Mathematical Sciences
a SpringerOpen Journal

ORIGINAL RESEARCH **Open Access**

# Common fixed points of six mappings in partially ordered *G*-metric spaces

Vahid Parvaneh[1*], Abdolrahman Razani[2] and Jamal Rezaei Roshan[3]

**Abstract**

The aim of this paper is to present some common fixed point results for six selfmappings satisfying generalized weakly $(\psi, \varphi)$-contractive condition in the setup of partially ordered *G*-metric spaces. Our results extend and generalize the comparable results in the work of Abbas from the context of ordered metric spaces to the setup of ordered *G*-metric spaces. Also, our results are supported by an example.

**Keywords:** Common fixed point; Generalized weakly contraction; Generalized metric space; Partially ordered set; Weak annihilator map; Dominating map

## Introduction and preliminaries

Alber and Guerre-Delabrere [1] defined weakly contractive mappings on Hilbert spaces as follows:

**Definition 1.1.** *A mapping $f : X \to X$ is said to be a weakly contractive mapping if*

$$d(fx, fy) \leq d(x, y) - \varphi(d(x, y)),$$

*where $x, y \in X$ and $\varphi : [0, \infty) \to [0, \infty)$ is a continuous and nondecreasing function such that $\varphi(t) = 0$ if and only if $t = 0$.*

**Theorem 1.2.** *[2] Let $(X, d)$ be a complete metric space and $f : X \to X$ be a weakly contractive mapping. Then $f$ has a unique fixed point.*

Recently, Zhang and Song [3] have introduced the concept of a generalized $\varphi$-weak contractive condition and obtained a common fixed point for two maps.

**Definition 1.3.** *Two mappings $T, S : X \to X$ are called generalized $\varphi$-weak contractions if there exists a lower semicontinuous function $\varphi : [0, \infty) \to [0, \infty)$ with $\varphi(0) = 0$ and $\varphi(t) > 0$ for all $t > 0$ such that*

$$d(Tx, Sy) \leq N(x, y) - \varphi(N(x, y)),$$

*for all $x, y \in X$, where*

$$N(x, y) = \max\{d(x, y), d(x, Tx), d(y, Sy), \frac{1}{2}[d(x, Sy)+d(y, Tx)]\}.$$

Zhang and Song proved the following theorem.

**Theorem 1.4.** *Let $(X, d)$ be a complete metric space and $T, S : X \to X$ be generalized $\varphi$-weak contractive mappings where $\varphi : [0, \infty) \to [0, \infty)$ is a lower semicontinuous function with $\varphi(0) = 0$ and $\varphi(t) > 0$ for all $t > 0$. Then, there exists a unique point $u \in X$ such that $u = Tu = Su$.*

Dorić [4], Moradi et al. [5], Abbas and Dorić [6], and Razani et al. [7] obtained some common fixed point theorems which are extensions of the result of Zhang and Song in the framework of complete metric spaces. Also, in these years many authors have focused on different contractive conditions in complete metric spaces with a partially order and have obtained some common fixed point theorems. For more details on fixed point theory, its applications, comparison of different contractive conditions and related results in ordered metric spaces we refer the reader to [8-15] and the references mentioned therein.

The concept of a generalized metric space, or a *G*-metric space, was introduced by Mustafa and Sims [16]. In recent years, many authors have obtained different fixed point theorems for mappings satisfying various contractive conditions on *G*-metric spaces (see e.g., [9,17-34]).

*Correspondence: vahid.parvaneh@kiau.ac.ir
[1]Department of Mathematics, Gilan-E-Gharb Branch, Islamic Azad University, Gilan-E-Gharb 67871-54699, Iran
Full list of author information is available at the end of the article

**Definition 1.5.** *[16] (G-metric space) Let $X$ be a nonempty set and $G : X \times X \times X \to R^+$ be a function satisfying the following properties:*

*(G1)* $G(x, y, z) = 0$ *iff* $x = y = z$;
*(G2)* $0 < G(x, x, y)$, *for all* $x, y \in X$ *with* $x \neq y$;
*(G3)* $G(x, x, y) \leq G(x, y, z)$, *for all* $x, y, z \in X$ *with* $z \neq y$;
*(G4)* $G(x, y, z) = G(x, z, y) = G(y, z, x) = \cdots$, *(symmetry in all three variables);*
*(G5)* $G(x, y, z) \leq G(x, a, a) + G(a, y, z)$, *for all* $x, y, z, a \in X$ *(rectangle inequality).*

*Then the function $G$ is called a G-metric on $X$ and the pair $(X, G)$ is called a G-metric space.*

**Definition 1.6.** *[16] Let $(X, G)$ be a G-metric space and let $\{x_n\}$ be a sequence of points in X. A point $x \in X$ is said to be the limit of the sequence $\{x_n\}$ and if $\lim_{n,m\to\infty} G(x, x_n, x_m) = 0$ and one says that the sequence $\{x_n\}$ is G-convergent to x. Thus, if $x_n \to x$ in a G-metric space $(X, G)$, then for any $\varepsilon > 0$, there exists a positive integer N such that $G(x, x_n, x_m) < \varepsilon$, for all $n, m \geq N$.*

**Definition 1.7.** *[16] Let $(X, G)$ be a G-metric space. A sequence $\{x_n\}$ is called G-Cauchy if for every $\varepsilon > 0$, there is a positive integer N such that $G(x_n, x_m, x_l) < \varepsilon$, for all $n, m, l \geq N$, that is, if $G(x_n, x_m, x_l) \to 0$, as $n, m, l \to \infty$.*

**Lemma 1.8.** *[16] Let $(X, G)$ be a G-metric space. Then the following are equivalent:*

*(1)* $\{x_n\}$ *is G-convergent to x.*
*(2)* $G(x_n, x_n, x) \to 0$, as $n \to \infty$.
*(3)* $G(x_n, x, x) \to 0$, as $n \to \infty$.

**Lemma 1.9.** *[35] If $(X, G)$ is a G-metric space, then $\{x_n\}$ is a G-Cauchy sequence if and only if for every $\varepsilon > 0$, there exists a positive integer N such that $G(x_n, x_m, x_m) < \varepsilon$, for all $m > n \geq N$.*

**Definition 1.10.** *[16] A G-metric space $(X, G)$ is said to be G-complete if every G-Cauchy sequence in $(X, G)$ is convergent in X.*

**Definition 1.11.** *[16] Let $(X, G)$ and $(X', G')$ be two G-metric spaces. A function $f : X \to X'$ is G-continuous at a point $x \in X$ if and only if it is G-sequentially continuous at x, that is, whenever $\{x_n\}$ is G-convergent to x, $\{f(x_n)\}$ is $G'$-convergent to $f(x)$.*

**Definition 1.12.** *A G-metric on X is said to be symmetric if $G(x, y, y) = G(y, x, x)$, for all $x, y \in X$.*

The concept of an altering distance function was introduced by Khan et al. [36] as follows.

**Definition 1.13.** *The function $\psi : [0, \infty) \to [0, \infty)$ is called an altering distance function if the following conditions hold:*

1. *$\psi$ is continuous and nondecreasing.*
2. *$\psi(t) = 0$ if and only if $t = 0$.*

**Definition 1.14.** *[8] Let $(X, \preceq)$ be a partially ordered set. A mapping f is called a dominating map on X if $x \preceq fx$, for each x in X.*

**Example 1.15.** *[8] Let $X = [0, 1]$ be endowed with the usual ordering. Let $f : X \to X$ be defined by $fx = x^{\frac{1}{3}}$. Then, $x \leq x^{\frac{1}{3}} = fx$, for all $x \in X$. Thus, f is a dominating map.*

**Example 1.16.** *[8] Let $X = [0, \infty)$ be endowed with the usual ordering. Let $f : X \to X$ be defined by $fx = \sqrt[n]{x}$ for $x \in [0, 1)$ and $fx = x^n$ for $x \in [1, \infty)$, for any positive integer n. Then for all $x \in X$, $x \leq fx$; that is, f is a dominating map.*

A subset $W$ of a partially ordered set $X$ is said to be well ordered if every two elements of $W$ be comparable [8].

