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Remarks on maximal open sets in *m*-spaces

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Abstract

This paper deals with maximal *m*-open sets. The *m*-closure and the *m*-interior of maximal *m*-open sets and their properties are investigated. Further, the behaviors of maximal *m*-open sets in *m*-homeomorphic *m*-spaces and product *m*-spaces are inspected. Our results are supported by some examples and counterexamples.

Keywords: Topology, *m*-structure, Maximal open set, Homeomorphism, Product space

MSC: 54A05; 54B05

Introduction

The concepts of minimal open sets and maximal open sets in topological spaces are introduced and considered by Nakaoka and Oda in [1-3]. More precisely, in 2001, Nakaoka and Oda [2] characterized minimal open sets and proved that any subset of a minimal open set is pre-open. Also, as an application of a theory of minimal open sets, they obtained a sufficient condition for a locally finite space to be a pre-Hausdorff space. The authors in [3] obtained fundamental properties of maximal open sets such as decomposition theorem for a maximal open set and established basic properties of intersections of maximal open sets, such as the law of radical closure. By a dual concept of minimal open sets and maximal open sets, the authors in [1] introduced the concepts of minimal closed sets and maximal closed sets, and obtained some results easily by dualizing the known results regarding minimal open sets and maximal open sets.

Several authors have used these new notions in many directions. For instance, maximal and minimal θ-open sets and their properties are considered by Caldas et al. [4]. Also, θ-generalized open sets are investigated by Caldas et al. [5]. The concept of minimal γ-open sets are introduced and considered by Hussain and Ahmad [6]. Moreover, Bhattacharya [7,8] introduced the new concepts of generalized minimal closed sets and IF generalized minimal closed sets. Finally, Al Ghour [9,10] has applied the notion of minimality and maximality of open sets to the fuzzy case.

Generalized topological concepts play important roles in almost all branches of pure and applied mathematics. One of the inspiration sources is the notion of *m*-structure and *m*-space introduced by Maki et al. [11]. In [12], the notion of maximal *m*-open set is introduced, and its properties are investigated. Some results about the existence of maximal *m*-open sets are given. Moreover, the relations between maximal *m*-open sets in an *m*-space and maximal open sets in the corresponding generated topology are considered.

This paper is organized as following. In the '*m*-structure and *m*-space' section, the concepts of *m*-structure and *m*-space are introduced, and some of their main properties are collected. Section 'Existence results' is devoted to consider the notion of maximal *m*-open sets, and some existence results are given. In the '*m*-closure, *m*-interior and maximal *m*-open sets' section, the *m*-closure and the *m*-interior of a maximal *m*-open set and their properties are investigated. Section 'Forming new maximal *m*-open sets from old ones' deals with the behavior of maximal *m*-open sets in *m*-homeomorphic *m*-spaces and product *m*-spaces.

m-structure and *m*-space

The concepts of *m*-structure and *m*-spaces, as generalizations of topology and topological spaces were introduced in [11]. For easy understanding of the materials incorporated in this paper, we recall some basic definitions and results. For details and more results on the following notions, we refer to [11,13-22] and the references cited therein.

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Let $\mathcal{P}(X)$ denote the set of all nonempty subsets of X. A family $\mathcal{M} \subseteq \mathcal{P}(X)$ is said to be an *m-structure* on X if $\emptyset, X \in \mathcal{M}$. In this case, $(X, \mathcal{M})$ is called an *m-space*. For examples in this setting, see [21]. In an m-space $(X, \mathcal{M})$, $A \in \mathcal{P}(X)$ is said to be an *m-open set* if $A \in \mathcal{M}$ and also $B \in \mathcal{P}(X)$ is an *m-closed set* if $B^c \in \mathcal{M}$. We set m-$\mathrm{Int}(A) = \bigcup\{U : U \subseteq A, U \in \mathcal{M}\}$ and m-$\mathrm{Cl}(A) = \bigcap\{F : A \subseteq F, F^c \in \mathcal{M}\}$. For any $x \in X$, $N(x)$ is said to be an *m-neighborhood* of x, if for any $z \in N(x)$ there is an m-open subset $G_z \subseteq N(x)$ such that $z \in G_z$.

