Mathematical Sciences
a SpringerOpen Journal

ORIGINAL RESEARCH Open Access

Some iterative schemes for generalized vector equilibrium problems and relatively nonexpansive mappings in Banach spaces

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Abstract

In this paper, we introduce two iterative schemes for finding a common solution of a generalized vector equilibrium problem and relatively nonexpansive mappings in a real Banach space. We study the strong and weak convergence of the sequences generated by the proposed iterative schemes. The results presented in this paper are the supplement, extension, and generalization of the previously known results in this area.

Keywords: Generalized vector equilibrium problem, Fixed-point problem, Relatively nonexpansive mappings, Iterative schemes

2000 MSC: 49J30; 47H10; 47H17; 90C99

Introduction

Throughout the paper unless otherwise stated, let E be a real Banach space with its dual space E^*, let $\langle .,. \rangle$ denote the duality pairing between E and E^*, and let $\|.\|$ denote the norm of E as well as of E^*. Let C be a nonempty, closed, and convex subset of E, and let 2^E denote the set of all nonempty subsets of E. Let Y be a Hausdorff topological space, and let P be a pointed, proper, closed, and convex cone of Y with $\text{int}P \neq \emptyset$. We denote the strong convergence and the weak convergence of a sequence $\{x_n\}$ to x in E by $x_n \to x$ and $x_n \rightharpoonup x$, respectively.

The normalized duality mapping $J : E \to 2^{E^*}$ is defined by

$$J(x) = \{x^* \in E^* : \langle x, x^* \rangle = \|x\|^2 = \|x^*\|^2\}$$

for every $x \in E$. It follows from the Hahn-Banach theorem that $J(x)$ is nonempty. A Banach space E is said to be strictly convex if $\frac{\|x+y\|}{2} < 1$ for $x, y \in E$ with $\|x\| = \|y\| = 1$ and $x \neq y$. It is also said to be uniformly convex if for each $\epsilon \in (0, 2]$, there exists $\delta > 0$ such that $\frac{\|x+y\|}{2} \leq 1 - \delta$ for $x, y \in E$ with $\|x\| = \|y\| = 1$ and $\|x - y\| \geq \epsilon$. The space E is said to be smooth if the limit $\lim_{t\to 0} \frac{\|x+ty\|-\|x\|}{t}$ exists for all $x, y \in M(E) = \{z \in E : \|z\| = 1\}$. It is also said to be uniformly smooth if the limit exists uniformly in $x, y \in M(E)$. We note that if E is smooth, strictly convex, and reflexive, then the normalized duality mapping J is single-valued, one-to-one, and onto. The normalized duality mapping J is said to be weakly sequentially continuous if $x_n \rightharpoonup x$ implies that $Jx_n \rightharpoonup Jx$.

In 1994, Blum and Oettli [1] introduced and studied the following equilibrium problem (EP): Find $x \in C$ such that

$$F(x, y) \geq 0, \quad \forall y \in C, \tag{1.1}$$

where $F : C \times C \to \mathbb{R}$ is a bifunction.

The EP(1.1) includes variational inequality problems, optimization problems, Nash equilibrium problems, saddle point problems, fixed point problems, and complementary problems as special cases. In other words, EP(1.1) is a unified model for several problems arising in science, engineering, optimization, economics, etc.

In the last two decades, EP(1.1) has been generalized and extensively studied in many directions due to its importance (see, for example, [2-6] and references therein for the literature on the existence of solution of the various generalizations of EP(1.1)). Some iterative methods have been studied for solving various classes of equilibrium problems (see, for example, [7-17] and references therein).

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In this paper, we introduce and study the following generalized vector equilibrium problem (GVEP). Let $F : C \times C \to Y$ be a nonlinear bimapping, and let $\psi : C \to Y$ be a nonlinear mapping; then, GVEP is to find $x^* \in C$ such that

$$F(x^*, x) + \psi(x) - \psi(x^*) \in P, \ \ \forall x \in C. \tag{1.2}$$

The solution set of GVEP(1.2) is denoted by Sol(GVEP(1.2)).

Example 1.1. Let $E = \mathbb{R}$, the set of all real numbers, with the inner product defined by $\langle x, y \rangle = xy$, $\forall x, y \in \mathbb{R}$. Let $Y = \mathbb{R}$, then $P = [0, +\infty)$ and let $C = [0, 2]$. Let F and ψ be defined by $F(x, y) = x^2 - y$ and $\psi(x) = x^2 \ \ \forall x, y \in C$, respectively; then, it is observed that Sol($GVEP$(1.2))=[1, 2]$\neq \emptyset$.

If $\psi = \mathbf{0}$, then GVEP(1.2) reduces to the strong vector equilibrium problem (SVEP): Find $x^* \in C$ such that

$$F(x^*, x) \in P, \ \ \forall x \in C, \tag{1.3}$$

which has been studied by Kazmi and Khan [18]. It is well known that the vector equilibrium problem provides a unified model of several problems, for example, vector optimization, vector variational inequality, vector complementary problem, and vector saddle point problem [5,6]. In recent years, the vector equilibrium problem has been intensively studied by many authors (see, for example, [2,4-6,18,19] and the references therein).

If $Y = \mathbb{R}$, then $P = [0, +\infty)$, and hence, GVEP(1.2) reduces to the following generalized equilibrium problem (GEP): Find $x \in C$ such that

$$F(x^*, x) + \psi(x) - \psi(x^*) \geq 0, \ \ \forall x \in C, \tag{1.4}$$

where $\psi : C \to \mathbb{R} \cup \{+\infty\}$ be a proper extended real-valued function. GEP(1.4) has been studied by Ceng and Yao [7].

Next, we recall that a mapping $T : C \to C$ is said to be nonexpansive if $\|Tx - Ty\| \leq \|x - y\|$, $\forall x, y \in C$.

The *fixed point problem* (FPP) for a nonexpansive mapping T is to

$$\text{Find } x \in C \text{ such that } x \in \text{Fix}(T), \tag{1.5}$$

where Fix(T) is the fixed point set of the nonexpansive mapping T. It is well known that Fix(T) is closed and convex.

Let E be a smooth, strictly convex, and reflexive Banach space.

Following Takahashi and Zembayashi [17], a point $p \in C$ is said to be an asymptotic fixed point of T if C contains a sequence $\{x_n\}$ which converges weakly to p such that $\lim_{n\to\infty} \|x_n - Tx_n\| = 0$. The set of asymptotic fixed points of T is denoted by $\widehat{\text{Fix}}(T)$. A mapping T from C into itself is said to be relatively nonexpansive if Fix(T) $\neq \emptyset$, $\widehat{\text{Fix}}(T) = \text{Fix}(T)$, and $\phi(p, Tx) \leq \phi(p, x)$ for all $x \in C$ and $p \in \text{Fix}(T)$, where $\phi : E \times E \to \mathbb{R}_+$ is the Lyapunov functional defined by

$$\phi(x, y) = \|x\|^2 - 2\langle x, Jy \rangle + \|y\|^2, \quad \forall x, y \in E. \tag{1.6}$$

In 2007, Tada and Takahashi [15] and Takahashi and Takahashi [16] proved weak and strong convergence theorems for finding a common solution of EP(1.1) and FPP(1.5) of a nonexpansive mapping in a Hilbert space (for further related work, see Ceng and Yao [7] and Shan and Huang [19]).

In 2009, Takahashi and Zembayashi [17] proved weak and strong convergence theorems for finding a common solution of EP(1.1) and FPP(1.5) of a relatively nonexpansive mapping in real Banach space. Further, Petrot et al. [20] extended and generalized some results of Takahashi and Zembayashi [17].

Motivated by the work of Takahashi and Zembayashi [17], Shan and Haung [19], and Petrot et al. [20] and by the ongoing research in this direction, we introduce and study two iterative schemes for finding a common solution of GVEP(1.2) and FPPs for two relatively nonexpansive mappings in real Banach space. We study the strong and weak convergence of the sequences generated by the proposed iterative schemes. The results presented in this paper extend and generalize many previously known results in this research area (see, for instance, [17,20]).

