to\infty} d(y_{n_k-1}, y_{m_k+1}) \leq \varepsilon. \tag{2.12}$$

Since $m(k) \geq n(k) - 1$, so from (2.4), $x_{m(k)} \succeq x_{n(k)-1}$ and $y_{m(k)} \preceq y_{n(k)-1}$, we have by (2.2)

$$\begin{aligned}
&\Phi_1(d(x_{m(k)+1}, x_{n(k)})) \\
&= \Phi_1(d(F(x_{m(k)}, y_{m(k)}), F(x_{n(k)-1}, y_{n(k)-1}))) \\
&\leq \psi_1(d(x_{m(k)}, x_{n(k)-1}), d(y_{m(k)}, y_{n(k)-1}), d(x_{m(k)}, F(x_{m(k)}, y_{m(k)})), d(x_{n(k)-1}, F(x_{n(k)-1}, y_{n(k)-1})), \\
&\quad \frac{1}{2}[\, d(x_{m(k)}, F(x_{n(k)-1}, y_{n(k)-1})) + d(x_{n(k)-1}, F(x_{m(k)}, y_{m(k)}))]\,) \\
&\quad - \psi_2(d(x_{m(k)}, x_{n(k)-1}), d(y_{m(k)}, y_{n(k)-1}), d(x_{m(k)}, F(x_{m(k)}, y_{m(k)})), d(x_{n(k)-1}, F(x_{n(k)-1}, y_{n(k)-1})), \\
&\quad \frac{1}{2}[\, d(x_{m(k)}, F(x_{n(k)-1}, y_{n(k)-1})) + d(x_{n(k)-1}, F(x_{m(k)}, y_{m(k)}))]\,).
\end{aligned} \tag{2.13}$$

In addition, we have

$$\begin{aligned}
&\Phi_1(d(y_{n(k)}, y_{m(k)+1})) \\
&= \Phi_1(d(F(y_{n(k)-1}, x_{n(k)-1}), F(y_{m(k)}, x_{m(k)}))) \\
&\leq \psi_1(d(y_{n(k)-1}, y_{m(k)}), d(x_{n(k)-1}, x_{m(k)}), d(y_{n(k)-1}, F(y_{n(k)-1}, x_{n(k)-1})), d(y_{m(k)}, F(y_{m(k)}, x_{m(k)})), \\
&\quad \frac{1}{2}[\, d(y_{n(k)-1}, F(y_{m(k)}, x_{m(k)})) + d(y_{m(k)}, F(y_{n(k)-1}, x_{n(k)-1}))]\,) \\
&\quad - \psi_2(d(y_{n(k)-1}, y_{m(k)}), d(x_{n(k)-1}, x_{m(k)}), d(y_{n(k)-1}, F(y_{n(k)-1}, x_{n(k)-1})), d(y_{m(k)}, F(y_{m(k)}, x_{m(k)})), \\
&\quad \frac{1}{2}[\, d(y_{n(k)-1}, F(y_{m(k)}, x_{m(k)})) + d(y_{m(k)}, F(y_{n(k)-1}, x_{n(k)-1}))]\,).
\end{aligned} \tag{2.14}$$

Taking the lim sup as $k \to \infty$, using (2.10 to 2.12) and the continuity of ψ_1 and ψ_2 in (2.13), we get

$$\begin{aligned}
\Phi_1(\varepsilon) &\leq \psi_1(\varepsilon, \varepsilon, 0, 0, \varepsilon) - \psi_2(\varepsilon, \varepsilon, 0, 0, \frac{1}{2} \liminf_{k\to\infty} [\, d(x_{n(k)-1}, F(x_{m(k)}, x_{m(k)})) + d(x_{m(k)}, F(x_{n(k)-1}, y_{n(k)-1}))]\,) \\
&\leq \Phi(\varepsilon) - \psi_2(\varepsilon, \varepsilon, 0, 0, \frac{1}{2} \liminf_{k\to\infty} [\, d(x_{n(k)-1}, F(x_{m(k)}, y_{m(k)})) + d(x_{m(k)}, F(x_{n(k)-1}, y_{n(k)-1}))]\,),
\end{aligned}$$

which implies that $\psi_2(\varepsilon, \varepsilon, 0, 0, \frac{1}{2} \liminf_{k\to\infty} [\, d(y_{n(k)-1}, F(y_{m(k)}, x_{m(k)})) + d(y_{m(k)}, F(y_{n(k)-1}, x_{n(k)-1}))]\,) = 0$, is a contradiction since $\varepsilon > 0$. We deduce that $\{x_n\}$ is a Cauchy sequence.