Definition 2.1. We say that the m-space $(X, \mathcal{M})$ enjoys the following:

(a) property $\mathfrak{I}$, if any finite intersection of m-open sets is m-open;
(b) property $\mathfrak{F}$, if any finite union of m-open sets is m-open;
(c) property $\mathfrak{U}$, if any arbitrary union of m-open sets is m-open.

Proposition 2.2. *[22] For an m-structure $\mathcal{M}$ on a set X, the following are equivalent:*

(a) *$\mathcal{M}$ has the property $\mathfrak{U}$.*
(b) *If m-$\mathrm{Int}(A) = A$, then $A \in \mathcal{M}$.*
(c) *If m-$\mathrm{Cl}(B) = B$, then $B^c \in \mathcal{M}$.*

Proposition 2.3. *[21] For any two sets A and B,*

(a) *m-$\mathrm{Int}(A) \subseteq A$ and m-$\mathrm{Int}(A) = A$ if A is an m-open set.*
(b) *$A \subseteq m$-$\mathrm{Cl}(A)$ and $A = m$-$\mathrm{Cl}(A)$ if A is an m-closed set.*
(c) *m-$\mathrm{Int}(A) \subseteq m$-$\mathrm{Int}(B)$ and m-$\mathrm{Cl}(A) \subseteq m$-$\mathrm{Cl}(B)$ if $A \subseteq B$.*
(d) *m-$\mathrm{Int}(A \cap B) \subseteq (m$-$\mathrm{Int}(A)) \cap (m$-$\mathrm{Int}(B))$ and $(m$-$\mathrm{Int}(A)) \cup (m$-$\mathrm{Int}(B)) \subseteq m$-$\mathrm{Int}(A \cup B)$.*
(e) *m-$\mathrm{Cl}(A \cup B) \supseteq (m$-$\mathrm{Cl}(A)) \cup (m$-$\mathrm{Cl}(B))$ and m-$\mathrm{Cl}(A \cap B) \subseteq (m$-$\mathrm{Cl}(A)) \cap (m$-$\mathrm{Cl}(B))$.*
(f) *m-$\mathrm{Int}(m$-$\mathrm{Int}(A)) = m$-$\mathrm{Int}(A)$ and m-$\mathrm{Cl}(m$-$\mathrm{Cl}(B)) = m$-$\mathrm{Cl}(B)$.*
(g) *$(m$-$\mathrm{Cl}(A))^c = m$-$\mathrm{Int}(A^c)$ and $(m$-$\mathrm{Int}(A))^c = m$-$\mathrm{Cl}(A^c)$.*

Existence results

Definition 3.1. [12] Let $(X, \mathcal{M})$ be an m-space. A nonempty proper m-open subset A of X is said to be *maximal m-open* if any m-open set which contains A is X or A. We denote the set of all maximal m-open sets of an m-space $(X, \mathcal{M})$ by $\max(X, \mathcal{M})$.

First, we represent an existence theorem of maximal m-open sets in a special case. Recall that a *cofinite* subset is a subset which it's complement is finite.

Theorem 3.2. *[12] Let $(X, \mathcal{M})$ be an m-space and B a nonempty proper cofinite m-open set. Then there exists at least one (cofinite) maximal m-open set A such that $B \subseteq A$.*

Proof. For the sake of completeness, we add the proof. If B is a maximal m-open set, put $A = B$. Otherwise, there exists an (cofinite) m-open set B_1 in which $B \subsetneq B_1 \neq X$. If B_1 is a maximal m-open set, we may put $A = B_1$. If B_1 is not maximal m-open, then there exists an (cofinite) m-open set B_2 such that $B \subsetneq B_1 \subsetneq B_2 \neq X$. By continuing this process, we have a sequence of m-open sets

$$B \subsetneq B_1 \subsetneq B_2 \subsetneq \cdots \subsetneq B_k \subsetneq \cdots \subsetneq X.$$

Since B is a cofinite set, this process will stop somewhere. Then, finally we will find a maximal m-open set $A = B_n$ for some $n \in \mathbb{N}$. □

Example 3.3. [12] Let $X = \mathbb{N}$, $B = \{1, 3\}$, and $A = \{1, 3, 5\}$. Set $\mathcal{M} = \{\emptyset, A, B, \mathbb{N}\} \cup \{C_n : n \in \mathbb{N}\}$, where $C_n = \{2, 4, 6, \ldots, 2n\}$. Clearly, B is not cofinite, while it has a maximal m-open extension A. This shows that a set which is not cofinite may has a maximal m-open extension. Moreover, C_n's are not cofinite and also they do not have any maximal m-open extension. This means that Theorem 3.2 may not hold, when the set is not cofinite.