Preliminaries

We recall some concepts and results which are needed in sequel.

Following Alber [21], the generalized projection Π_C from E onto C is defined by

$$\Pi_C(x) = \inf_{y \in C} \phi(x, y), \quad \forall x \in E,$$

where $\phi(x, y)$ is obtained by (1.6).

Lemma 2.1. [21,22]. *Let E be a smooth, strictly convex, and reflexive Banach space, and let C be a nonempty closed convex subset of E. Then, the following conclusions hold:*

(i) $\phi(x, \Pi_C y) + \phi(\Pi_C y, y) \leq \phi(x, y), \ \ \forall x \in C, y \in E;$
(ii) *Let* $x \in E$ *and* $z \in C$*, then*

$$z = \Pi_C(x) \Leftrightarrow \langle z - y, Jx - Jz \rangle \geq 0, \ \ \forall y \in C.$$

Remark 2.1. [17]

(i) From the definition of ϕ, we have

$$(\|x\| - \|y\|)^2 \leq \phi(x, y) \leq (\|x\| + \|y\|)^2, \ \forall x, y \in E.$$

(ii) If E is a real Hilbert space H, then $\phi(x, y) = (\|x\| - \|y\|)^2$, and Π_C is the metric projection P_C of H onto C.

(iii) If E is a smooth, strictly convex, and reflexive Banach space, then for $x, y \in E$, $\phi(x, y) = 0$ if and only if $x = y$.

Lemma 2.2. [23]. *Let C be a nonempty, closed, and convex subset of a smooth, strictly convex, and reflexive Banach space E, and let T be a relatively nonexpansive mapping from C into itself. Then, Fix*(T) *is closed and convex.*

Lemma 2.3. [22]. *Let E be a smooth and uniformly convex Banach space, and let* $\{x_n\}$ *and* $\{y_n\}$ *be sequences in E such that either* $\{x_n\}$ *or* $\{y_n\}$ *is bounded. If* $\lim_{n\to\infty} \phi(x_n, y_n) = 0$*, then* $\lim_{n\to\infty} \|x_n - y_n\| = 0$.

Lemma 2.4. [24,25]. *Let E be a uniformly convex Banach space, and let* $r > 0$*. Then, there exists a strictly increasing, continuous, and convex function* $g : [0, 2r] \to \mathbb{R}$ *such that* $g(0) = 0$ *and*

$$\|tx+(1-t)y\|^2 \leq t\|x\|^2+(1-t)\|y\|^2-t(1-t)g(\|x-y\|)$$

for all $x, y \in B_r$ *and* $t \in [0, 1]$*, where* $B_r = \{z \in E : \|z\| \leq r\}$.

Lemma 2.5. [22]. *Let E be a smooth and uniformly convex Banach space, and let* $r > 0$*. Then, there exists a strictly increasing, continuous, and convex function* $g : [0, 2r] \to \mathbb{R}$ *such that* $g(0) = 0$ *and*

$$g(\|x - y\|) \leq \phi(x, y), \quad \forall x, y \in B_r.$$

Definition 2.1. [26,27]. Let X and Y be two Hausdorff topological spaces, and let D be a nonempty, convex subset of X and P be a pointed, proper, closed, and convex cone of Y with int$P \neq \emptyset$. Let 0 be the zero point of Y, $\mathbb{U}(0)$ be the neighborhood set of 0, $\mathbb{U}(x_0)$ be the neighborhood set of x_0, and $f : D \to Y$ be a mapping.

(i) If, for any $V \in \mathbb{U}(0)$ in Y, there exists $U \in \mathbb{U}(x_0)$ such that

$$f(x) \in f(x_0) + V + P, \ \forall x \in U \cap D,$$

then f is called *upper P-continuous* on x_0. If f is *upper P-continuous* for all $x \in D$, then f is called *upper P-continuous* on D;

(ii) If, for any $V \in \mathbb{U}(0)$ in Y, there exists $U \in \mathbb{U}(x_0)$ such that

$$f(x) \in f(x_0) + V - P, \ \forall x \in U \cap D,$$

then f is called *lower P-continuous* on x_0. If f is *lower P-continuous* for all $x \in D$, then f is called *lower P-continuous* on D;

(iii) If, for any $x, y \in D$ and $t \in [0, 1]$, the mapping f satisfies

$$f(x) \in f(tx+(1-t)y)+P \text{ or } f(y) \in f(tx+(1-t)y)+P,$$

then f is called *proper P-quasiconvex*;

(iv) If, for any $x_1, x_2 \in D$ and $t \in [0, 1]$, the mapping f satisfies

$$tf(x_1) + (1 - t)f(x_2) \in f(tx + (1 - t)y) + P,$$

then f is called *P-convex.*

Lemma 2.6. [28]. *Let X and Y be two real Hausdorff topological spaces; D is a nonempty, compact, and convex subset of X, and P is a pointed, proper, closed, and convex cone of Y with intP* $\neq \emptyset$*. Assume that* $g : D \times D \to Y$ *and* $\Phi : D \to Y$ *are two nonlinear mappings. Suppose that g and* Φ *satisfy*

(i) $g(x, x) \in P, \ \forall x \in D;$
(ii) Φ *is upper P-continuous on D;*
(iii) $g(., y)$ *is lower P-continuous,* $\forall x \in D;$
(iv) $g(x, .) + \Phi(.)$ *is proper P-quasiconvex,* $\forall x \in D.$

Then, there exists a point $x \in D$ *which satisfies*

$$G(x, y) \in P \setminus \{0\}, \ \forall y \in D,$$

where

$$G(x, y) = g(x, y) + \Phi(y) - \Phi(x), \ \forall x, y \in D.$$

Let $F : C\times C \to Y$ and $\psi : C \to Y$ be two mappings. For any $z \in E$, define a mapping $G_z : C \times C \to Y$ as follows:

$$G_z(x, y) = F(x, y)+\psi(y)-\psi(x)+\frac{e}{r}\langle y-x, Jx-Jz\rangle, \quad (2.1)$$

where r is a positive real number and $e \in$ int$P$.

Assumption 2.1. *Let* G_z*,* F*,* ψ *satisfy the following conditions:*

(i) *For all* $x \in C$*,* $F(x, x) = 0;$
(ii) F *is monotone, i.e.,* $F(x, y) + F(y, x) \in -P, \ \forall x, y \in C;$
(iii) $F(., y)$ *is continuous,* $\forall y \in C;$
(iv) $F(x, .)$ *is weakly continuous and P-convex, i.e.,*

$$tF(x, y_1) + (1 - t)F(x, y_2) \in F(x, ty_1 + (1 - t)y_2) + P, \ \forall x, y_1, y_2 \in C, \ \forall t \in [0, 1];$$

(v) $G_z(., y)$ *is lower P-continuous,* $\forall y \in C$ *and* $z \in E;$
(vi) $\psi(.)$ *is P-convex and weakly continuous;*
(vii) $G_z(x, .)$ *is proper P-quasiconvex,* $\forall x \in C$ *and* $z \in E.$

Results

First, we prove the following technical result:

Theorem 3.1. *Let E be a uniformly smooth and uniformly convex Banach space, and let C be a nonempty, compact, and convex subset of E. Assume that P is a pointed, proper, closed, and convex cone of a real Hausdorff topological space Y with intP* $\neq \emptyset$*. Let* $G_z : C \times C \to Y$ *be defined* *by (2.1). Let $F : C \times C \to Y$, $\psi : C \to Y$ and G_z satisfy Assumption 2.1. Define a mapping $T_r(z) : E \to C$ as follows:*

$$T_r(z) = \{x \in C : F(x,y) + \psi(y) - \psi(x) + \frac{e}{r}\langle y - x, Jx - Jz\rangle \in P, \quad \forall y \in C\},$$

where $e \in intP$, and r is a positive real number. Then,

(i) $T_r(z) \neq \emptyset, \quad \forall z \in E$;
(ii) T_r *is single-valued;*
(iii) T_r *is a firmly nonexpansive-type mapping, i.e., for all* $z_1, z_2 \in E$,

$$\langle T_r z_1 - T_r z_2, JT_r z_1 - JT_r z_2\rangle \leq \langle T_r z_1 - T_r z_2, Jz_1 - Jz_2\rangle;$$

(iv) *Fix*(T_r) =*Sol(GVEP(1.2));*
(v) *Sol(GVEP(1.2)) is closed and convex.*

Proof.