**Definition 1.17.** *[8] Let $(X, \preceq)$ be a partially ordered set. A mapping f is called a weak annihilator of g if $fgx \preceq x$ for all $x \in X$.*

Jungck in [37] introduced the following definition.

**Definition 1.18.** *[37] Let $(X, d)$ be a metric space and $f, g : X \to X$ be two mappings. The pair $(f, g)$ is said to be compatible if and only if $\lim_{n\to\infty} d(fgx_n, gfx_n) = 0$, whenever $\{x_n\}$ is a sequence in X such that $\lim_{n\to\infty} fx_n = \lim_{n\to\infty} gx_n = t$, for some $t \in X$.*

**Definition 1.19.** *[38,39] Let $(X, G)$ be a G-metric space and $f, g : X \to X$ be two mappings. The pair $(f, g)$ is said to be compatible if and only if $\lim_{n\to\infty} G(fgx_n, fgx_n, gfx_n) = 0$, whenever $\{x_n\}$ is a sequence in X such that $\lim_{n\to\infty} fx_n = \lim_{n\to\infty} gx_n = t$, for some $t \in X$.*

**Definition 1.20.** *[40] Let f and g be two self mappings of a metric space $(X, d)$. The f and g are said to be weakly compatible if for all $x \in X$, the equality $fx = gx$ implies $fgx = gfx$.*

Let $X$ be a non-empty set and $f : X \to X$ be a given mapping. For every $x \in X$, let $f^{-1}(x) = \{u \in X : fu = x\}$.

**Definition 1.21.** *Let $(X, \preceq)$ be a partially ordered set and $f, g, h : X \to X$ be mappings such that $fX \subseteq hX$ and $gX \subseteq hX$. The ordered pair $(f, g)$ is said to be partially weakly increasing with respect to h if for all $x \in X$, $fx \preceq gy$, for all $y \in h^{-1}(fx)$ [41].*

Since we are motivated by the work in [8] in this paper, we prove some common fixed point theorems for nonlinear generalized $(\psi, \varphi)$-weakly contractive mappings in partially ordered $G$-metric spaces.

## Main results

Abbas et al. [8] proved the following theorem.

**Theorem 2.1.** *Let $(X, \preceq, d)$ be an ordered complete metric space. Let f, g, S and T be selfmaps on X, $(T, f)$ and $(S, g)$ be partially weakly increasing with $f(X) \subseteq T(X)$ and $g(X) \subseteq S(X)$, dominating maps f and g be weak annihilators of T and S, respectively. Suppose that there exist altering distance functions $\psi$ and $\varphi$ such that for every two comparable elements $x, y \in X$,*

$$\psi(d(fx, gy)) \leq \psi(M(x, y)) - \varphi(M(x, y)),$$

*is satisfied where*

$$M(x, y) = \max\{d(Sx, Ty), d(fx, Sx), d(gy, Ty), \frac{d(Sx, gy) + d(fx, Ty)}{2}\}.$$

*If for a nondecreasing sequence $\{x_n\}$ with $x_n \preceq y_n$ for all n, $y_n \to u$ implies that $x_n \preceq u$ and either of the following:*

*(a) $(f, S)$ are compatible, f or S is continuous, and $(g, T)$ are weakly compatible or*
*(b) $(g, T)$ are compatible, g or T is continuous, and $(f, S)$ are weakly compatible,*

*then f, g, S, and T have a common fixed point. Moreover, the set of common fixed points of f, g, S and T is well ordered if and only if f, g, S, and T have one and only one common fixed point.*

Let $(X, \preceq, G)$ be an ordered $G$-metric space and $f, g, h, R, S, T : X \to X$ be six self mappings. In the rest of this paper, unless otherwise stated, for all $x, y, z \in X$, let

$$M(x, y, z) = \max\{G(Tx, Ry, Sz), G(Tx, Ry, gy), G(Ry, Sz, hz), G(Sz, Tx, fx), \frac{G(Tx, fx, gy) + G(Ry, gy, hz) + G(Sz, hz, fx)}{3}\}.$$

Our first result is the following.

**Theorem 2.2.** *Let $(X, \preceq, G)$ be a partially ordered complete G-metric space. Let $f, g, h, R, S, T : X \to X$ be the six mappings such that $f(X) \subseteq R(X)$, $g(X) \subseteq S(X)$, $h(X) \subseteq T(X)$ and dominating maps f, g, and h are weak annihilators of R, S, and T, respectively. Suppose that for every three comparable elements $x, y, z \in X$,*

$$\psi\left(G(fx, gy, hz)\right) \leq \psi\left(M(x, y, z)\right) - \varphi\left(M(x, y, z)\right), \quad (1)$$

*where $\psi, \varphi : [0, \infty) \to [0, \infty)$ are altering distance functions. Then, f, g, h, R, S, and T have a common fixed point in X provided that for a nondecreasing sequence $\{x_n\}$ with $x_n \preceq y_n$ for all n, $y_n \to u$ implies that $x_n \preceq u$ and either of the following:*

*(i) One of g or R and one of f or T are continuous, the pairs $(f, T)$ and $(g, R)$ are compatible, and the pair $(h, S)$ is weakly compatible or*
*(ii) One of h or S and one of f or T are continuous, the pairs $(f, T)$ and $(h, S)$ are compatible, and the pair $(g, R)$ is weakly compatible or*
*(iii) One of g or R and one of h or S are continuous, the pairs $(g, R)$ and $(h, S)$ are compatible, and the pair $(f, T)$ is weakly compatible.*

*Moreover, the set of common fixed points of f, g, h, R, S, and T is well ordered if and only if f, g, h, R, S, and T have one and only one common fixed point.*

**Proof 2.3.** *Let $x_0 \in X$ be an arbitrary point. Since $f(X) \subseteq R(X)$, we can choose $x_1 \in X$ such that $fx_0 = Rx_1$. Since $g(X) \subseteq S(X)$, we can choose $x_2 \in X$ such that $gx_1 = Sx_2$. Also, as $h(X) \subseteq T(X)$, we can choose $x_3 \in X$ such that $hx_2 = Tx_3$.*

*Continuing this process, we can construct a sequence $\{z_n\}$ defined by*

$$z_{3n+1} = Rx_{3n+1} = fx_{3n},$$
$$z_{3n+2} = Sx_{3n+2} = gx_{3n+1},$$

*and*

$$z_{3n+3} = Tx_{3n+3} = hx_{3n+2},$$

*for all $n \geq 0$.*

*Now, since f, g and h are dominating and f, g, and h are weak annihilators of R, S and T, we obtain that*

$$x_0 \preceq fx_0 = Rx_1 \preceq fRx_1 \preceq x_1 \preceq gx_1 = Sx_2 \preceq gSx_2 \preceq x_2 \preceq hx_2 = Tx_3 \preceq hTx_3 \preceq x_3.$$

*By continuing this process, we get*

$$x_1 \preceq x_2 \preceq x_3 \preceq \cdots \preceq x_k \preceq x_{k+1} \preceq \cdots.$$

*We will complete the proof in three steps.*

*Step I. We will prove that* $\lim_{k\to\infty} G(z_k, z_{k+1}, z_{k+2}) = 0$.