For a nonempty proper cofinite m-open set in an m-space, the maximal m-open extension is not always unique. As in the following example, it is possible that an m-open set has many maximal m-open extensions.

Example 3.4. [12] Let $X = \mathbb{N}$, $C_1 = \{1, 3, 5\}$, $C_2 = \{1, 3\}$, $C_3 = \{2, 4, 6\}$, $C_4 = \{2\}$, $C_5 = \{4\}$, and $C_6 = \{6\}$. Set $\mathcal{M} = \{\emptyset, B_1, A_2, B_3, A_4, A_5, A_6, \mathbb{N}\}$, where $B_i = \mathbb{N} \setminus C_i$ for $i = 1, 3$ and $A_j = \mathbb{N} \setminus C_j$ for $j = 2, 4, 5, 6$. Evidently, B_1 and B_3 are cofinite, and B_1 has a unique maximal m-open extension A_2, whereas B_3 has three maximal m-open extensions A_4, A_5, and A_6.

Theorem 3.5. *[12] Suppose that $(X, \mathcal{M})$ is an m-space with the property of $\mathfrak{F}$, let S be a nonempty proper m-open set such that each element of it's complement is contained in a finite m-closed set. Then, there exists at least one (cofinite) maximal m-open set A with $S \subseteq A$.*

Example 3.6. [12] Let $X = \mathbb{N}$ and $U_{2n+1} = \mathbb{N} \setminus \{2n+1, 2n+3, \ldots\}$ for each $n \in \mathbb{N}$. Consider the m-structure $\mathcal{M} = \{\emptyset, \mathbb{N} \setminus \{1\}, \mathbb{N} \setminus \{2\}, \mathbb{N} \setminus \{1, 2\}, U_3^c, \mathbb{N}\} \cup \{U_{2n+1} : n \in \mathbb{N}\}$ on X. Clearly, $(X, \mathcal{M})$ is an m-space with the property of $\mathfrak{F}$. Set $S_1 = \mathbb{N} \setminus \{1, 2\}$. It is easy to see that S_1 satisfies in all conditions of Theorem 3.5, and so it has two extensions $\mathbb{N} \setminus \{1\}$ and $\mathbb{N} \setminus \{2\}$. Now, imagine $S_2 = U_3$, for $5 \in U_3^c$, there is no finite m-closed set containing 5. Note that S_2 does not have any maximal m-open extension because of the following chain:

$$U_3 \subsetneq U_5 \subsetneq \cdots \subsetneq U_{2n+1} \subsetneq \cdots \subsetneq \mathbb{N}.$$

Finally, set $S_3 = U_3^c$. For $4 \in S_3^c$, there is no finite m-closed set containing 4, while S_3 has two maximal extensions $\mathbb{N} \setminus \{1\}$ and $\mathbb{N} \setminus \{2\}$.

Corollary 3.7. *[12] Suppose that $(X, \mathcal{M})$ is an m-space with the property of $\mathfrak{F}$, let each element of X be contained in a finite m-closed set. Then, for any nonempty proper m-open set S, there exists at least one (cofinite) maximal m-open set A with $S \subseteq A$.*

Remark 3.8. [12] Let X, $\mathcal{M}$, and U_{2n+1} be the same as in Example 3.6. One can find that Theorem 3.5 is stronger than Corollary 3.7.