(i) Let $g(x,y) = G_z(x,y)$ and $\Phi(y) = 0$ for all $x, y \in C$ and $z \in E$. It is easy to observe that $g(x,y)$ and $\Phi(y)$ satisfy all the conditions of Lemma 2.6. Then, there exists a point $x \in C$ such that

$$G_z(x,y) + \Phi(y) - \Phi(x) \in P, \quad \forall y \in C,$$

and thus $T_r(z) \neq \emptyset$, $\forall z \in E$.

(ii) For each $z \in E$, $T_r(z) \neq \emptyset$, let $x_1, x_2 \in T_r(z)$. Then,

$$F(x_1,y) + \psi(y) - \psi(x_1) + \frac{e}{r}\langle y - x_1, Jx_1 - Jz\rangle \in P, \quad \forall y \in C \tag{3.1}$$

and

$$F(x_2,y) + \psi(y) - \psi(x_2) + \frac{e}{r}\langle y - x_2, Jx_2 - Jz\rangle \in P, \forall y \in C. \tag{3.2}$$

Letting $y = x_2$ in (3.1) and $y = x_1$ in (3.2), and then adding, we have

$$F(x_1,x_2) + F(x_2,x_1) + \frac{e}{r}\langle x_2 - x_1, Jx_1 - Jx_2\rangle \in P.$$

Since F is monotone, $e \in \text{int}P$, $r > 0$ and P is a closed and convex cone, we have

$$\langle x_2 - x_1, Jx_1 - Jx_2\rangle \geq 0.$$

Since E is strictly convex, the preceding inequality implies $x_1 = x_2$. Hence, T_r is single-valued.

(iii) For any $z_1, z_2 \in E$, let $x_1 = T_r(z_1)$ and $x_2 = T_r(z_2)$. Then,

$$F(x_1,y) + \psi(y) - \psi(x_1) + \frac{e}{r}\langle y - x_1, Jx_1 - Jz_1\rangle \in P, \quad \forall y \in C \tag{3.3}$$

and

$$F(x_2,y) + \psi(y) - \psi(x_2) + \frac{e}{r}\langle y - x_2, Jx_2 - Jz_2\rangle \in P, \quad \forall y \in C. \tag{3.4}$$

Letting $y = x_2$ in (3.3) and $y = x_1$ in (3.4), and then adding, we have

$$F(x_1,x_2) + F(x_2,x_1) + \frac{e}{r}\langle x_2 - x_1, Jx_1 - Jx_2 - Jz_1 + Jz_2\rangle \in P.$$

Again, since F is monotone, $e \in \text{int}P$, $r > 0$ and P is closed and convex cone, we have

$$\langle x_2 - x_1, Jx_2 - Jx_1\rangle \leq \langle x_2 - x_1, Jz_2 - Jz_1\rangle,$$

or

$$\langle T_r(z_1) - T_r(z_2), JT_r(z_1) - JT_r(z_2)\rangle \leq \langle T_r(z_1) - T_r(z_2), Jz_1 - Jz_2\rangle. \tag{3.5}$$

Hence, T_r is firmly nonexpansive-type mapping.

(iv) Let $x \in \text{Fix}(T_r)$. Then,

$$F(x,y) + \psi(y) - \psi(x) + \frac{e}{r}\langle y - x, Jx - Jx\rangle \in P, \forall y \in C$$

and so

$$F(x,y) + \psi(y) - \psi(x) \in P, \quad \forall y \in C.$$

Thus, $x \in \text{Sol}(GVEP(1.2))$.
Let $x \in \text{Sol}(GVEP(1.2))$. Then,

$$F(x,y) + \psi(y) - \psi(x) \in P, \quad \forall y \in C$$

and so

$$F(x,y) + \psi(y) - \psi(x) + \frac{e}{r}\langle y - x, Jx - Jx\rangle \in P, \forall y \in C.$$

Hence, $x \in \text{Fix}(T_r)$. Thus, $\text{Fix}(T_r) = \text{Sol}(GVEP(1.2))$.

(v) As in the proof of Lemma 2.8 in [17], we have

$$\phi(T_r(z_1), T_r(z_2)) + \phi(T_r(z_2), T_r(z_1)) \leq \phi(T_r z_1, z_2) + \phi(T_r z_2, z_1),$$

for $z_1, z_2 \in C$. Taking $z_2 = u \in \text{Fix}(T_r)$, we have

$$\phi(u, T_r z_1) \leq \phi(u, z_1).$$

Next, we show that $\widehat{\mathrm{Fix}}(T_r) = \mathrm{Sol}(GVEP(1.2))$. Indeed, let $p \in \widehat{\mathrm{Fix}}(T_r)$. Then, there exists $\{z_n\} \subset E$ such that $z_n \rightharpoonup p$ and $\lim_{n\to\infty}(z_n - T_r z_n) = 0$. Moreover, we get $T_r z_n \rightharpoonup p$. Hence, we have $p \in C$. Since J is uniformly continuous on bounded sets, we have

$$\lim_{n\to\infty} \frac{\|Jz_n - JT_r z_n\|}{r} = 0, \quad r > 0. \tag{3.6}$$

From the definition of T_r, we have

$$F(T_r z_n, y) + \psi(y) - \psi(T_r z_n) + \frac{e}{r}\langle y - T_r z_n, JT_r z_n - Jz_n\rangle \in P, \forall y \in C$$

$$0 \in F(y, T_r z_n) - (\psi(y) - \psi(T_r z_n)) - \frac{e}{r}\langle y - T_r z_n, JT_r z_n - Jz_n\rangle + P, \quad \forall y \in C.$$

Let $y_t = (1-t)p + ty$, $\forall t \in (0,1]$. Since $y \in C$ and $p \in C$, we get $y_t \in C$ and hence

$$\begin{aligned} 0 &\in F(y_t, T_r z_n) - (\psi(y_t) - \psi(T_r z_n)) - \frac{e}{r}\langle y_t - T_r z_n, JT_r z_n - Jz_n\rangle + P \\ &= F(y_t, T_r z_n) - (\psi(y_t) - \psi(T_r z_n)) - e\langle y_t - T_r z_n, \frac{JT_r z_n - Jz_n}{r}\rangle + P. \end{aligned} \tag{3.7}$$

Since $F(x,.)$ and $\psi(.)$ are weakly continuous for all $x \in C$, then it follows from (3.6) and (3.7) that

$$0 \in F(y_t, p) - \psi(y_t) + \psi(p) + P. \tag{3.8}$$

Further, it follows from Assumption 2.1 (i), (iv), (vi) that $tF(y_t, y) + (1-t)F(y_t, p) + t\psi(y) + (1-t)\psi(p) - \psi(y_t)$

$$\begin{aligned} &\in F(y_t, y_t) + \psi(y_t) - \psi(y_t) + P \\ &\in P, \end{aligned}$$

or

$$-t[F(y_t, y) + \psi(y) - \psi(y_t)] - (1-t)[F(y_t, p) + \psi(p) - \psi(y_t)] \in -P. \tag{3.9}$$

Using (3.8) in (3.9), we have

$$\begin{aligned} &-t[F(y_t, y) + \psi(y) - \psi(y_t)] \in -P \\ &F(y_t, y) + \psi(y) - \psi(y_t) \in P. \end{aligned}$$

Letting $t \to 0$, we obtain

$$F(p, y) + \psi(y) - \psi(p) \in P, \quad \forall y \in C,$$

i.e., $p \in \mathrm{Sol}(GVEP(1.2))$. So, we get $\mathrm{Fix}(T_r) = \mathrm{Sol}(GVEP(1.2)) = \widehat{\mathrm{Fix}}(T_r)$. Therefore, T_r is a relatively nonexpansive mapping. Further, it follows from Lemma 2.2 that $\mathrm{Sol}(GVEP(1.2)) = \mathrm{Fix}(T_r)$ is closed and convex. This completes the proof. □

Next, we have the following lemma whose proof is on the similar lines of the proof of Lemma 2.9 [17] and hence omitted.