Similarly, taking the lim sup as $k \to +\infty$ and using the continuity of φ and ψ in (2.14), we get

$$\psi_2(\varepsilon, \varepsilon, 0, 0, \frac{1}{2} \liminf_{k\to\infty} [\, d(y_{n(k)-1}, F(y_{m(k)}, x_{m(k)})) + d(y_{m(k)}, F(y_{n(k)-1}, x_{n(k)-1}))]\,) = 0,$$

which implies that $\varepsilon = 0$, is a contradiction since $\varepsilon > 0$. We deduce that $\{y_n\}$ is a Cauchy sequence. Since (X, d) is a complete metric space, so there exist points x and y in X such that

$$\lim_{n\to\infty} d(x_n, x) = 0 \textit{ and } \lim_{n\to\infty} d(y_n, y) = 0. \tag{2.15}$$

Now, we prove that $x = F(x, y)$, which becomes

$$\begin{aligned} d(F(x,y), x) &\le d(F(x,y), x_{n+1}) + d(x_{n+1}, x) \\ &\le d(F(x,y), x_{n+1}) + d(x_{n+1}, x). \end{aligned}$$

It follows by (2.15) that

$$d(F(x,y), x) \le \limsup_{n\to\infty} d(F(x,y), x_{n+1}).$$

Then, since Φ_1 is nondecreasing and continuous, we get that

$$\begin{aligned} \Phi_1(d(F(x,y), x)) &\le \limsup_{n\to\infty} \Phi_1(d(F(x,y), x_{n+1})) \\ &= \Phi_1(\limsup_{n\to\infty} d(F(x,y), x_{n+1})). \end{aligned} \tag{2.16}$$

Now, from (2.2)

$$\begin{aligned} &\Phi_1(d(F(x,y), x_{n+1})) \\ &= \Phi_1(d(F(x,y), F(x_n, y_n))) \\ &\le \psi_1(d(x, x_n), d(y, y_n), d(x, F(x,y)), d(x_n, F(x_n, y_n)), \\ &\frac{1}{2}[\, d(x, F(x_n, y_n)) + d(x_n, F(x,y))]\,) \\ &\quad - \psi_2(d(x, x_n), d(y, y_n), d(x, F(x,y)), d(x_n, F(x_n, y_n)), \\ &\frac{1}{2}[\, d(x, F(x_n, y_n)) + d(x_n, F(x,y))]\,). \end{aligned} \tag{2.17}$$

Passing to the upper limit as $n \to \infty$ in (2.17), we obtain using the continuity of ψ_1, ψ_2 that is

$$\begin{aligned} &\limsup_{n\to\infty} \Phi_1(d(F(x,y), x_{n+1})) \\ &\le \psi_1(0, 0, d(x, F(x,y)), 0, \frac{1}{2} d(x, F(x,y))) \\ &- \psi_2(0, 0, d(x, F(x,y)), 0, \frac{1}{2} d(x, F(x,y))). \end{aligned}$$

Therefore, from (2.16) we have

$$\begin{aligned} \Phi_1(d(F(x,y), x)) &\le \Phi_1(d(F(x,y), x)) \\ &\quad - \psi_2(0, 0, d(x, F(x,y)), 0, \frac{1}{2} d(x, F(x,y))), \end{aligned}$$

which implies that $d(x, F(x,y)) = 0$. Thus, we deduce that

$$F(x,y) = x. \tag{2.18}$$

Similarly, we may show that $F(y,x) = y$. Thus (x,y) is a coupled fixed point of F. Suppose that assumption 2 holds. Since $\{x_n\}$ is a nondecreasing sequence that converges to x in (X, d); by the assumption on X, we get that $x_n \preceq x$ for all $n \in \mathbb{N}$. Similarly, $\{y_n\}$ is a nonincreasing sequence convergent to y in (X, d); by the assumption on X, we get that $y_n \succeq y$ for all $n \in \mathbb{N}$. By (2.2), we obtain