Theorem 3.9. *If the m-space $(X, \mathcal{M})$ has the property of $\mathfrak{F}$, then Theorems 3.2 and 3.5 are equivalent.*

Proof. First, we prove that Theorem 3.2 implies Theorem 3.5. To see this, since S is a proper subset of X, there exists an element x of S^c. By the assumption that there exists a finite m-closed set F such that $x \in F$, one can easily check that $S \cup F^c$ is a nonempty proper cofinite m-open set. Therefore, by Theorem 3.2, we can find a maximal m-open set A satisfying $S \cup F^c \subseteq A$. Evidently, A is a (cofinite) maximal m-open extension of S. For the converse, suppose that B is a nonempty proper cofinite m-open set, B^c is a finite m-closed set which satisfies in all conditions of Theorem 3.5. Then, there exists at least one (cofinite) maximal m-open set A such that $B \subseteq A$. □

Proposition 3.10. *[12] Let $(X, \mathcal{M})$ be an m-space with the property of $\mathfrak{F}$, and let $A, B \in \max(X, \mathcal{M})$ and W be an m-open set. Then*

(a) $A \cup W = X$ *or* $W \subseteq A$,
(b) $A \cup B = X$ *or* $A = B$.

If the space does not have the property of $\mathfrak{F}$, then the last proposition is not true in general as you can see in the following example:

Example 3.11. [12] Let $X = \{x, y, z, t\}$, $A = \{x, y\}$, $B = \{y, z\}$, and $W = \{z\}$. Put $\mathcal{M} = \{\emptyset, A, B, W, X\}$, then clearly $A, B \in \max(X, \mathcal{M})$, and W is an m-open set, whereas

(a) $A \cup W \neq X$ and $W \not\subseteq A$.
(b) $A \cup B \neq X$ and $A \neq B$.

Also, it is possible that Proposition 3.10 holds, whereas the space does not enjoy the property of $\mathfrak{F}$. The following example shows that.

Example 3.12. [12] Let $X = \{x, y, z, t\}$, $A = \{x, y, z\}$, $B = \{y, z, t\}$, $W_1 = \{x\}$, and $W_2 = \{t\}$. Put $\mathcal{M} = \{\emptyset, A, B, W_1, W_2, X\}$, then clearly $A, B \in \max(X, \mathcal{M})$. It is not hard to check that the results of Proposition 3.10 hold here, whereas the space does not enjoy the property of $\mathfrak{F}$.

Corollary 3.13. *[12] Let $(X, \mathcal{M})$ be an m-space with the property of $\mathfrak{F}$ and $A \in \max(X, \mathcal{M})$. If $x \in A$, then $A \cup W = X$ or $W \subseteq A$ for any m-open neighborhood W of x.*

m-closure, *m*-interior and maximal *m*-open sets

Theorem 4.1. *Let $(X, \mathcal{M})$ be an m-space with the property of $\mathfrak{F}$, $A \in \max(X, \mathcal{M})$, and $x \in A^c$. Then, $A^c \subseteq W$ for any m-open neighborhood W of x.*

Proof. Suppose W is an m-open neighborhood of x. Since $x \in A^c$, we have $W \not\subseteq A$. It follows from part (a) of Proposition 3.10 that $A \cup W = X$. Therefore, $A^c \subseteq W$. □

Example 4.2. Let $X = \{x, y, z, t\}$, $A_1 = \{x, y\}$, $A_2 = \{t\}$, and $W = \{x, z\}$. Put $\mathcal{M} = \{\emptyset, A_1, A_2, W, X\}$. Clearly, $A_1, A_2 \in \max(X, \mathcal{M})$. One can easily check that the result of Theorem 4.1 is not true for A_1 and A_2. Now, let $B_1 = \{x, y, t\}$, $B_2 = \{y, z, t\}$, $B_3 = \{x, z\}$. Put $\mathcal{N} = \{\emptyset, \{y\}, B_1, B_2, B_3, X\}$. The result of Theorem 4.1 is true for B_1 and B_2 while it is not correct about B_3 for $y \in B_3^c$. Note that in these cases, considered spaces do not have the property of $\mathfrak{F}$.