Lemma 3.1. *Let E, C, F, ψ, G_z be same as in Theorem 3.1, and let $r > 0$. Then, for $x \in E$ and $q \in Fix(T_r)$, we have*

$$\phi(q, T_r x) + \phi(T_r x, x) \le \phi(q, x).$$

Now, we prove a strong convergence theorem for finding a common solution of GVEP(1.2) and the fixed point problems of two relatively nonexpansive mappings in a Banach space.

Theorem 3.2. *Let E be a uniformly smooth and uniformly convex Banach space, and let C be a nonempty, compact, and convex subset of E. Assume that P is a pointed, proper, closed, and convex cone of a real Hausdorff topological space Y with $intP \neq \emptyset$. Let the mappings $F : C \times C \to Y$ and $\psi : C \to Y$ satisfy Assumption 2.1, and let S, T be relatively nonexpansive mappings from C into itself such that $\Gamma := Fix(T) \cap Fix(S) \cap Sol(GVEP(1.2)) \neq \emptyset$. Let $\{x_n\}$ be a sequence generated by the scheme:*

$$\begin{aligned} x_0 &= x \in C, \\ y_n &= J^{-1}(\delta_n Jx_n + (1-\delta_n)JTz_n), \\ z_n &= J^{-1}(\alpha_n Jx_n + (1-\alpha_n)JSx_n), \end{aligned}$$

$$u_n \in C \text{ such that } F(u_n, y) + \psi(y) - \psi(u_n) + \frac{e}{r}\langle y - u_n, Ju_n - Jy_n\rangle \in P, \ \forall y \in C, \tag{3.10}$$

$$\begin{aligned} H_n &= \{z \in C : \phi(z, u_n) \le \phi(z, x_n)\}, \\ W_n &= \{z \in C : \langle x_n - z, Jx - Jx_n\rangle \ge 0\}, \\ x_{n+1} &= \prod\nolimits_{H_n \cap W_n} x, \quad \text{for every } n \in N \cup \{0\}, \end{aligned}$$

where $e \in \mathrm{int}P$, J is the normalized duality mapping on E, and $r \in [a, \infty)$ for some $a > 0$. Assume that $\{\alpha_n\}$ and $\{\delta_n\}$ are sequences in [0,1] satisfying the conditions:

(i) $\limsup_{n\to\infty} \delta_n < 1$;
(ii) $0 < \liminf_{n\to\infty} \alpha_n \le \limsup_{n\to\infty} \alpha_n < 1$.

Then, $\{x_n\}$ converges strongly to $\prod_\Gamma x$, where $\prod_\Gamma x$ is the generalized projection of E onto Γ.

Proof. Since S and T are relatively nonexpansive mappings from C into itself, it follows from Lemma 2.2 and Theorem 3.1(v) that Γ is closed and convex. Now, we show that $H_n \cap W_n$ is closed and convex. From the definition of

W_n, it is obvious that W_n is closed and convex. Further, from the definition of ϕ, we observe that H_n is closed and

$$\phi(z,u_n) \le \phi(z,x_n) \Leftrightarrow \|u_n\|^2 - \|x_n\|^2 - 2\langle z, Ju_n - Jx_n\rangle \ge 0,$$

and hence H_n is convex. So, $H_n \cap W_n$ is a closed convex subset of E for all $n \in N \cup \{0\}$.

Let $u \in \Gamma$. It follows from Theorem 3.1 that (3.10) is equivalent to $u_n = T_r y_n$ for all $n \in N \cup \{0\}$, and T_r is relatively nonexpansive. Since S and T are relatively nonexpansive, we have

$$\begin{aligned}\phi(u,u_n) &= \phi(u, T_r y_n)\\ &\le \phi(u,y_n)\\ &\le \phi(u, J^{-1}(\delta_n Jx_n + (1-\delta_n)JTz_n))\\ &= \|u\|^2 - 2\langle u, \delta_n Jx_n + (1-\delta_n)JTz_n\rangle\\ &\quad + \|\delta_n Jx_n + (1-\delta_n)JTz_n\|^2\\ &\le \|u\|^2 - 2\delta_n\langle u, Jx_n\rangle - 2(1-\delta_n)\langle u, JTz_n\rangle\\ &\quad + \delta_n\|x_n\|^2 + (1-\delta_n)\|Tz_n\|^2\\ &\le \delta_n\phi(u,x_n) + (1-\delta_n)\phi(u,Tz_n)\\ &\le \delta_n\phi(u,x_n) + (1-\delta_n)\phi(u,Tz_n)\\ &\le \delta_n\phi(u,x_n) + (1-\delta_n)\phi(u,z_n),\end{aligned} \tag{3.11}$$

and

$$\begin{aligned}\phi(u,z_n) &= \phi(u, J^{-1}(\alpha_n Jx_n + (1-\alpha_n)JSx_n))\\ &= \|u\|^2 - 2\langle u, \alpha_n Jx_n + (1-\alpha_n)JSx_n\rangle\\ &\quad + \|\alpha_n Jx_n + (1-\alpha_n)JSx_n\|^2\\ &\le \|u\|^2 - 2\alpha_n\langle u, Jx_n\rangle - 2(1-\alpha_n)\langle u, JSx_n\rangle\\ &\quad + \alpha_n\|x_n\|^2 + (1-\alpha_n)\|Sx_n\|^2\\ &\le \alpha_n\phi(u,x_n) + (1-\alpha_n)\phi(u,Sx_n)\\ &\le \alpha_n\phi(u,x_n) + (1-\alpha_n)\phi(u,x_n)\\ &\le \phi(u,x_n).\end{aligned} \tag{3.12}$$

Using (3.12) in (3.11), we have

$$\begin{aligned}\phi(u,u_n) &\le \delta_n\phi(u,x_n) + (1-\delta_n)\phi(u,x_n)\\ &\le \phi(u,x_n).\end{aligned}$$

Hence, we have $u \in H_n$. This implies that $\Gamma \subset H_n$, $\forall n \in N \cup \{0\}$.

Next, we show by induction that $\Gamma \subset H_n \cap W_n$, $\forall n \in N \cup \{0\}$. From $W_0 = C$, we have $\Gamma \subset H_0 \cap W_0$. Suppose that $\Gamma \subset H_k \cap W_k$, for some $k \in N \cup \{0\}$. Then, there exists $x_{k+1} \in H_k \cap W_k$ such that $x_{k+1} = \prod_{H_k \cap W_k} x$. From the definition of x_{k+1}, we have, for all $z \in H_k \cap W_k$,

$$\langle x_{k+1} - z, Jx - Jx_{k+1}\rangle \ge 0.$$

Since $\Gamma \subset H_k \cap W_k$, we have

$$\langle x_{k+1} - z, Jx - Jx_{k+1}\rangle \ge 0, \quad \forall z \in \Gamma, \tag{3.13}$$

and hence $z \in W_{k+1}$. So, we have $\Gamma \subset W_{k+1}$. Therefore, we have $\Gamma \subset H_{k+1} \cap W_{k+1}$.