*Define* $G_k = G(z_k, z_{k+1}, z_{k+2})$. *Suppose* $G_{k_0} = 0$ *for some* $k_0$. *Then,* $z_{k_0} = z_{k_0+1} = z_{k_0+2}$. *Consequently, the sequence* $\{z_k\}$ *is constant, for* $k \geq k_0$. *Indeed, let* $k_0 = 3n$. *Then* $z_{3n} = z_{3n+1} = z_{3n+2}$, *and we obtain from (1),*

$$\begin{aligned}\psi\left(G(z_{3n+1}, z_{3n+2}, z_{3n+3})\right) &= \psi\left(G(fx_{3n}, gx_{3n+1}, hx_{3n+2})\right)\\ &\leq \psi\left(M(x_{3n}, x_{3n+1}, x_{3n+2})\right)\\ &\quad - \varphi\left(M(x_{3n}, x_{3n+1}, x_{3n+2})\right),\end{aligned} \tag{2}$$

*where*

$$\begin{aligned}&M(x_{3n}, x_{3n+1}, x_{3n+2})\\ &= \max\{G(Tx_{3n}, Rx_{3n+1}, Sx_{3n+2}), G(Tx_{3n}, Rx_{3n+1}, gx_{3n+1}),\\ &\quad G(Rx_{3n+1}, Sx_{3n+2}, hx_{3n+2}), G(Sx_{3n+2}, Tx_{3n}, fx_{3n}),\\ &\quad \frac{G(Tx_{3n}, fx_{3n}, gx_{3n+1}) + G(Rx_{3n+1}, gx_{3n+1}, hx_{3n+2})}{3}\\ &\quad + \frac{G(Sx_{3n+2}, hx_{3n+2}, fx_{3n})}{3}\}\\ &= \max\{G(z_{3n}, z_{3n+1}, z_{3n+2}), G(z_{3n}, z_{3n+1}, z_{3n+2}),\\ &\quad G(z_{3n+1}, z_{3n+2}, z_{3n+3}), G(z_{3n+2}, z_{3n}, z_{3n+1}),\\ &\quad \frac{G(z_{3n}, z_{3n+1}, z_{3n+2}) + G(z_{3n+1}, z_{3n+2}, z_{3n+3})}{3}\\ &\quad + \frac{G(z_{3n+2}, z_{3n+3}, z_{3n+1})}{3}\}\\ &= \max\{0, 0, G(z_{3n+1}, z_{3n+2}, z_{3n+3}), 0,\\ &\quad \frac{0 + G(z_{3n+1}, z_{3n+2}, z_{3n+3}) + G(z_{3n+2}, z_{3n+3}, z_{3n+1})}{3}\}.\end{aligned}$$

*Now from (2),*

$$\begin{aligned}\psi(G(z_{3n+1}, z_{3n+2}, z_{3n+3})) &\leq \psi(G(z_{3n+1}, z_{3n+2}, z_{3n+3}))\\ &\quad - \varphi(G(z_{3n+1}, z_{3n+2}, z_{3n+3})),\end{aligned}$$

*and so,* $\varphi(G(z_{3n+1}, z_{3n+2}, z_{3n+3})) = 0$, *that is,* $z_{3n+1} = z_{3n+2} = z_{3n+3}$.

*Similarly, if* $k_0 = 3n + 1$ *or* $k_0 = 3n + 2$, *one can easily obtain that* $z_{3n+2} = z_{3n+3} = z_{3n+4}$ *and* $z_{3n+3} = z_{3n+4} = z_{3n+5}$, *and so the sequence* $\{z_k\}$ *is constant (for* $k \geq k_0$*), and* $z_{k_0}$ *is a common fixed point of* $R$, $S$, $T$, $f$, $g$, *and* $h$.

*Suppose*

$$G_k = G(z_k, z_{k+1}, z_{k+2}) > 0 \tag{3}$$

*for all k. We prove that for each* $k = 1, 2, 3, \cdots$

$$\begin{aligned}G(z_{k+1}, z_{k+2}, z_{k+3}) &\leq M(x_k, x_{k+1}, x_{k+2})\\ &= G(z_k, z_{k+1}, z_{k+2}).\end{aligned} \tag{4}$$

*Let* $k = 3n$. *Since* $x_{k-1} \preceq x_k$, *using (1) we obtain that*

$$\begin{aligned}\psi\left(G(z_{3n+1}, z_{3n+2}, z_{3n+3})\right) &= \psi\left(G(fx_{3n}, gx_{3n+1}, hx_{3n+2})\right)\\ &\leq \psi\left(M(x_{3n}, x_{3n+1}, x_{3n+2})\right)\\ &\quad - \varphi\left(M(x_{3n}, x_{3n+1}, x_{3n+2})\right)\\ &\leq \psi\left(M(x_{3n}, x_{3n+1}, x_{3n+2})\right),\end{aligned} \tag{5}$$

*where*

$$\begin{aligned}&M(x_{3n}, x_{3n+1}, x_{3n+2})\\ &= \max\{G(Tx_{3n}, Rx_{3n+1}, Sx_{3n+2}), G(Tx_{3n}, Rx_{3n+1}, gx_{3n+1}),\\ &\quad G(Rx_{3n+1}, Sx_{3n+2}, hx_{3n+2}), G(Sx_{3n+2}, Tx_{3n}, fx_{3n}),\\ &\quad \frac{G(Tx_{3n}, fx_{3n}, gx_{3n+1}) + G(Rx_{3n+1}, gx_{3n+1}, hx_{3n+2})}{3}\\ &\quad + \frac{G(Sx_{3n+2}, hx_{3n+2}, fx_{3n})}{3}\}\\ &= \max\{G(z_{3n}, z_{3n+1}, z_{3n+2}), G(z_{3n}, z_{3n+1}, z_{3n+2}),\\ &\quad G(z_{3n+1}, z_{3n+2}, z_{3n+3}), G(z_{3n+2}, z_{3n}, z_{3n+1}),\\ &\quad \frac{G(z_{3n}, z_{3n+1}, z_{3n+2}) + G(z_{3n+1}, z_{3n+2}, z_{3n+3})}{3}\\ &\quad + \frac{G(z_{3n+2}, z_{3n+3}, z_{3n+1})}{3}\}.\end{aligned}$$

*Since* $\psi$ *is a nondecreasing function from (5), we get*

$$G(z_{3n+1}, z_{3n+2}, z_{3n+3}) \leq M(x_{3n}, x_{3n+1}, x_{3n+2}). \tag{6}$$

*If for an* $n \geq 0$, $G(z_{3n+1}, z_{3n+2}, z_{3n+3}) > G(z_{3n}, z_{3n+1}, z_{3n+2}) > 0$, *then*

$$M(x_{3n}, x_{3n+1}, x_{3n+2}) = G(z_{3n+1}, z_{3n+2}, z_{3n+3}).$$

*Therefore, (5) implies that*

$$\begin{aligned}\psi(G(z_{3n+1}, z_{3n+2}, z_{3n+3})) &\leq \psi(G(z_{3n+1}, z_{3n+2}, z_{3n+3}))\\ &\quad - \varphi(G(z_{3n+1}, z_{3n+2}, z_{3n+3})),\end{aligned}$$