Corollary 4.3. *Let $(X, \mathcal{M})$ be an m-space with the property of $\mathfrak{F}$ and $A \in \max(X, \mathcal{M})$. Then, one of the following statements holds:*

(a) *For each $x \in A^c$ and each m-open neighborhood W of x, $W = X$;*
(b) *There exists an m-open set W such that $A^c \subseteq W$ and $W \subsetneq X$.*

Proof. Suppose (a) does not hold, then there exists $x \in A^c$ and an m-open neighborhood W of x in which $W \subsetneq X$. By Theorem 4.1, we have $A^c \subseteq W$ which means that (b) is true. □

Corollary 4.4. *Suppose $(X, \mathcal{M})$ is an m-space with the property of $\mathfrak{F}$ and $A \in \max(X, \mathcal{M})$, then one of the following holds:*

(a) *For each $x \in A^c$ and each m-open neighborhood W of x, we have $A^c \subsetneq W$.*
(b) *There exists an m-open set W such that $A^c = W \neq X$.*

Proof. Suppose that (b) does not hold. According to Theorem 4.1, since $A \in \max(X, \mathcal{M})$, we have $A^c \subseteq W$ for each $x \in A^c$ and any m-open neighborhood W of x.

Now, since (b) does not hold, we have $A^c \subsetneq W$, i.e., (a) is true. □

Theorem 4.5. *Suppose $(X, \mathcal{M})$ is an m-space with the property of $\mathfrak{F}$, $A \in \max(X, \mathcal{M})$ and B is a proper m-closed set containing A, then $A = B$. Indeed, there is no proper m-closed set properly containing A.*

Proof. On the contrary, suppose B is a proper m-closed set containing A in which $A \neq B$, therefore, B^c is a nonempty m-open set contained in A^c. So, $A \subsetneq A \cup B^c \subsetneq X$. This is a contradiction with the maximality of A, since $(X, \mathcal{M})$ is an m-space with the property of $\mathfrak{F}$. □

Corollary 4.6. *Suppose $(X, \mathcal{M})$ is an m-space with the property of $\mathfrak{F}$ and $A \in \max(X, \mathcal{M})$, then $m\text{-Cl}(A) = X$ or $m\text{-Cl}(A) = A$.*

Proof. If A is an m-closed set, then we have $m\text{-Cl}(A) = A$. Otherwise, according to Theorem 4.5, there is no proper m-closed set properly containing A, so $m\text{-Cl}(A) = X$. □

Remark 4.7. Corollary 4.6 is an extension of Theorem 3.4 in [3] to m-space and improves it's proof in a more straight way.

Example 4.8. Let $X = \{x, y, z, t\}$, $A_1 = \{x, y\}$, $U_1 = \{t\}$, $A_2 = \{x, z, t\}$, $U_2 = \{y\}$, and $A_3 = \{y, z, t\}$. Put $\mathcal{M} = \{\emptyset, A_1, A_2, A_3, U_1, U_2, X\}$. Clearly, $A_1, A_2, A_3 \in \max(X, \mathcal{M})$. One can easily verify that $m\text{-Cl}(A_1) = U_1^c$; hence, A_1 does not satisfy in the conclusion of Theorem 4.6. Besides $m\text{-Cl}(A_2) = A_2$ and $m\text{-Cl}(A_3) = X$ which imply that the result of Theorem 4.6 satisfies for A_2 and A_3 in different ways. Note that the space does not have the property of $\mathfrak{F}$.

Theorem 4.9. *Suppose $(X, \mathcal{M})$ is an m-space with the property of $\mathfrak{F}$ and $A \in \max(X, \mathcal{M})$, then $m\text{-Int}(A^c) = A^c$ or $m\text{-Int}(A^c) = \emptyset$.*

Proof. It is a straightforward consequence of Theorem 4.6 and part (g) of Proposition 2.3. □

Theorem 4.10. *Suppose $(X, \mathcal{M})$ is an m-space with the property of $\mathfrak{F}$ and $A \in \max(X, \mathcal{M})$, then*

(a) *$m\text{-Int}(B) = A$ for any proper subset B of X containing A;*
(b) *$m\text{-Cl}(S) = A^c$ for each nonempty subset S of A^c;*
(c) *$\left(m\text{-Cl}(S)\right)^c = m\text{-Int}(S^c) = A$ for each nonempty subset S of A^c.*

Proof. Let U be any m-open subset of B, maximality of A, and property of $\mathfrak{F}$, imply that $U \subseteq A$. Now, the definition of m-interior and Proposition 2.3 imply that $m\text{-Int}(B) = A$; i.e., (a) is proved. On the other hand, suppose S is a nonempty subset of A^c, then S^c is a proper subset of X with $A \subseteq S^c$. Therefore, (a) implies that $m\text{-Int}(S^c) = A$; hence by part (g) of Proposition 2.3, we obtain $m\text{-Cl}(S) = A^c$. Finally, for (c), according to hypothesis $A \subseteq S^c \subsetneq X$, now, (a) together with (b) imply (c). □