Thus, we have that $\Gamma \subset H_n \cap W_n$ for all $n \in N \cup \{0\}$. This means that $\{x_n\}$ is well defined. From the definition of W_n, we have $x_n = \prod_{W_n} x$.

Using $x_n = \prod_{W_n} x$, from Lemma 2.1, we have $\phi(x_n, x) = \phi(\prod_{W_n} x, x) \le \phi(u,x) - \phi(u, \prod_{W_n} x) \le \phi(u,x)$, $\forall u \in \Gamma \subset W_n$.

Then, $\{\phi(x_n,x)\}$ is bounded. Therefore, $\{x_n\}$ and $\{Sx_n\}$ are bounded.

Since $x_{n+1} = \prod_{H_n \cap W_n} x \in H_n \cap W_n \subset W_n$ and $x_n = \prod_{W_n} x$, from the definition of $\prod_{W_n}$, we have

$$\phi(x_n,x) \le \phi(x_{n+1},x), \quad \forall n \in N \cup \{0\}.$$

Thus, $\{\phi(x_n,x)\}$ is nondecreasing. So, the limit of $\{\phi(x_n,x)\}$ exists. By the construction of W_n, we have $W_m \subset W_n$ and $x_m = \prod_{W_m} x \in W_n$ for any positive integer $m \ge n$. It follows that

$$\begin{aligned}\phi(x_m,x_n) &= \phi\left(x_m, \prod\nolimits_{W_n} x\right)\\ &\le \phi(x_m,x) - \phi\left(\prod\nolimits_{W_n} x, x\right)\\ &= \phi(x_m,x) - \phi(x_n,x).\end{aligned} \tag{3.14}$$

Letting $m,n \to \infty$ in (3.14), we have $\phi(x_m,x_n) \to 0$. It follows from Lemma 2.3 that $\|x_m - x_n\| \to 0$ as $m,n \to \infty$. Hence, $\{x_n\}$ is a Cauchy sequence. Since E is a Banach space and C is closed and convex, one can assume that $x_n \to \hat{x} \in C$ as $n \to \infty$. From (3.14), we have

$$\phi(x_{n+1},x_n) \le \phi(x_{n+1},x) - \phi(x_n,x), \quad \forall n \in N \cup \{0\}$$

which implies

$$\lim_{n\to\infty} \phi(x_{n+1},x_n) = 0.$$

Further, from $x_{n+1} = \prod_{H_n \cap W_n} x \in H_n$, we have

$$\phi(x_{n+1},u_n) \le \phi(x_{n+1},x_n), \quad \forall n \in N \cup \{0\}$$

and hence

$$\lim_{n\to\infty} \phi(x_{n+1},u_n) = 0.$$

Since

$$\lim_{n\to\infty} \phi(x_{n+1},x_n) = \lim_{n\to\infty} \phi(x_{n+1},u_n) = 0,$$

and E is uniformly convex and smooth, then from Lemma 2.3, we have

$$\lim_{n\to\infty} \|x_{n+1} - x_n\| = \|x_{n+1} - u_n\| = 0,$$

and hence, we have

$$\lim_{n\to\infty} \|x_n - u_n\| = 0.$$

Since J is uniformly norm-to-norm continuous on bounded sets, we have

$$\lim_{n\to\infty} \|Jx_n - Ju_n\| = 0$$

because E is a uniformly smooth Banach space and E^* is a uniformly convex Banach space.

Since $\{x_n\}$ and $\{Sx_n\}$ are bounded and $z_n = J^{-1}(\alpha_n Jx_n + (1-\alpha_n)JSx_n)$, then we can easily see that $\{z_n\}$ is a bounded sequence, and hence, $\{Tz_n\}$ is bounded.

Let $r = \sup_{n\in N\cup\{0\}}\{\|x_n\|, \|Tz_n\|, \|Sx_n\|\}$. From Lemma 2.4, we have

$$\begin{aligned}
\phi(u,z_n) &= \phi(u, J^{-1}(\alpha_n Jx_n + (1-\alpha_n)JSx_n)) \\
&= \|u\|^2 - 2\langle u, \alpha_n Jx_n + (1-\alpha_n)JSx_n\rangle \\
&\quad + \|\alpha_n Jx_n + (1-\alpha_n)JSx_n\|^2 \\
&\le \|u\|^2 - 2\alpha_n\langle u, Jx_n\rangle - 2(1-\alpha_n)\langle u, JSx_n\rangle \\
&\quad + \alpha_n\|x_n\|^2 + (1-\alpha_n)\|Sx_n\|^2 \\
&\quad - \alpha_n(1-\alpha_n)g(\|Jx_n - JSx_n\|) \\
&\le \alpha_n\phi(u,x_n) + (1-\alpha_n)\phi(u,Sx_n) \\
&\quad - \alpha_n(1-\alpha_n)g(\|Jx_n - JSx_n\|) \\
&\le \alpha_n\phi(u,x_n) + (1-\alpha_n)\phi(u,x_n) \\
&\quad - \alpha_n(1-\alpha_n)g(\|Jx_n - JSx_n\|) \\
&\le \phi(u,x_n) - \alpha_n(1-\alpha_n)g(\|Jx_n - JSx_n\|).
\end{aligned}$$

It follows from (3.11) that

$$\begin{aligned}
\phi(u,u_n) &\le \delta_n\phi(u,x_n) + (1-\delta_n)\phi(u,z_n) \\
&\le \delta_n\phi(u,x_n) + (1-\delta_n)[\phi(u,x_n) \\
&\quad - \alpha_n(1-\alpha_n)g(\|Jx_n - JSx_n\|)] \\
&\le \phi(u,x_n) - \alpha_n(1-\alpha_n)(1-\delta_n)g(\|Jx_n - JSx_n\|),
\end{aligned}$$

or

$$\alpha_n(1-\alpha_n)(1-\delta_n)g(\|Jx_n - JSx_n\|) \le \phi(u,x_n) - \phi(u,u_n). \tag{3.15}$$

Further, we have

$$\begin{aligned}
\phi(u,x_n) - \phi(u,u_n) &= \|x_n\|^2 - \|u_n\|^2 - 2\langle u, Jx_n - Ju_n\rangle \\
&\le |\|x_n\|^2 - \|u_n\|^2| + 2\|\langle u, Jx_n - Ju_n\rangle\| \\
&\le |\|x_n\| - \|u_n\||(\|x_n\| + \|u_n\|) \\
&\quad + 2\|u\|\|Jx_n - Ju_n\| \\
&\le \|x_n - u_n\|(\|x_n\| + \|u_n\|) \\
&\quad + 2\|u\|\|Jx_n - Ju_n\|.
\end{aligned}$$

and hence, it follows from $\lim_{n\to\infty}\|x_n - u_n\| = 0$ and $\lim_{n\to\infty}\|Jx_n - Ju_n\| = 0$ that

$$\lim_{n\to\infty}(\phi(u,x_n) - \phi(u,u_n)) = 0. \tag{3.16}$$

Using conditions (i) and (ii) and (3.16) in (3.15), we have

$$\lim_{n\to\infty} g(\|Jx_n - JSx_n\|) = 0.$$

Further, it follows from the property of g that

$$\lim_{n\to\infty} \|Jx_n - JSx_n\| = 0.$$

Since J^{-1} is uniformly norm-to-norm continuous on bounded sets, we have

$$\lim_{n\to\infty} \|x_n - Sx_n\| = 0. \tag{3.17}$$

Next, we have

$$\begin{aligned}
\phi(u,u_n) &\le \phi(u,y_n) \\
&\le \phi(u, J^{-1}(\delta_n Jx_n + (1-\delta_n)JTz_n)) \\
&= \|u\|^2 - 2\langle u, \delta_n Jx_n + (1-\delta_n)JTz_n\rangle + \|\delta_n Jx_n \\
&\quad + (1-\delta_n)JTz_n\|^2 \\
&\le \|u\|^2 - 2\delta_n\langle u, Jx_n\rangle - 2(1-\delta_n)\langle u, JTz_n\rangle \\
&\quad + \delta_n\|x_n\|^2 + (1-\delta_n)\|Tz_n\|^2 \\
&\quad - \delta_n(1-\delta_n)g(\|Jx_n - JTz_n\|) \\
&\le \delta_n\phi(u,x_n) + (1-\delta_n)\phi(u, Tz_n) \\
&\quad - \delta_n(1-\delta_n)g(\|Jx_n - JTz_n\|) \\
&\le \delta_n\phi(u,x_n) + (1-\delta_n)\phi(u,z_n) \\
&\quad - \delta_n(1-\delta_n)g(\|Jx_n - JTz_n\|) \\
&\le \delta_n\phi(u,x_n) + (1-\delta_n)\phi(u,x_n) \\
&\quad - \delta_n(1-\delta_n)g(\|Jx_n - JTz_n\|) \\
&\le \phi(u,x_n) - \delta_n(1-\delta_n)g(\|Jx_n - JTz_n\|),
\end{aligned}$$