*which is only possible when* $G(z_{3n+1}, z_{3n+2}, z_{3n+3}) = 0$. *This is a contradiction to (3). Hence,*$G(z_{3n+1}, z_{3n+2}, z_{3n+3}) \leq G(z_{3n}, z_{3n+1}, z_{3n+2})$ *and*

$$M(x_{3n}, x_{3n+1}, x_{3n+2}) = G(z_{3n}, z_{3n+1}, z_{3n+2}).$$

*Therefore, (4) is proved for* $k = 3n$. *Similarly, it can be shown that*

$$\begin{aligned}G(z_{3n+2}, z_{3n+3}, z_{3n+4}) &\leq M(x_{3n+1}, x_{3n+2}, x_{3n+3})\\ &= G(z_{3n+1}, z_{3n+2}, z_{3n+3}),\end{aligned} \tag{7}$$

*and*

$$\begin{aligned}G(z_{3n+3}, z_{3n+4}, z_{3n+5}) &\leq M(x_{3n+2}, x_{3n+3}, x_{3n+4})\\ &= G(z_{3n+2}, z_{3n+3}, z_{3n+4}).\end{aligned} \tag{8}$$

*Hence, we conclude that* $\{G(z_k, z_{k+1}, z_{k+2})\}$ *is a nondecreasing sequence of nonnegative real numbers. Thus, there is an* $r \geq 0$ *such that*

$$\lim_{k\to\infty} G(z_k, z_{k+1}, z_{k+2}) = r. \tag{9}$$

*Since*

$$G(z_{k+1}, z_{k+2}, z_{k+3}) \leq M(x_k, x_{k+1}, x_{k+2}) \leq G(z_k, z_{k+1}, z_{k+2}), \tag{10}$$

*letting $k \to \infty$ in (10), we get*

$$\lim_{k\to\infty} M(x_k, x_{k+1}, x_{k+2}) = r. \tag{11}$$

*Letting $n \to \infty$ in (5) and using (9) and (11) and the continuity of $\psi$ and $\varphi$, we get $\psi(r) \leq \psi(r) - \varphi(r) \leq \psi(r)$ and hence $\varphi(r) = 0$. This gives us*

$$\lim_{k\to\infty} G(x_k, x_{k+1}, x_{k+2}) = 0, \tag{12}$$

*from our assumptions about $\varphi$. Also, from Definition 1.5, part (G3), we have*

$$\lim_{k\to\infty} G(x_k, x_{k+1}, x_{k+1}) = 0. \tag{13}$$

*Step II. We will show that $\{z_n\}$ is a G-Cauchy sequence in X. Therefore, we will show that for every $\varepsilon > 0$, there exists a positive integer k such that for all $m, n \geq k$, $G(z_m, z_n, z_n) < \varepsilon$. Suppose the above statement is false. Then there exists $\varepsilon > 0$ for which we can find subsequences $\{z_{m(k)}\}$ and $\{z_{n(k)}\}$ of $\{z_n\}$ such that $n(k) > m(k) \geq k$ and*

(a) $m(k) = 3t$ and $n(k) = 3t' + 1$, where $t$ and $t'$ are nonnegative integers.

(b)

$$G(z_{m(k)}, z_{n(k)}, z_{n(k)}) \geq \varepsilon. \tag{14}$$

(c) *$n(k)$ is the smallest number such that the condition (b) holds; i.e.,*

$$G(z_{m(k)}, z_{n(k)-1}, z_{n(k)-1}) < \varepsilon. \tag{15}$$

*From rectangle inequality and (15), we have*

$$\begin{aligned} G(z_{m(k)}, z_{n(k)}, z_{n(k)}) &\leq G(z_{m(k)}, z_{n(k)-1}, z_{n(k)-1}) + G(z_{n(k)-1}, z_{n(k)}, z_{n(k)}) \\ &< \varepsilon + G(z_{n(k)-1}, z_{n(k)}, z_{n(k)+1}). \end{aligned} \tag{16}$$

*Making $k \to \infty$ in (16) from (12) and (15), we conclude that*

$$\lim_{k\to\infty} G(z_{m(k)}, z_{n(k)}, z_{n(k)}) = \varepsilon. \tag{17}$$

*Again, from rectangle inequality,*

$$\begin{aligned} G(z_{m(k)}, z_{n(k)}, z_{n(k)+1}) &\leq G(z_{m(k)}, z_{n(k)}, z_{n(k)}) + G(z_{n(k)}, z_{n(k)}, z_{n(k)+1}) \\ &\leq G(z_{m(k)}, z_{n(k)}, z_{n(k)}) + G(z_{n(k)}, z_{n(k)+1}, z_{n(k)+2}), \end{aligned} \tag{18}$$

*and*

$$G(z_{m(k)}, z_{n(k)}, z_{n(k)}) \leq G(z_{m(k)}, z_{n(k)}, z_{n(k)+1}). \tag{19}$$

*Hence, in (18) and (19), if $k \to \infty$, using (12), (14), and (17), we have*

$$\lim_{k\to\infty} G(z_{m(k)}, z_{n(k)}, z_{n(k)+1}) = \varepsilon. \tag{20}$$

*On the other hand,*

$$G(z_{m(k)}, z_{n(k)+1}, z_{n(k)+1}) \leq G(z_{m(k)}, z_{n(k)}, z_{n(k)}) + G(z_{n(k)}, z_{n(k)+1}, z_{n(k)+1}), \tag{21}$$

*and*

$$G(z_{n(k)}, z_{n(k)+1}, z_{m(k)}) \leq G(z_{n(k)}, z_{n(k)+1}, z_{n(k)+1}) + G(z_{n(k)+1}, z_{n(k)+1}, z_{m(k)}). \tag{22}$$

*Hence, in (21) and (22), if $k \to \infty$ is from (13), (17), and (20), we have*

$$\lim_{k\to\infty} G(z_{m(k)}, z_{n(k)+1}, z_{n(k)+1}) = \varepsilon. \tag{23}$$

*In a similar way, we have*

$$\begin{aligned} G(z_{m(k)+1}, z_{n(k)}, z_{n(k)+1}) &\leq G(z_{m(k)+1}, z_{m(k)}, z_{m(k)}) + G(z_{m(k)}, z_{n(k)}, z_{n(k)+1}) \\ &\leq 2G(z_{m(k)}, z_{m(k)+1}, z_{m(k)+1}) + G(z_{m(k)}, z_{n(k)}, z_{n(k)+1}), \end{aligned} \tag{24}$$

*and*

$$G(z_{m(k)}, z_{n(k)}, z_{n(k)+1}) \leq G(z_{m(k)}, z_{m(k)+1}, z_{m(k)+1}) + G(z_{m(k)+1}, z_{n(k)}, z_{n(k)+1}), \tag{25}$$

*and therefore, from (24) and (25) by taking limit when $k \to \infty$, using (13) and (20), we get that*

$$\lim_{k\to\infty} G(z_{m(k)+1}, z_{n(k)}, z_{n(k)+1}) = \varepsilon. \tag{26}$$

*Also,*

$$G(z_{m(k)}, z_{n(k)+1}, z_{n(k)+1}) \leq G(z_{m(k)}, z_{m(k)+1}, z_{n(k)+1}), \tag{27}$$

*and*

$$\begin{aligned} G(z_{m(k)}, z_{m(k)+1}, z_{n(k)+1}) &\leq G(z_{m(k)}, z_{m(k)+1}, z_{m(k)+1}) + G(z_{m(k)+1}, z_{m(k)+1}, z_{n(k)+1}) \\ &\leq G(z_{m(k)}, z_{m(k)+1}, z_{m(k)+1}) + G(z_{m(k)+1}, z_{n(k)}, z_{n(k)+1}). \end{aligned} \tag{28}$$