Example 4.11. Let $X = \{x, y, z, t, r\}$, $A = \{x, y\}$, and $U = \{y, r\}$. Put $\mathcal{M} = \{\emptyset, A, U, X\}$. Consider $B = \{x, y, z, t\}$ and $B' = \{x, y, t, r\}$. Clearly, $A \in \max(X, \mathcal{M})$, B and B' are proper subset X containing A. It is easy to verify that $m\text{-Int}(B) = A$ and $m\text{-Int}(B') = \{x, y, r\} \supsetneq A$. Moreover, let $S = \{r\}$ and $S' = \{z\}$. We see that S and S' are nonempty subsets of A^c. Also, $m\text{-Cl}(S) = A^c$ and $m\text{-Cl}(S') = \{z, t\} \subsetneq A^c$.

Corollary 4.12. *Suppose $(X, \mathcal{M})$ is an m-space with the property of $\mathfrak{F}$, $A \in \max(X, \mathcal{M})$ and B a nonempty subset of X, with $A \subsetneq B$. Then $m\text{-Cl}(B) = X$.*

Proof. We have $A \subsetneq B \subseteq X$, so $B \setminus A \neq \emptyset$ and $B = A \cup (B \setminus A)$. Therefore, using Proposition 2.3 and part (b) of Theorem 4.10, we get

$$m\text{-Cl}(B) = m\text{-Cl}(A \cup (B \setminus A)) \supseteq m\text{-Cl}(A) \cup m\text{-Cl}(B \setminus A) \supseteq A \cup A^c = X.$$

Hence, $m\text{-Cl}(B) = X$. □

Corollary 4.13. *Suppose $(X, \mathcal{M})$ is an m-space with the property of $\mathfrak{F}$, $A \in \max(X, \mathcal{M})$ and suppose that A^c has at least two elements, then $m\text{-Cl}(X \setminus \{a\}) = X$ for any element a of A^c.*

Proof. According to the assumption, we have $A \subsetneq X \setminus \{a\}$, so we can deduce the result by Corollary 4.12. □

Example 4.14. Let $X = \mathbb{N}$, $A = \mathbb{N} \setminus \{1, 2\}$, and $\mathcal{M} = \{\emptyset, A, X\} \cup \{\{4, 5, \ldots, n\} : n \in \mathbb{N}\}$. Clearly, the m-space $\mathcal{M}$ has the property of $\mathfrak{F}$, $A \in \max(\mathbb{N}, \mathcal{M})$. One can easily see $m\text{-Cl}(\mathbb{N} \setminus \{1\}) = \mathbb{N}$ and $m\text{-Cl}(\mathbb{N} \setminus \{2\}) = \mathbb{N}$.

Theorem 4.15. *Suppose $(X, \mathcal{M})$ is an m-space with the property of $\mathfrak{F}$, $A \in \max(X, \mathcal{M})$, and B a subset of X with $A \subseteq B$, then $B \subseteq m\text{-Int}(m\text{-Cl}(B))$.*

Proof. In case $B = A$, we have B is an m-open set. Hence, it follows from Proposition 2.3 that $B = m\text{-Int}(B) \subseteq m\text{-Int}(m\text{-Cl}(B))$. Otherwise, $A \subsetneq B$, and consequently by using Corollary 4.12, we get $m\text{-Int}(m\text{- Cl}(B)) = m\text{-Int}(X) = X \supseteq B$. So, we have the result. □

Example 4.16. Let $X = \{x, y, z, t\}$, $A = \{x, y\}$, $B = \{x, y, z\}$, and $C = \{x, z, t\}$. Put $\mathcal{M} = \{\emptyset, A, C, X\}$. Then, it is easy to see that not only $A \in \max(X, \mathcal{M})$ and $A \subseteq B$ but also $B \subseteq m\text{-Int}(m\text{-Cl}(B))$. Therefore, the result of Theorem 4.15 holds here, whereas the space does not enjoy the property of $\mathfrak{F}$. Now suppose $A' = \{x, y\}$, $B' = \{x, y, z\}$, and $\mathcal{M}' = \{\emptyset, A', \{t\}, X\}$. Then one can easily deduce that $A' \in \max(X, \mathcal{M}')$ *and* $A' \subseteq B'$, while $B' \nsubseteq m\text{-Int}(m\text{-Cl}(B'))$. We see that the result of Theorem 4.15 may not hold when the space does not have the property of $\mathfrak{F}$.