or

$$\begin{aligned}
\delta_n(1-\delta_n)g(\|Jx_n - JTz_n\|) &\le \phi(u,x_n) \\
&\quad - \phi(u,u_n) \to 0 \text{as } n\to\infty.
\end{aligned}$$

Thus,

$$\lim_{n\to\infty} g(\|Jx_n - JTz_n\|) = 0.$$

It follows from the property of g that

$$\lim_{n\to\infty} \|Jx_n - JTz_n\| = 0,$$

and hence

$$\lim_{n\to\infty} \|x_n - Tz_n\| = 0. \tag{3.18}$$

Now,

$$\begin{aligned}
\|Jx_n - Jz_n\| &= \|Jx_n - (\alpha_n Jx_n + (1-\alpha_n)JSx_n)\| \\
&= \|(1-\alpha_n)(Jx_n - JSx_n)\| \\
&= (1-\alpha_n)\|Jx_n - JSx_n\|.
\end{aligned}$$

Since $\lim_{n\to\infty}\|Jx_n - JSx_n\| = 0$, the preceding equality implies that

$$\lim_{n\to\infty} \|Jx_n - Jz_n\| = 0,$$

and hence

$$\lim_{n\to\infty} \|x_n - z_n\| = 0. \tag{3.19}$$

It follows from (3.18), (3.19), and the inequality

$$\|z_n - Tz_n\| \leq \|z_n - x_n\| + \|x_n - Tz_n\|$$

that

$$\lim_{n\to\infty} \|z_n - Tz_n\| = 0.$$

Since $x_n \to \hat{x}$, it follows from (3.17), (3.18), and (3.20) that $\hat{x}$ is a fixed point of S and T, i.e., $\hat{x} \in \mathrm{Fix}(T) \cap \mathrm{Fix}(S)$.

On the same lines of the proof of Theorem 3.1(v) with (3.10), we can easily prove that $\hat{x} \in \mathrm{Sol}(GVEP(1.2))$. Then, $\hat{x} \in \Gamma$.

Finally, we prove that $\hat{x} = \prod_\Gamma x$. By taking the limit in (3.13), we have

$$\langle \hat{x} - z, Jx - J\hat{x}\rangle \geq 0, \quad \forall z \in \Gamma. \tag{3.20}$$

Further, in view of Lemma 2.1, we see that $\hat{x} = \prod_\Gamma x$. This completes the proof. □

Now, we prove the weak convergence theorem for finding the common solution for GVEP (1.2) and the fixed point problems of two relatively nonexpansive mappings. First, we prove the following proposition:

Proposotion 3.1. *Let E be a uniformly smooth and uniformly convex Banach space, and let C be a non-empty, compact, and convex subset of E. Assume that P is a pointed, proper, closed, and convex cone of a real Hausdorff topological space Y with $intP \neq \emptyset$. Let $F : C \times C \to Y$ and $\psi : C \to Y$ satisfy Assumption 2.1, and let S, T be relatively nonexpansive mappings from C into itself such that $\Gamma \neq \emptyset$. Let $\{x_n\}$ be a sequence generated by the following scheme:*

$$\begin{aligned} &z_1 \in E,\\ &x_n \in C \text{ such that}\\ &F(x_n, y) + \psi(y) - \psi(x_n) + \frac{e}{r}\langle y - x_n, Jx_n - Jz_n\rangle \in P, \quad \forall y \in C,\\ &y_n = J^{-1}(\alpha_n Jx_n + (1-\alpha_n)JSx_n),\\ &z_{n+1} = J^{-1}(\delta_n Jx_n + (1-\delta_n)JTy_n), \end{aligned}$$

for every $n \in N$, where $e \in intP$, J is the normalized duality mapping on E, and $r \in [a, \infty)$ for some $a > 0$. Assume that $\{\alpha_n\}$ and $\{\delta_n\}$ are sequences in [0,1] satisfying the conditions (i) and (ii) of Theorem 3.2. Then, $\{\prod_\Gamma x_n\}$ converges strongly to $z \in \Gamma$.

Proof. Let $u \in \Gamma$. Since $x_n = T_r z_n$ and T_r, S, T are relatively nonexpansive, we have

$$\begin{aligned} \phi(u, x_{n+1}) &= \phi(u, T_r z_{n+1})\\ &\leq \phi(u, z_{n+1})\\ &\leq \phi(u, J^{-1}(\delta_n Jx_n + (1-\delta_n)JTy_n))\\ &= \|u\|^2 - 2\langle u, \delta_n Jx_n + (1-\delta_n)JTy_n\rangle\\ &\quad + \|\delta_n Jx_n + (1-\delta_n)JTy_n\|^2\\ &\leq \|u\|^2 - 2\delta_n\langle u, Jx_n\rangle - 2(1-\delta_n)\langle u, JTy_n\rangle\\ &\quad + \delta_n\|x_n\|^2 + (1-\delta_n)\|Ty_n\|^2\\ &\leq \delta_n\phi(u, x_n) + (1-\delta_n)\phi(u, Ty_n)\\ &\leq \delta_n\phi(u, x_n) + (1-\delta_n)\phi(u, y_n), \end{aligned} \tag{3.21}$$

and

$$\begin{aligned} \phi(u, y_n) &= \phi(u, J^{-1}(\alpha_n Jx_n + (1-\alpha_n)JSx_n))\\ &= \|u\|^2 - 2\langle u, \alpha_n Jx_n + (1-\alpha_n)JSx_n\rangle\\ &\quad + \|\alpha_n Jx_n + (1-\alpha_n)JSx_n\|^2\\ &\leq \|u\|^2 - 2\alpha_n\langle u, Jx_n\rangle - 2(1-\alpha_n)\langle u, JSx_n\rangle\\ &\quad + \alpha_n\|x_n\|^2 + (1-\alpha_n)\|Sx_n\|^2\\ &\leq \alpha_n\phi(u, x_n) + (1-\alpha_n)\phi(u, Sx_n)\\ &\leq \alpha_n\phi(u, x_n) + (1-\alpha_n)\phi(u, x_n)\\ &\leq \phi(u, x_n). \end{aligned} \tag{3.22}$$

Using (3.22) in (3.21), we have

$$\begin{aligned} &\phi(u, x_{n+1}) \leq \delta_n\phi(u, x_n) + (1-\delta_n)\phi(u, x_n)\\ &\phi(u, x_{n+1}) \leq \phi(u, x_n). \end{aligned} \tag{3.23}$$

Therefore, $\lim_{n\to\infty}\phi(u, x_n)$ exists, and hence, $\phi(u, x_n)$ is bounded. This implies that $\{x_n\}$ and $\{Sx_n\}$ are bounded. Further, it follows from (3.22) that $\phi(u, y_n)$ is also bounded, and hence, $\{y_n\}$ and $\{Ty_n\}$ are bounded.