*Hence in (27) and (28), if $k \to \infty$ from (13), (23), and (25), we have*

$$\lim_{k\to\infty} G(z_{m(k)}, z_{m(k)+1}, z_{n(k)+1}) = \varepsilon. \tag{29}$$

*Also,*

$$G(z_{m(k)+1}, z_{n(k)+1}, z_{n(k)+1}) \leq G(z_{m(k)+1}, z_{n(k)+1}, z_{n(k)}) \tag{30}$$

*and*

$$G(z_{m(k)+1}, z_{n(k)}, z_{n(k)+1}) \leq G(z_{m(k)+1}, z_{n(k)+1}, z_{n(k)+1}) + G(z_{n(k)+1}, z_{n(k)+1}, z_{n(k)}). \tag{31}$$

*So from (13), (26), (29), and (30), we have*

$$\lim_{k\to\infty} G(z_{m(k)+1}, z_{n(k)+1}, z_{n(k)+1}) = \varepsilon. \tag{32}$$

*Finally,*

$$\begin{aligned} G(z_{m(k)+1}, z_{n(k)+1}, z_{n(k)+2}) &\leq G(z_{m(k)+1}, z_{n(k)+1}, z_{n(k)+1}) \\ &\quad + G(z_{n(k)+1}, z_{n(k)+1}, z_{n(k)+2}) \\ &\leq G(z_{m(k)+1}, z_{n(k)+1}, z_{n(k)+1}) \\ &\quad + G(z_{n(k)}, z_{n(k)+1}, z_{n(k)+2}), \end{aligned} \tag{33}$$

*and*

$$G(z_{m(k)+1}, z_{n(k)+1}, z_{n(k)+1}) \leq G(z_{m(k)+1}, z_{n(k)+1}, z_{n(k)+2}). \tag{34}$$

*Hence in (33) and (34), if $k \to \infty$ and by using (12) and (32), we have*

$$\lim_{k\to\infty} G(z_{m(k)+1}, z_{n(k)+1}, z_{n(k)+2}) = \varepsilon. \tag{35}$$

*Since $x_{m(k)} \preceq x_{n(k)} \preceq x_{n(k)+1}$, putting $x = x_{m(k)}$, $y = x_{n(k)}$, and $z = x_{n(k)+1}$ in (1) for all $k \geq 0$, we have*

$$\begin{aligned} \psi\big(G(z_{m(k)+1}, z_{n(k)+1}, z_{n(k)+2})\big) &= \psi\big(G(fx_{m(k)}, gx_{n(k)}, hx_{n(k)+1})\big) \\ &\leq \psi\big(M(x_{m(k)}, x_{n(k)}, x_{n(k)+1})\big) \\ &\quad - \varphi\big(M(x_{m(k)}, x_{n(k)}, x_{n(k)+1})\big), \end{aligned} \tag{36}$$

*where*

$$\begin{aligned} M(x_{m(k)}, x_{n(k)}, x_{n(k)+1}) &= \max\{G(Tx_{m(k)}, Rx_{n(k)}, Sx_{n(k)+1}), G(Tx_{m(k)}, Rx_{n(k)}, gx_{n(k)}), \\ &\quad G(Rx_{n(k)}, Sx_{n(k)+1}, hx_{n(k)+1}), G(Sx_{n(k)+1}, Tx_{m(k)}, fx_{m(k)}), \\ &\quad \frac{G(Tx_{m(k)}, fx_{m(k)}, gx_{n(k)}) + G(Rx_{n(k)}, gx_{n(k)}, hx_{n(k)+1}) + G(Sx_{n(k)+1}, hx_{n(k)+1}, fx_{m(k)})}{3}\} \\ &= \max\{G(z_{m(k)}, z_{n(k)}, z_{n(k)+1}), G(z_{m(k)}, z_{n(k)}, z_{n(k)+1}), \\ &\quad G(z_{n(k)}, z_{n(k)+1}, z_{n(k)+2}), G(z_{n(k)+1}, z_{m(k)}, z_{m(k)+1}), \\ &\quad \frac{G(z_{m(k)}, z_{m(k)+1}, z_{n(k)+1}) + G(z_{n(k)}, z_{n(k)+1}, z_{n(k)+2}) + G(z_{n(k)+1}, z_{n(k)+2}, z_{m(k)+1})}{3}\}. \end{aligned}$$

*Now, from (13), (19), (26), and (35), if $k \to \infty$ in (36), we have*

$$\psi(\varepsilon) \leq \psi(\varepsilon) - \varphi(\varepsilon). \tag{37}$$

*Hence, $\varepsilon = 0$, which is a contradiction. Consequently, $\{z_n\}$ is a G-Cauchy sequence.*

*Step III. We will show that f, g, h, R, S, and T have a common fixed point.*

*Since $\{z_n\}$ is a G-Cauchy sequence in the complete G-metric space X, there exists $z \in X$ such that*

$$\begin{aligned} \lim_{n\to\infty} G(z_{3n+1}, z_{3n+1}, z) &= \lim_{n\to\infty} G(Rx_{3n+1}, Rx_{3n+1}, z) \\ &= \lim_{n\to\infty} G(fx_{3n}, fx_{3n}, z) = 0, \end{aligned} \tag{38}$$

$$\begin{aligned} \lim_{n\to\infty} G(z_{3n+2}, z_{3n+2}, z) &= \lim_{n\to\infty} G(Sx_{3n+2}, Sx_{3n+2}, z) \\ &= \lim_{n\to\infty} G(gx_{3n+1}, gx_{3n+1}, z) = 0, \end{aligned} \tag{39}$$

*and*

$$\begin{aligned} \lim_{n\to\infty} G(z_{3n+3}, z_{3n+3}, z) &= \lim_{n\to\infty} G(Tx_{3n+3}, Tx_{3n+3}, z) \\ &= \lim_{n\to\infty} G(hx_{3n+2}, hx_{3n+2}, z) = 0. \end{aligned} \tag{40}$$

*Let (i) holds. Assume that R and T are continuous and let the pairs $(f, T)$ and $(g, R)$ are compatible. This implies that*

$$\lim_{n\to\infty} G(Tfx_{3n}, fTx_{3n}, fTx_{3n}) = \lim_{n\to\infty} G(Tz, fTx_{3n}, fTx_{3n}) = 0, \tag{41}$$

*and*

$$\begin{aligned} &\lim_{n\to\infty} G(Rgx_{3n+1}, gRx_{3n+1}, gRx_{3n+1}) \\ &\quad = \lim_{n\to\infty} G(Rz, gRx_{3n+1}, gRx_{3n+1}) = 0. \end{aligned} \tag{42}$$

*Since*

$$\begin{aligned} Rx_{3n+1} &\preceq fRx_{3n+1} \preceq x_{3n+1} \preceq gx_{3n+1} \\ &= Sx_{3n+2} \preceq gSx_{3n+2} \preceq x_{3n+2} \\ &\preceq hx_{3n+2} = Tx_{3n+3}, \end{aligned}$$

*by using (1) we obtain that*

$$\psi\left(G(fTx_{3n+3}, gRx_{3n+1}, hx_{3n+2})\right) \leq \psi\left(M(Tx_{3n+3}, Rx_{3n+1}, x_{3n+2})\right) - \varphi\left(M(Tx_{3n+3}, Rx_{3n+1}, x_{3n+2})\right), \tag{43}$$