Corollary 4.17. *[2] Suppose (X, τ) is a topological space, $A \in \max(X, \tau)$, and B a subset of X with $A \subseteq B$, then B is a preopen set.*

Corollary 4.18. *Suppose $(X, \mathcal{M})$ is an m-space with the property of $\mathfrak{F}$, $A \in \max(X, \mathcal{M})$, then $X \setminus \{a\} \subseteq m\text{-Int}(m\text{-Cl}(X \setminus \{a\}))$ for any element a of A^c.*

Proof. According to hypothesis $A \subseteq X \setminus \{a\}$, so by Theorem 4.15, the result is clear. □

Forming new maximal *m*-open sets from old ones

Definition 5.1. Two *m*-spaces $(X, \mathcal{M})$ and $(Y, \mathcal{N})$ are called *m-homeomorphic* if there exists a bijective function $f : X \to Y$ for which f and f^{-1} are *m*-continuous. In this case, f is called an *m-homeomorphism*.

Theorem 5.2. *Suppose $(X, \mathcal{M})$ and $(Y, \mathcal{N})$ are two m-spaces and $f : (X, \mathcal{M}) \to (Y, \mathcal{N})$ is an m-homeomorphism function, then $A \in \max(X, \mathcal{M})$ if and only if $f(A) \in \max(Y, \mathcal{N})$.*

Proof. Let $A \in \max(X, \mathcal{M})$, *m*-continuity of f^{-1} guaranties that $f(A) \in \mathcal{N}$. We have to prove that $f(A) \in \max(Y, \mathcal{N})$. Suppose this is not the case, then there is $U \in \mathcal{N}$ in which $f(A) \subsetneq U \subsetneq Y$. Since f is bijective, we have $A \subsetneq f^{-1}(U) \subsetneq X$. Since f is *m*-continuous, $f^{-1}(U) \in \mathcal{M}$. This is a contradiction with the maximality of A; hence, we get $f(A) \in \max(Y, \mathcal{N})$. The converse follows from the fact that $f^{-1} : (Y, \mathcal{N}) \to (X, \mathcal{M})$ is *m*-homeomorphism. □

Example 5.3. Let $X = \{1, 2\}$, $\mathcal{M} = \{\emptyset, \{1\}, X\}$, $Y = \{1\}$ and $\mathcal{N} = \{\emptyset, Y\}$. Suppose $f : X \to Y$ is defined by $f(x) = 1$ for each $x \in X$, it is easy to see that f is *m*-continuous, *m*-open, and surjective, but it is not one to one. Clearly, $A = \{1\}$ is maximal *m*-open in X, whereas $f(A) = \{1\}$ is not maximal *m*-open in Y. This shows that the hypothesis of 'f is one to one' is a necessary condition for Theorem 5.2.

Example 5.4. Let $X = \{1, 2, 3\}$, $\mathcal{M} = \{\emptyset, \{1, 2\}, X\}$, $Y = \{1, 2, 3, 4\}$, and $\mathcal{N} = \{\emptyset, \{1, 2\}, \{1, 2, 3\}, Y\}$. Let $f : X \to Y$ be defined by $f(x) = x$ for each $x \in X$. It is easy to see that f is *m*-continuous, *m*-open and one to one but it is not surjective. Clearly, $A = \{1, 2\}$ is maximal *m*-open in X, whereas $f(A) = \{1, 2\}$ is not maximal *m*-open in Y. Also, $U = \{1, 2, 3\}$ is not maximal *m*-open in X, while $f(U) = \{1, 2, 3\}$ is maximal *m*-open in Y. This shows that the hypothesis of 'f is surjective' is a necessary condition for Theorem 5.2.