$$\begin{aligned} &\Phi_1(d(F(x,y), x_{n+1})) \\ &= \Phi_1(d(F(x,y), F(x_n, y_n))) \\ &\le \psi_1(d(x, x_n), d(y, y_n), d(x, F(x,y)), d(x_n, F(x_n, y_n)), \\ &\qquad \frac{1}{2}[\, d(x, F(x_n, y_n)) + d(x_n, F(x,y))]\,) \\ &- \psi_2(d(x, x_n), d(y, y_n), d(x, F(x,y)), d(x_n, F(x_n, y_n)), \\ &\qquad \frac{1}{2}[\, d(x, F(x_n, y_n)) + d(x_n, F(x,y))]\,) \\ &= \psi_1(d(x, x_n), d(y, y_n), d(x, F(x,y)), d(x_n, x_{n+1}), \\ &\qquad \frac{1}{2}[\, d(x, x_{n+1}) + d(x_n, F(x,y))]\,) \\ &- \psi_2(d(x, x_n), d(y, y_n), d(x, F(x,y)), d(x_n, x_{n+1}), \\ &\qquad \frac{1}{2}[\, d(x, x_{n+1}) + d(x_n, F(x,y))]\,). \end{aligned} \tag{2.19}$$

Passing to the limit as $n \to \infty$ in (2.19) and using the continuity of ψ_1, ψ_2, we obtain

$$\begin{aligned} \Phi_1(d(F(x,y), x)) &\le \psi_1(0, 0, d(x, F(x,y)), 0, \frac{1}{2} d(x, F(x,y))) \\ &\quad - \psi_2(0, 0, d(x, F(x,y)), 0, \frac{1}{2} d(x, F(x,y))) \\ &\le \Phi_1(d(F(x,y), x)) - \psi_2(0, 0, d(x, F(x,y)), \\ &\quad 0, \frac{1}{2} d(x, F(x,y))), \end{aligned}$$

which implies that

$$d(x, F(x,y) = 0 \text{ and so } \quad F(x,y) = x. \tag{2.20}$$

Similarly, we may show that $F(y,x) = y$. Thus, (x,y) is a coupled fixed point of F. □

A number of coupled fixed point results may be obtained by assuming different forms for the functions ψ_1 and ψ_2. In particular, fixed point results under various contractive conditions may be derived from the above theorems.

Here, for example, we derive the following corollaries from our Theorem 11.

Corollary 2.1. *Let $(X, \preceq)$ be a partially ordered set and suppose that there exists a metric d on X such that (X, d) is a complete metric space. Let $F : X \times X \to X$ be a given* *mapping having the mixed monotone property. Assume there exists $k \in [0,1)$ such that*

$$d(F(x,y),F(u,v)) \le \frac{k}{5}[\,d(x,u)+d(y,v)+d(x,F(x,y)) + d(u,F(u,v))+\frac{1}{2}[\,d(x,F(u,v)) + d(u,F(x,y))]\,] \tag{2.21}$$

for $x,y,u,v \in X$ with $x \succeq u$ and $y \preceq v$. Assume either F is continuous, or X has the following properties:

(i) If a nondecreasing sequence $x_n \to x$, then $x_n \preceq x$ for all n.
(ii) If a nonincreasing sequence $y_n \to y$, then $y \preceq y_n$ for all n.

If there exist $x_0, y_0 \in X$ such that $x_0 \preceq F(x_0,y_0)$ and $y_0 \succeq F(y_0,x_0)$. Then, F has a coupled fixed point, that is, there exists $(x,y) \in X$ such that $F(x,y)=x$ and $F(y,x)=y$.