*where*

$$\begin{aligned} &M(Tx_{3n+3}, Rx_{3n+1}, x_{3n+2}) \\ &= \max\{G(TTx_{3n+3}, RRx_{3n+1}, Sx_{3n+2}), \\ &\quad G(TTx_{3n+3}, RRx_{3n+1}, gRx_{3n+1}), \\ &\quad G(RRx_{3n+1}, Sx_{3n+2}, hx_{3n+2}), \\ &\quad G(Sx_{3n+2}, TTx_{3n+3}, fTx_{3n+3}), \\ &\quad \frac{G(TTx_{3n+3}, fTx_{3n+3}, gRx_{3n+1})}{3} \\ &\quad + \frac{G(RRx_{3n+1}, gRx_{3n+1}, hx_{3n+2})}{3} \\ &\quad + \frac{G(Sx_{3n+2}, hx_{3n+2}, fTx_{3n+3})}{3}\} \\ &\to \max\{G(Tz, Rz, z), G(Tz, Rz, Rz), G(Rz, z, z), G(z, Tz, Tz), \\ &\quad \frac{G(Tz, Tz, Rz) + G(Rz, Rz, z) + G(z, z, Tz)}{3}\}, \end{aligned}$$

*as $n \to \infty$.*

*On taking the limit as $n \to \infty$ in (43), we obtain that*

$$\psi\left(G(Tz, Rz, z)\right) \leq \psi\left(G(Tz, Rz, z)\right) - \varphi\left(G(Tz, Rz, z)\right), \tag{44}$$

*and hence, $Tz = Rz = z$.*

*Since $x_{3n+1} \preceq x_{3n+2} \preceq hx_{3n+2}$ and $hx_{3n+2} \to z$, as $n \to \infty$, we have $x_{3n+1} \preceq x_{3n+2} \preceq z$. Therefore, from (1),*

$$\psi\left(G(fz, gx_{3n+1}, hx_{3n+2})\right) \leq \psi\left(M(z, x_{3n+1}, x_{3n+2})\right) - \varphi\left(M(z, x_{3n+1}, x_{3n+2})\right), \tag{45}$$

*where*

$$\begin{aligned} &M(z, x_{3n+1}, x_{3n+2}) \\ &= \max\{G(Tz, Rx_{3n+1}, Sx_{3n+2}), G(Tz, Rx_{3n+1}, gx_{3n+1}), \\ &\quad G(Rx_{3n+1}, Sx_{3n+2}, hx_{3n+2}), G(Sx_{3n+2}, Tz, fz), \\ &\quad \frac{G(Tz, fz, gx_{3n+1}) + G(Rx_{3n+1}, gx_{3n+1}, hx_{3n+2})}{3}\} \\ &\quad + \frac{G(Sx_{3n+2}, hx_{3n+2}, fz)}{3} \\ &\to \max\{G(Tz, z, z), G(Tz, z, z), G(z, z, z), G(z, Tz, fz), \\ &\quad \frac{G(Tz, fz, z) + G(z, z, z) + G(z, z, fz)}{3}\} = G(z, z, fz), \end{aligned}$$

*as $n \to \infty$.*

*If in (45) $n \to \infty$, we obtain that*

$$\psi\left(G(fz, z, z)\right) \leq \psi\left(G(fz, z, z)\right) - \varphi\left(G(fz, z, z)\right), \tag{46}$$

*hence $fz = z$.*

*Since $x_{3n+2} \preceq hx_{3n+2}$ and $hx_{3n+2} \to z$, as $n \to \infty$, we have $x_{3n+2} \preceq z$. Hence from(1),*

$$\psi\left(G(fz, gz, hx_{3n+2})\right) \leq \psi\left(M(z, z, x_{3n+2})\right) - \varphi\left(M(z, z, x_{3n+2})\right), \tag{47}$$

*where*

$$\begin{aligned} &M(z, z, x_{3n+2}) \\ &= \max\{G(Tz, Rz, Sx_{3n+2}), G(Tz, Rz, gz), \\ &\quad G(Rz, Sx_{3n+2}, hx_{3n+2}), G(Sx_{3n+2}, Tz, fz), \\ &\quad \frac{G(Tz, fz, gz) + G(Rz, gz, hx_{3n+2}) + G(Sx_{3n+2}, hx_{3n+2}, fz)}{3}\} \\ &\to \max\{G(Tz, Rz, z), G(Tz, Rz, gz), G(Rz, z, z), G(z, Tz, fz), \\ &\quad \frac{G(Tz, fz, gz) + G(Rz, gz, z) + G(z, z, fz)}{3}\} = G(z, z, gz) \end{aligned}$$

*as $n \to \infty$.*

*Making $n \to \infty$ in (47), we obtain that*

$$\psi\left(G(z, gz, z)\right) \leq \psi\left(G(z, z, gz)\right) - \varphi\left(G(z, z, gz)\right), \tag{48}$$

*which implies that $gz = z$.*

*Since $g(X) \subseteq S(X)$, there exists a point $w \in X$ such that $z = gz = Sw$. Suppose that $hw \neq Sw$. Since $z \preceq gz = Sw \preceq gSw \preceq w$, we have $z \preceq w$. Hence, from (1), we obtain that*

$$\psi\left(G(fz, gz, hw)\right) \leq \psi\left(M(z, z, w)\right) - \varphi\left(M(z, z, w)\right), \tag{49}$$

*where*

$$\begin{aligned} M(z, z, w) &= \max\{G(Tz, Rz, Sw), G(Tz, Rz, gz), \\ &\quad G(Rz, Sw, hw), G(Sw, Tz, fz), \\ &\quad \frac{G(Tz, fz, gz) + G(Rz, gz, hw) + G(Sw, hw, fz)}{3}\} \\ &\to \max\{G(z, z, z), G(z, z, z), G(z, z, hw), G(z, z, z), \\ &\quad \frac{G(z, z, z) + G(z, z, hw) + G(z, hw, z)}{3}\} \\ &= G(z, z, hw), \end{aligned}$$

*as $n \to \infty$.*

*On taking the limit as $n \to \infty$ in (49), we obtain that*

$$\psi\left(G(z, z, hw)\right) \leq \psi\left(G(z, z, hw)\right) - \varphi\left(G(z, z, hw)\right), \tag{50}$$

*which yields that $hw = z$.*

*Now, Since $h$ and $S$ are weakly compatible, we have $hz = hSw = Shw = Sz$. Thus, $z$ is a coincidence point of $h$ and $S$.*

*Now, we are ready to show that $hz = z$.*

*Since $x_{3n} \preceq fx_{3n}$ and $fx_{3n} \to z$, as $n \to \infty$, we have $x_{3n} \preceq z$. Hence, from (1),*

$$\psi\left(G(fx_{3n}, gz, hz)\right) \leq \psi\left(M(x_{3n}, z, z)\right) - \varphi\left(M(x_{3n}, z, z)\right), \tag{51}$$

*where*

$$\begin{aligned}M(x_{3n},z,z) &= \max\{G(Tx_{3n},Rz,Sz),G(Tx_{3n},Rz,gz),\\ &\quad G(Rz,Sz,hz),G(Sz,Tx_{3n},fx_{3n}),\\ &\quad \frac{G(Tx_{3n},fx_{3n},gz)+G(Rz,gz,hz)+G(Sz,hz,fx_{3n})}{3}\}\\ &\to \max\{G(z,z,z),G(z,z,z),G(z,z,hz),G(z,z,z),\\ &\quad \frac{G(z,z,z)+G(z,z,hz)+G(z,hz,z)}{3}\}\\ &= G(z,z,hz),\end{aligned}$$

*as* $n\to\infty$.

*Letting* $n\to\infty$ *in (51), we obtain that*

$$\psi\left(G(z,z,hz)\right)\le\psi\left(G(z,z,hz)\right)-\varphi\left(G(z,z,hz)\right),\quad (52)$$

*hence* $hz=z$. *Therefore,* $fz=gz=hz=Rz=Sz=Tz=z$.