Example 5.5. Let $X = Y = \{1, 2, 3\}$, $\mathcal{M} = \{\emptyset, \{1\}, X\}$, and $\mathcal{N} = \{\emptyset, \{1\}, \{1, 2\}, Y\}$. Suppose $f : (X, \mathcal{M}) \to (X, \mathcal{N})$ be the identity mapping. It is easy to see that f is *m*-open and bijective, but it is not *m*-continuous. Clearly, $A = \{1\}$ is maximal *m*-open in $(X, \mathcal{M})$, whereas $f(A) = \{1\}$ is not maximal *m*-open in $(X, \mathcal{N})$. Also, $U = \{1, 2\}$ is not maximal *m*-open in $(X, \mathcal{M})$, while $f(U) = \{1, 2\}$ is maximal *m*-open in $(X, \mathcal{N})$. This shows that the hypothesis of 'f is *m*-continuous' is a necessary condition for Theorem 5.2. Now in this example, let $g = f^{-1}$ be *m*-continuous and bijective but it is not *m*-open. By this, one can easily deduce that the hypothesis of 'to be *m*-open' is a necessary condition for Theorem 5.2.

Corollary 5.6. *Suppose $(X, \mathcal{M})$ and $(Y, \mathcal{N})$ are m-homeomorphic, then $\max(X, \mathcal{M})$ and $\max(Y, \mathcal{N})$ have the same cardinal.*

Proof. It is an immediate consequence of Theorem 5.2. □

The following example shows that the converse of the above result may not hold.

Example 5.7. Let $X = \{1, 2, 3\}$, $\mathcal{M} = \{\emptyset, \{1, 2\}, \{2, 3\}, X\}$, $Y = \{1, 2, 3, 4\}$, and $\mathcal{N} = \{\emptyset, \{1, 2\}, \{2, 3\}, Y\}$. Then, $\max(X, \mathcal{M})$ and $\max(Y, \mathcal{N})$ have the same cardinal, while $(X, \mathcal{M})$ and $(Y, \mathcal{N})$ are not *m*-homeomorphic because there is no bijective function between X and Y.

Theorem 5.8. *Suppose $(X, \mathcal{M})$ and $(Y, \mathcal{N})$ are two m-spaces. The following statements are equivalent.*

(a) $A \in \max(X, \mathcal{M})$;
(b) $A \times Y \in \max(X \times Y, \mathcal{M} \times \mathcal{N})$;
(c) $Y \times A \in \max(Y \times X, \mathcal{N} \times \mathcal{M})$.

Proof. For (a)⇔(b), it suffices to prove that A is not maximal *m*-open if and only if $A \times Y$ is not maximal *m*-open. Suppose A is not maximal *m*-open, then there exists an *m*-open set U in X such that $A \subsetneq U \subsetneq X$. Thus, $A \times Y \subsetneq U \times Y \subsetneq X \times Y$ which implies that $A \times Y$ is not maximal *m*-open. Conversely, suppose $A \times Y$ is not maximal *m*-open, so there exists an *m*-open set $U \in \mathcal{M}$ such that $A \times Y \subsetneq U \times Y \subsetneq X \times Y$ which implies that $A \subsetneq U \subsetneq X$. Then, A is not maximal *m*-open. Finally, it is easy to see that the function $f : (X \times Y, \mathcal{M} \times \mathcal{N}) \to (Y \times X, \mathcal{N} \times \mathcal{M})$ defined by $f(x, y) = (y, x)$ for all $(x, y) \in X \times$

Y is an m-homeomorphism. Now, (b)⇔(c) follows from Theorem 5.2. □

Theorem 5.9. *Suppose* $(X, \mathcal{M})$ *and* $(Y, \mathcal{N})$ *are two m-spaces, S is a nonempty proper cofinite m-open subset of X, and T is a nonempty proper cofinite m-open subset of Y, then there exist at least two (cofinite) maximal m-open sets* $A \times Y$ *and* $X \times B$ *in product m-space such that* $S \times T \subseteq A \times Y$ *and* $S \times T \subseteq X \times B$.

Proof. By Theorem 3.2, the proof is clear. □

Competing interest
The authors declare that they have no competing interests.

Authors' contributions
All authors contributed equally and significantly in writing this article. All authors read and approved the final manuscript.

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