Proof. Let $\psi_1(t_1,t_2,t_3,t_4,t_5)=\frac{1}{5}[\,t_1+t_2+t_3+t_4+t_5]$ and $\psi_2(t_1,t_2,t_3,t_4,t_5)=\frac{1-k}{5}[\,t_1+t_2+t_3+t_4+t_5]$, where $k \in [0,1)$. Then $\Phi_1(t)=t$. Now, the corollary follows from Theorem 11. □

Corollary 2.2. *Let $(X,\preceq)$ be a partially ordered set and suppose that there exists a metric d on X such that (X,d) is a complete metric space. Let $F: X\times X \to X$ be a given mapping having the mixed monotone property such that there exists $k \in [0,1)$,*

$$d(F(x,y),F(u,v)) \le k\max\{d(x,u),d(y,v),d(x,F(x,y)), d(u,F(u,v)),\frac{1}{2}[\,d(x,F(u,v))+d(u,F(x,y))]\,)\}$$

for $x,y,u,v \in X$ with $x \succeq u$ and $y \preceq v$. Assume either F is continuous, or X has the following properties:

(i) If a nondecreasing sequence $x_n \to x$, then $x_n \preceq x$ for all n.
(ii) If a nonincreasing sequence $y_n \to y$, then $y \preceq y_n$ for all n.

If there exist $x_0, y_0 \in X$ such that $x_0 \preceq F(x_0,y_0)$ and $y_0 \succeq F(y_0,x_0)$. Then, F has a coupled fixed point, that is, there exists $(x,y) \in X$ such that $F(x,y)=x$ and $F(y,x)=y$.

Proof. It suffices to take

$$\psi_1(t_1,t_2,t_3,t_4,t_5)=\max\{t_1,t_2,t_3,t_4,t_5\}\ and \psi_2 (t_1,t_2,t_3,t_4,t_4)=(1-k)\psi_1(t_1,t_2,t_3,t_4,t_5),$$

where $k \in (0,1)$. Then $\Phi_1(t)=t$. □

Now, a consequence of Corollary 2.2 by taking $F(x,y)=fx$ where $f: X \to X$, is the following:

Corollary 2.3. *Let $(X,\preceq)$ be a partially ordered set and suppose that there exists a metric d on X such that (X,d) is a complete metric space. Let $f: X \to X$ be a nondecreasing given mapping such that*

$$d(fx,fu) \le k\max\{d(x,y),d(x,fx),d(y,fy), \frac{1}{2}[\,d(x,fy)+d(y,fx)]\,)\}$$

for $x,y \in X$ with $x \succeq y$ and $k \in [0,1)$. Assume either f is continuous, or X has the following property if a nondecreasing sequence $x_n \to x$, then $x_n \preceq x$ for all n.
If there exists $x_0 \in X$ such that $x_0 \preceq fx_0$, then f has a fixed point.

Remark 12. Corollary 2.3 is the ordered version of Ćirić's Theorem [14].

Now, we illustrate our results by an example.

Example 2.4. Let $X=[0,+\infty)$ be endowed with its Euclidian metric $d(x,y)=|x-y|$ and its usual ordering order $\le$. Take $F: X\times X \to X$ defined by

$$F(x,y)=\begin{cases}\frac{x-3y}{5} & \text{if } x \ge 3y\\ 0 & \text{if not.}\end{cases}$$

X satisfies the properties (i) and (ii) in Corollary 2.1. Take $k=\frac{5}{6}$. We claim that (2.21) holds for each $x \ge u$ and $y \le v$. We divide the proof into the following four cases:

- If $x \ge 3y$ and $u \ge 3v$, here we have $F(x,y)=\frac{x-3y}{5}$ and $F(u,v)=\frac{u-3v}{5}$,

$$\begin{aligned}
d(F(x,y),F(u,v)) &= |\frac{x-3y}{5}-\frac{u-3v}{5}|\\
&= \frac{x-u}{5}+\frac{3(v-y)}{5}\\
&= \frac{x-u}{6}+\frac{v-y}{6}+\frac{x-u}{30}+\frac{12(v-y)}{30}+\frac{v-y}{30}\\
&\le \frac{x-u}{6}+\frac{v-y}{6}+\frac{x}{30}+\frac{4u}{30}+\frac{v}{30}\text{since} v \le \frac{u}{3}\\
&\le \frac{1}{6}[\,(x-u)+(v-y)+(\frac{4x+3y}{5})+\frac{4u}{5}+\frac{3v}{5}]\\
&= \frac{k}{5}\left[d(x,u)+d(y,v)+d(x,F(x,y))+d(u,F(u,v))\right]\\
&\le \frac{k}{5}\left[d(x,u)+d(y,v)+d(x,F(x,y)) +d(u,F(u,v))+\frac{1}{2}[\,d(x,F(u,v))+d(u,F(x,y))]\right].
\end{aligned}$$