*Similarly, the result follows when (ii) or (iii) hold.*

*Suppose that the set of common fixed points of f, g, h, R, S, and T is well ordered. We claim that common fixed point of f, g, h, R, S, and T is unique. Assume on contrary that* $fu=gu=hu=Ru=Su=Tu=u$, $fv=gv=hv=Rv=Sv=Tv=v$, *and* $u\neq v$. *By using (1), we obtain*

$$\psi\left(G(fu,gv,hv)\right)\le\psi\left(M(u,v,v)\right)-\varphi\left(M(u,v,v)\right),\quad (53)$$

*where*

$$\begin{aligned}M(u,v,v) &= \max\{G(Tu,Rv,Sv),G(Tu,Rv,gv),\\ &\quad G(Rv,Sv,hv),G(Sv,Tu,fu),\\ &\quad \frac{G(Tu,fu,gv)+G(Rv,gv,hv)+G(Sv,hv,fu)}{3}\}\\ &= \max\{G(u,v,v),G(v,u,u)\}.\end{aligned}$$

*On the other hand, as v and u are comparable,*

$$\psi\left(G(fv,gu,hu)\right)\le\psi\left(M(v,u,u)\right)-\varphi\left(M(v,u,u)\right),\quad (54)$$

*where*

$$\begin{aligned}M(v,u,u) &= \max\{G(Tv,Ru,Su),G(Tv,Ru,gu),\\ &\quad G(Ru,Su,hu),G(Su,Tv,fv),\\ &\quad \frac{G(Tv,fv,gu)+G(Ru,gu,hu)+G(Su,hu,fv)}{3}\}\\ &= \max\{G(v,u,u),G(u,v,v)\}.\end{aligned}$$

*From (53) and (54),*

$$\begin{aligned}&\psi\left(\max\{G(u,v,v),G(v,u,u)\}\right)\\ &\quad=\max\{\psi\left(G(u,v,v)\right),\psi\left(G(v,u,u)\right)\}\\ &\quad\le\psi\left(\max\{G(u,v,v),G(v,u,u)\}\right)\\ &\qquad-\varphi\left(\max\{G(u,v,v),G(v,u,u)\}\right).\end{aligned}\quad (55)$$

*Therefore,* $\varphi\left(\max\{G(u,v,v),G(v,u,u)\}\right)=0$ *which yields that* $u=v$ *is a contradiction. Conversely, if f, g, h, R, S, and T have only one common fixed point then, clearly, the set of common fixed points of f, g, h, R, S, and T is well ordered.*

We assume that

$$\begin{aligned}M_1(x,y,z) &= \max\{G(Tx,Ry,Sz),G(Tx,Ry,fy),\\ &\quad G(Ry,Sz,fz),G(Sz,Tx,fx),\\ &\quad \frac{G(Tx,fx,fy)+G(Ry,fy,fz)+G(Sz,fz,fx)}{3}\}.\end{aligned}$$

Taking $f=g=h$ in Theorem 2.2, we obtain the following common fixed point result in corollary.

**Corollary 2.4.** *Let* $(X,\preceq,G)$ *be a partially ordered complete G-metric space. Let* $f,R,S,T:X\to X$ *be four mappings such that* $f(X)\subseteq R(X)\cup S(X)\cup T(X)$ *and dominating map f is a weak annihilator of R, S, and T. Suppose that for every three comparable elements* $x,y,z\in X$,

$$\psi\left(G(fx,fy,fz)\right)\le\psi\left(M_1(x,y,z)\right)-\varphi\left(M_1(x,y,z)\right),\quad (56)$$

*where* $\psi,\varphi:[0,\infty)\to[0,\infty)$ *are altering distance functions. Then, f, R, S, and T have a common fixed point in X provided that for a nondecreasing sequence* $\{x_n\}$ *with* $x_n\preceq y_n$ *for all n,* $y_n\to u$ *implies that* $x_n\preceq u$ *and either of the following:*

*(i)* *One of f or R and one of f or T are continuous, the pairs* $(f,T)$, *and* $(f,R)$ *are compatible, and the pair* $(f,S)$ *is weakly compatible or*
*(ii)* *One of f or S and one of f or T are continuous, the pairs* $(f,T)$, *and* $(f,S)$ *are compatible, and the pair* $(f,R)$ *is weakly compatible or*
*(iii)* *One of f or R and one of f or S are continuous, the pairs* $(f,R)$, *and* $(f,S)$ *are compatible, and the pair* $(f,T)$ *is weakly compatible.*

*Moreover, the set of common fixed points of f, R, S, and T is well ordered if and only if f, R, S, and T have one and only one common fixed point.*

Let

$$\begin{aligned}M_2(x,y,z) &= \max\{G(Tx,Ty,Tz),G(Tx,Ty,gy),\\ &\quad G(Ty,Tz,hz),G(Tz,Tx,fx),\\ &\quad \frac{G(Tx,fx,gy)+G(Ty,gy,hz)+G(Tz,hz,fx)}{3}\}.\end{aligned}$$

Taking $T=R=S$ in Theorem 2.2, we obtain the following common fixed point result.

**Corollary 2.5.** *Let* $(X,\preceq,G)$ *be a partially ordered complete G-metric space. Let* $f,g,h,T:X\to X$ *be four* *mappings such that $f(X) \cup g(X) \cup h(X) \subseteq T(X)$ and dominating maps f, g, and h are weak annihilators of T. Suppose that for every three comparable elements $x, y, z \in X$,*

$$\psi\left(G(fx, gy, hz)\right) \leq \psi\left(M_2(x, y, z)\right) - \varphi\left(M_2(x, y, z)\right), \tag{57}$$

*where $\psi, \varphi : [0, \infty) \to [0, \infty)$ are altering distance functions. Then, f, g, h, and T have a common fixed point in X provided that for a nondecreasing sequence $\{x_n\}$, with $x_n \preceq y_n$ for all n, $y_n \to u$ implies that $x_n \preceq u$ and either of the following:*

*(i) One of f or T and one of g or T are continuous, the pairs $(f, T)$ and $(g, T)$ are compatible, and the pair $(h, T)$ is weakly compatible or*
*(ii) One of f or T and one of h or T are continuous, the pairs $(f, T)$ and $(h, T)$ are compatible, and the pair $(g, T)$ is weakly compatible or*
*(iii) One of g or T and one of h or T are continuous, the pairs $(g, T)$ and $(h, T)$ are compatible, and the pair $(f, T)$ is weakly compatible.*

*Moreover, the set of common fixed points of f, g, h, and T is well ordered if and only if f, g, h, and T have one and only one common fixed point.*

Let

$$M_3(x, y, z) = \max\{G(Sx, Ry, Sz), G(Sx, Ry, gy), G(Ry, Sz, gz), G(Sz, Sx, fx), \frac{G(Sx, fx, gy) + G(Ry, gy, gz) + G(Sz, gz, fx)}{3}\}.$$

Taking $S = T$ and $g = h$ in Theorem 2.2, we obtain the following common fixed point result.