Define $w_n = \prod_\Gamma x_n$, for every $n \in N$. Then, from $w_n \in \Gamma$ and (3.23), we have

$$\phi(w_n, x_{n+1}) \leq \phi(w_n, x_n). \tag{3.24}$$

Since $\prod_\Gamma$ is the generalized projection, from Lemma 2.1, we have

$$\begin{aligned} \phi(w_{n+1}, x_{n+1}) &= \phi\left(\prod\nolimits_\Gamma x_{n+1}, x_{n+1}\right)\\ &\leq \phi(w_n, x_{n+1}) - \phi\left(w_n, \prod\nolimits_\Gamma x_{n+1}\right)\\ &= \phi(w_n, x_{n+1}) - \phi(w_n, w_{n+1})\\ &\leq \phi(w_n, x_{n+1}). \end{aligned} \tag{3.25}$$

Hence, from (3.24), we have

$$\phi(w_{n+1}, x_{n+1}) \leq \phi(w_n, x_n).$$

Therefore, $\{\phi(w_n, x_n)\}$ is a convergent sequence. We also have from (3.24) that, for all $m \in N$,

$$\phi(w_n, x_{n+m}) \leq \phi(w_n, x_n).$$

From $w_{n+m} = \prod_\Gamma x_{n+m}$ and Lemma 2.1, we have

$$\begin{aligned}\phi(w_n, w_{n+m}) + &\phi(w_{n+m}, x_{n+m}) \\ &\leq \phi(w_n, x_{n+m}) \leq \phi(w_n, x_n)\end{aligned}$$

and hence

$$\phi(w_n, w_{n+m}) \leq \phi(w_n, x_n) - \phi(w_{n+m}, x_{n+m}).$$

Let $r = \sup_{n\in N} \|w_n\|$. From Lemma 2.3, there exists a continuous, strictly increasing, and convex function g with $g(0) = 0$ such that $g(\|x - y\|) \leq \phi(x, y)$ for $x, y \in B_r$. So, we have

$$\begin{aligned}g(\|w_n - w_{n+m}\|) &\leq \phi(w_n, w_{n+m}) \leq \phi(w_n, x_n) \\ &- \phi(w_{n+m}, x_{n+m}).\end{aligned}$$

Since $\{\phi(w_n, x_n)\}$ is a convergent sequence, from the property of g, we have that $\{w_n\}$ is a Cauchy sequence. Since Γ is closed, $\{w_n\}$ converges strongly to $z \in \Gamma$. This completes the proof. □

Now, we are able to prove the following weak convergence theorem.

Theorem 3.3. *Let E be a uniformly smooth and uniformly convex Banach space, and let C be a non-empty, compact, and convex subset of E. Assume that P is a pointed, proper, closed, and convex cone of a real Hausdorff topological space Y with $intP \neq \emptyset$. Let $F : C \times C \to Y$ and $\psi : C \to Y$ satisfy Assumption 2.1, and let S, T be relatively nonexpansive mappings from C into itself such that $\Gamma \neq \emptyset$. Let $\{x_n\}$ be a sequence generated by the scheme:*

$$\begin{aligned}&z_1 \in E, \\ &x_n \in C \text{ such that} \\ &F(x_n, y) + \psi(y) - \psi(x_n) \\ &\quad + \frac{e}{r}\langle y - x_n, Jx_n - Jz_n\rangle \in P, \quad \forall y \in C, \\ &y_n = J^{-1}(\alpha_n Jx_n + (1 - \alpha_n)JSx_n), \\ &z_{n+1} = J^{-1}(\delta_n Jx_n + (1 - \delta_n)JTy_n),\end{aligned}$$

for every $n \in N$, where $e \in intP$, J is the normalized duality mapping on E, and $r \in [a, \infty)$ for some $a > 0$. Assume that $\{\alpha_n\}$ and $\{\delta_n\}$ are sequences in [0,1] satisfying the conditions (i) and (ii) of Theorem 3.2. If J is weakly sequentially continuous, then x_n converges weakly to $z \in \Gamma$, where $z = \lim_{n\to\infty} \prod_\Gamma x_n$.

Proof. As in the proof of Proposition 3.1, we have that $\{x_n\}$, $\{y_n\}$, $\{Sx_n\}$, and $\{Ty_n\}$ are bounded sequences. Let $r = \sup_{n\in N}\{\|x_n\|, \|y_n\|, \|Sx_n\|, \|Ty_n\|\}$. Let $u \in \Gamma$. Since $x_n = T_r z_n$ and T_r, S, T are relatively nonexpansive, using Lemma 2.3, we have

$$\begin{aligned}\phi(u, y_n) &= \phi(u, J^{-1}(\alpha_n Jx_n + (1 - \alpha_n)JSx_n)) \\ &= \|u\|^2 - 2\langle u, \alpha_n Jx_n + (1 - \alpha_n)JSx_n\rangle \\ &\quad + \|\alpha_n Jx_n + (1 - \alpha_n)JSx_n\|^2 \\ &\leq \|u\|^2 - 2\alpha_n\langle u, Jx_n\rangle - 2(1 - \alpha_n)\langle u, JSx_n\rangle \\ &\quad + \alpha_n\|x_n\|^2 + (1 - \alpha_n)\|Sx_n\|^2 \\ &\quad - \alpha_n(1 - \alpha_n)g(\|Jx_n - JSx_n\|) \\ &\leq \alpha_n\phi(u, x_n) + (1 - \alpha_n)\phi(u, Sx_n) \\ &\quad - \alpha_n(1 - \alpha_n)g(\|Jx_n - JSx_n\|) \\ &\leq \alpha_n\phi(u, x_n) + (1 - \alpha_n)\phi(u, x_n) \\ &\quad - \alpha_n(1 - \alpha_n)g(\|Jx_n - JSx_n\|) \\ &\leq \phi(u, x_n) - \alpha_n(1 - \alpha_n)g(\|Jx_n - JSx_n\|).\end{aligned} \tag{3.26}$$

Using (3.26) in (3.21), we have

$$\begin{aligned}\phi(u, x_{n+1}) &\leq \delta_n\phi(u, x_n) + (1 - \delta_n)[\phi(u, x_n) \\ &\quad - \alpha_n(1 - \alpha_n)g(\|Jx_n - JSx_n\|)] \\ &\leq \delta_n\phi(u, x_n) + (1 - \delta_n)\phi(u, x_n) \\ &\quad - \alpha_n(1 - \alpha_n)(1 - \delta_n)g(\|Jx_n - JSx_n\|) \\ &\leq \phi(u, x_n) - \alpha_n(1 - \alpha_n)(1 - \delta_n)g(\|Jx_n - JSx_n\|),\end{aligned}$$

or

$$\begin{aligned}&\alpha_n(1 - \alpha_n)(1 - \delta_n)g(\|Jx_n - JSx_n\|) \\ &\quad \leq \phi(u, x_n) - \phi(u, x_{n+1}).\end{aligned} \tag{3.27}$$