- If $x \geq 3y$ and $u < 3v$, here we have $F(x,y) = \frac{x-3y}{5}$ and $F(u,v) = 0$,

$$\begin{aligned} d(F(x,y),F(u,v)) &= \frac{x-3y}{5} \leq \frac{x}{5} \\ &= \frac{x-u}{6} + \frac{u}{6} + \frac{x}{30} \\ &\leq \frac{x-u}{6} + \frac{u}{6} + \frac{4x+3y}{30} \\ &= \frac{k}{5}\left[d(x,u) + d(u,F(u,v)) + d(x,F(x,y))\right] \\ &\leq \frac{k}{5}\left[d(x,u) + d(y,v) + d(x,F(x,y)) + d(u,F(u,v)) \right. \\ &\quad \left. + \frac{1}{2}[d(x,F(u,v)) + d(u,F(x,y))]\right]. \end{aligned}$$

- If $x < 3y$ and $u \geq 3v$, here we have $F(u,v) = \frac{u-3v}{5}$ and $F(x,y) = 0$,

$$\begin{aligned} d(F(x,y),F(u,v)) &= \frac{u-3v}{5} \\ &\leq \frac{u}{5} = \frac{u}{6} + \frac{u}{30} \\ &\leq \frac{x}{6} + \frac{4u+3v}{30} \\ &= \frac{k}{5}\left[d(x,F(x,y)) + d(u,F(u,v))\right] \\ &\leq \frac{k}{5}\left[d(x,u) + d(y,v) + d(x,F(x,y)) + d(u,F(u,v)) \right. \\ &\quad \left. + \frac{1}{2}[d(x,F(u,v)) + d(u,F(x,y))]\right]. \end{aligned}$$

- If $x < 3y$ and $u < 3v$, here we have $F(u,v) = F(x,y) = 0$, so (2.21) holds.

Moreover, it is easy that the other hypotheses of Corollary 2.1 are verified, so F has a coupled fixed point which is $(0,0)$.

On the other hand, Theorem 9 of Bhaskar and Lakshmikantham could not be applied in this case. Indeed, assume there exists $k \in [0,1)$ such that (1.2) holds for $x \geq u$ and $y \leq v$, that is,

$$d(F(x,y),F(u,v)) \leq \frac{k}{2}\left[d(x,u) + d(y,v)\right].$$

If we take $x = u = 7$, $y = 1$ and $v = 2$, that is, $3v \leq u = x$, $x \geq 3y$ and $y \leq v$, so we get that

$$\frac{3}{5} \leq \frac{k}{2};$$

hence, $k \geq \frac{6}{5} > 1$, it is a contradiction.

Now, as an application, it is easy to state a corollary of Theorem 11 involving a contraction of integral type.

Corollary 2.5. *Let F satisfy the conditions of Theorem 11, except that condition (2.2) is replaced by the following: there exists a positive Lebesgue integrable function ϕ on $\mathbb{R}_+$ such that $\int_0^{\varepsilon} \phi(t)dt > 0$ for each $\varepsilon > 0$ and that Then, F has a coupled fixed point.*

$$\begin{aligned} &\int_0^{\Phi_1(d(F(x,y),F(u,v)))} \phi(t)dt \\ &\leq \int_0^{\psi_1(d(x,u),d(y,v),d(x,F(x,y)),d(u,F(u,v)),\frac{1}{2}[d(x,F(u,v))+d(u,F(x,y))])} \phi(t)dt \\ &- \int_0^{\psi_2(d(x,u),d(y,v),d(x,F(x,y)),d(u,F(u,v)),\frac{1}{2}[d(x,F(u,v))+d(u,F(x,y))])} \phi(t)dt. \end{aligned} \tag{2.22}$$

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
HKN and HA have worked together on each section of the paper such as the literature review, results and examples. Both authors read and approved the final manuscript.

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