Since $\{\phi(u, x_n)\}$ is convergent and using conditions (i) and (ii) in (3.27), we have

$$\lim_{n\to\infty} g(\|Jx_n - JSx_n\|) = 0.$$

From the property of g, we have

$$\lim_{n\to\infty} \|Jx_n - JSx_n\| = 0.$$

Since J^{-1} is uniformly norm-to-norm continuous on bounded sets, we have

$$\lim_{n\to\infty} \|x_n - Sx_n\| = 0. \tag{3.28}$$

Next, we have

$$\begin{aligned}\phi(u,x_{n+1}) &\le \phi(u,z_{n+1})\\ &\le \phi(u,J^{-1}(\delta_n Jx_n+(1-\delta_n)JTy_n))\\ &= \|u\|^2-2\langle u,\delta_n Jx_n+(1-\delta_n)JTy_n\rangle\\ &\quad+\|\delta_n Jx_n+(1-\delta_n)JTy_n\|^2\\ &\le \|u\|^2-2\delta_n\langle u,Jx_n\rangle-2(1-\delta_n)\langle u,JTy_n\rangle\\ &\quad+\delta_n\|x_n\|^2+(1-\delta_n)\|Ty_n\|^2\\ &\quad-\delta_n(1-\delta_n)g(\|Jx_n-JTy_n\|)\\ &\le \delta_n\phi(u,x_n)+(1-\delta_n)\phi(u,Ty_n)\\ &\quad-\delta_n(1-\delta_n)g(\|Jx_n-JTy_n\|)\\ &\le \delta_n\phi(u,x_n)+(1-\delta_n)\phi(u,y_n)\\ &\quad-\delta_n(1-\delta_n)g(\|Jx_n-JTy_n\|)\\ &\le \delta_n\phi(u,x_n)+(1-\delta_n)\phi(u,x_n)\\ &\quad-\delta_n(1-\delta_n)g(\|Jx_n-JTy_n\|)\\ &\le \phi(u,x_n)-\delta_n(1-\delta_n)g(\|Jx_n-JTy_n\|),\end{aligned}$$

or

$$\delta_n(1-\delta_n)g(\|Jx_n-JTy_n\|)\le\phi(u,x_n)-\phi(u,x_{n+1}). \tag{3.29}$$

Since $\{\phi(u,x_n)\}$ is convergent and using condition (i) in (3.29), we have

$$\lim_{n\to\infty} g(\|Jx_n-JTy_n\|)=0.$$

From the property of g, we have

$$\lim_{n\to\infty}\|Jx_n-JTy_n\|=0,$$

and hence

$$\lim_{n\to\infty}\|x_n-Ty_n\|=0. \tag{3.30}$$

Now,

$$\begin{aligned}\|Jx_n-Jy_n\| &= \|Jx_n-(\alpha_n Jx_n+(1-\alpha_n)JSx_n)\|\\ &= \|(1-\alpha_n)(Jx_n-JSx_n)\|\\ &= (1-\alpha_n)\|Jx_n-JSx_n\|,\end{aligned}$$

which implies

$$\lim_{n\to\infty}\|Jx_n-Jy_n\|=0,$$

and hence

$$\lim_{n\to\infty}\|x_n-y_n\|=0. \tag{3.31}$$

It follows from (3.30), (3.31), and the inequality $\|y_n-Ty_n\|\le\|y_n-x_n\|+\|x_n-Ty_n\|$ that

$$\lim_{n\to\infty}\|y_n-Ty_n\|=0. \tag{3.32}$$

Since $\{x_n\}$ and $\{y_n\}$ are bounded and $\lim_{n\to\infty}\|x_n-y_n\|=0$, there exist subsequences $\{x_{n_k}\}$ and $\{y_{n_k}\}$ of $\{x_n\}$ and $\{y_n\}$, respectively, such that $x_{n_k}\rightharpoonup\hat{x}\in C$ and $y_{n_k}\rightharpoonup\hat{x}\in C$. It follows from (3.28) and (3.32) that $\hat{x}\in\widehat{\mathrm{Fix}}(S)\cap\widehat{\mathrm{Fix}}(T)=\mathrm{Fix}(S)\cap\mathrm{Fix}(T)$, i.e., $\hat{x}\in\mathrm{Fix}(S)\cap\mathrm{Fix}(T)$.

Next, we show that $\hat{x}\in\mathrm{Sol}(GVEP(1.2))$. Let $r=\sup_{n\in N}\{\|x_n\|,\|z_n\|\}$. From Lemma 2.4, there exists a continuous, strictly increasing, and convex function g_1 with $g_1(0)=0$ such that

$$g_1(\|x-y\|)\le\phi(x,y),\quad \forall x,y\in B_r.$$

Since $x_n=T_r z_n$, we have from Lemma 3.1 that, for $u\in\Gamma$,

$$\begin{aligned}g_1(\|x_n-z_n\|) &\le \phi(x_n,z_n)\\ &\le \phi(u,z_n)-\phi(u,x_n)\\ &\le \phi(u,x_{n-1})-\phi(u,x_n).\end{aligned}$$

Since $\{\phi(u,x_n)\}$ converges, we have

$$\lim_{n\to\infty} g_1(\|x_n-z_n\|)=0.$$

From the property of g_1, we have

$$\lim_{n\to\infty}\|x_n-z_n\|=0.$$

Since J is uniformly norm-to-norm continuous on bounded sets, we have

$$\lim_{n\to\infty}\|Jx_n-Jz_n\|=0.$$

From $r\ge a$, we have

$$\lim_{n\to\infty}\frac{\|Jx_n-Jz_n\|}{r}=0.$$

By $x_n=T_r z_n$, we have

$$\begin{aligned}&F(T_r(z_n),y)+\psi(y)-\psi(T_r z_n)\\ &\qquad+\frac{e}{r}\langle y-T_r z_n, JT_r z_n-Jz_n\rangle\in P,\\ &\qquad\forall y\in C\\ &0\in F(y,T_r(z_n))-\psi(y)-\psi(T_r z_n)\\ &\qquad+\frac{e}{r}\langle y-T_r z_n, JT_r z_n-Jz_n\rangle+P,\\ &\qquad\forall y\in C.\end{aligned}$$

Replacing n by n_i, we have

$$\begin{aligned}&0\in F(y,T_r(z_{n_i}))-\psi(y)-\psi(T_r z_{n_i})\\ &\quad+\frac{e}{r}\langle y-T_r z_{n_i}, JT_r z_{n_i}-Jz_{n_i}\rangle+P,\forall y\in C.\end{aligned}$$

As in the proof of Theorem 3.1, we have $\hat{x}\in\mathrm{Sol}(GVEP(1.2))$. Hence, $\hat{x}\in\Gamma$.

Let $w_n=\prod_\Gamma x_n$. From Lemma 2.1 and $\hat{x}\in\Gamma$, we have

$$\langle w_{n_k}-\hat{x}, Jx_{n_k}-Jw_{n_k}\rangle\ge 0.$$

It follows from Proposition 3.1 that $\{w_n\}$ converges strongly to $z\in\Gamma$. Since J is weakly sequentially continuous, we have

$$\langle z-\hat{x}, J\hat{x}-Jz\rangle\ge 0\ \text{ as }\ k\to\infty.$$

On the other hand, since J is monotone, we have

$$\langle z - \hat{x}, J\hat{x} - Jz \rangle \leq 0.$$

Hence, we have

$$\langle z - \hat{x}, J\hat{x} - Jz \rangle = 0.$$

From the strict convexity of E, we have $z = \hat{x}$. Therefore, $\{x_n\}$ converges weakly to $\hat{x} \in \Gamma$, where $\hat{x} = \lim_{n\to\infty} \prod_\Gamma x_n$. This completes the proof . □

Remark 3.1

(i) If we take $\psi = 0$, then Theorems 3.2 and 3.3 are reduced to the theorems of finding a common solution of SVEP(1.3) and fixed point problems for two relatively nonexpansive mappings.

(ii) If we take $Y = \mathbb{R}$, $P = [0, +\infty)$, $T = I$, identity mapping, and $\delta_n = 0$, $\forall n$, then the results presented in this paper are reduced to the corresponding results of Takahashi and Zembayashi [17].

(iii) The method presented in this paper can be used to extend the results of Shan and Huang [19] and Petrot et al. [20].

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
KRK and MF contributed equally. Both authors read and approved the final manuscript.

Acknowledgements
The authors would like to thank the anonymous referee for his careful reading of the paper.

Received: 17 March 2013 Accepted: 13 April 2013
Published: 3 May 2013

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doi:10.1186/2251-7456-7-19
Cite this article as: Kazmi and Farid: Some iterative schemes for generalized vector equilibrium problems and relatively nonexpansive mappings in Banach spaces. *Mathematical Sciences* 2013 **7**:19.