**Corollary 2.6.** *Let $(X, \preceq, G)$ be a partially ordered complete G-metric space. Let $f, g, R, S : X \to X$ be four mappings such that $f(X) \subseteq R(X)$ and $g(X) \subseteq S(X)$ and dominating maps f and g are weak annihilators of R and S, respectively. Suppose that for every three comparable elements $x, y, z \in X$,*

$$\psi\left(G(fx, gy, gz)\right) \leq \psi\left(M_3(x, y, z)\right) - \varphi\left(M_3(x, y, z)\right), \tag{58}$$

*where $\psi, \varphi : [0, \infty) \to [0, \infty)$ are altering distance functions. Then, f, g, R, and S have a common fixed point in X provided that for a nondecreasing sequence $\{x_n\}$ with $x_n \preceq y_n$ for all n, $y_n \to u$ implies that $x_n \preceq u$ and either of the following:*

*(i) One of g or R and one of f or S are continuous, the pairs $(f, S)$ and $(g, R)$ are compatible, and the pair $(g, S)$ is weakly compatible or*
*(ii) One of g or S and one of f or S are continuous, the pairs $(f, S)$ and $(g, S)$ are compatible, and the pair $(g, R)$ is weakly compatible or*
*(iii) One of g or R and one of g or S are continuous, the pairs $(g, R)$ and $(g, S)$ are compatible, and the pair $(f, S)$ is weakly compatible.*

*Moreover, the set of common fixed points of f, g, R and S is well ordered if and only if f, g, R and S have one and only one common fixed point.*

Let

$$M_4(x, y, z) = \max\{G(Tx, Ty, Tz), G(Tx, Ty, fy), G(Ty, Tz, fz), G(Tz, Tx, fx), \frac{G(Tx, fx, fy) + G(Ty, fy, fz) + G(Tz, fz, fx)}{3}\}.$$

Taking $R = S = T$ and $f = g = h$ in Theorem 2.2, we obtain the following common fixed point result:

**Corollary 2.7.** *Let $(X, \preceq, G)$ be a partially ordered complete G-metric space. Let $f, T : X \to X$ be two mappings such that $f(X) \subseteq T(X)$, dominating map f is a weak annihilator of T. Suppose that for every three comparable elements $x, y, z \in X$,*

$$\psi\left(G(fx, fy, fz)\right) \leq \psi\left(M_4(x, y, z)\right) - \varphi\left(M_4(x, y, z)\right), \tag{59}$$

*where $\psi, \varphi : [0, \infty) \to [0, \infty)$ are altering distance functions. Then, f and T have a common fixed point in X provided that for a nondecreasing sequence $\{x_n\}$, $x_n \preceq y_n$ for all n, and $y_n \preceq u$ implies that $x_n \preceq u$ and one f or T is continuous and the pair $(f, T)$ is compatible.*

*Moreover, the set of common fixed points of f and T is well ordered if and only if f and T have one and only one common fixed point.*

**Example 2.8.** *(see also [42]) Let $X = [0, \infty)$ and G on X be given by $G(x, y, z) = |x - y| + |y - z| + |x - z|$, for all $x, y, z \in X$. We define an ordering '$\preceq$' on X as follows:*

$$x \preceq y \iff y \leq x, \ \forall x, y \in X. \tag{60}$$

*Define self-maps f, g, h, S, T and R on X by*

$$\begin{aligned} fx &= \ln(1 + x), Rx = e^{3x} - 1, \\ gx &= \ln(1 + \frac{x}{2}), Sx = e^{2x} - 1, \\ hx &= \ln(1 + \frac{x}{3}), Tx = e^{6x} - 1. \end{aligned} \tag{61}$$

*For each $x \in X$, we have $1 + x \leq e^x$, $1 + \frac{x}{2} \leq e^x$ and $1 + \frac{x}{3} \leq e^x$. Hence, $fx = \ln(1 + x) \leq x$, $gx = \ln(1 + \frac{x}{2}) \leq x$, and $hx = \ln(1 + \frac{x}{3}) \leq x$, which yields that $x \preceq fx$, $x \preceq gx$, and $x \preceq hx$, so f, g, and h are dominating.*

*Also, for each $x \in X$, we have $fRx = \ln(1+Rx) = 3x \geq x$,*

$$gSx = \ln(1+\frac{Sx}{2}) = \ln(1+\frac{e^{2x}-1}{2}) = \ln(\frac{1+e^{2x}}{2}) = \ln(e^x\frac{e^{-x}+e^x}{2}) = x + \ln(\frac{e^{-x}+e^x}{2}) \geq x,$$

*and since $t^6 - 3t + 2 \geq 0$ for each $t \geq 1$, we have*

$$hTx = \ln(1+\frac{Tx}{3}) = \ln(1+\frac{e^{6x}-1}{3}) = \ln(\frac{2+e^{6x}}{3}) \geq x.$$

*Hence, $fRx \preceq x$, $gSx \preceq x$ and $hTx \preceq x$. Thus f, g, and h are weak annihilators of S, T, and R, respectively.*

*Furthermore, $fX = TX = gX = SX = hX = RX = [0,\infty)$ and the pairs $(f,T)$, $(g,R)$, and $(h,S)$ are compatible. For example, we will show that the pair $(f,T)$ is compatible. Let $\{x_n\}$ is a sequence in X such that for some $t \in X$, $\lim_{n\to\infty} G(t,fx_n,fx_n) = 0$ and $\lim_{n\to\infty} G(t,Tx_n,Tx_n) = 0$. Therefore, we have*

$$\lim_{n\to\infty} |fx_n - t| = \lim_{n\to\infty} |Tx_n - t| = 0.$$

*Since f and T are continuous, we have*

$$\begin{aligned}\lim_{n\to\infty} G(fTx_n,fTx_n,Tfx_n) &= 2\lim_{n\to\infty} |fTx_n - Tfx_n| \\ &= 2|ft - Tt| \\ &= 2\left|\ln(1+t) - e^{6t} + 1\right|.\end{aligned}$$

*On the other hand, $\left|\ln(1+t) - e^{6t} + 1\right| = 0 \Longleftrightarrow t = 0$.*

*Define control functions $\psi, \varphi : [0,\infty) \to [0,\infty)$ with $\psi(t) = bt$ and $\varphi(t) = (b-1)t$ for all $t \in [0,\infty)$, where $1 < b \leq 6$.*

*Now, we will show that f, g, h, R, S and T satisfy (1). Using the mean value theorem, we have*

$$\begin{aligned}&\psi(G(fx,gy,hz)) \\ &= b(|fx-gy| + |fx-hz| + |gy-hz|) \\ &= b(\left|\ln(x+1) - \ln(\frac{y}{2}+1)\right| + \left|\ln(x+1) - \ln(\frac{z}{3}+1)\right| \\ &\quad + \left|\ln(\frac{y}{2}+1) - \ln(\frac{z}{3}+1)\right|) \\ &\leq b(\frac{1}{2}|2x-y| + \frac{1}{3}|3x-z| + \frac{1}{6}|3y-2z|) \\ &= b\frac{(|6x-3y| + |6x-2z| + |3y-2z|)}{6} \\ &\leq \frac{b}{6}(|e^{6x}-e^{3y}| + |e^{3y}-e^{2z}| + |e^{2z}-e^{6x}|) \\ &\leq |Tx-Ry| + |Ry-Sz| + |Sz-Tx| \\ &= G(Tx,Ry,Sz) \leq M(x,y,z) \\ &= \psi(M(x,y,z)) - \varphi(M(x,y,z)).\end{aligned}$$

*Thus, (1) is satisfied for all $x,y,z \in X$. Therefore, all the conditions of the Theorem 2.2 are satisfied. Moreover, 0 is the unique common fixed point of f, g, h, R, S, and T.*

*Denoted by $\Lambda$, the set of all functions $\mu : [0+\infty) \to [0,+\infty)$, verifying the following conditions:*

*(I) $\mu$ is a positive Lebesgue integrable mapping on each compact subset of $[0,+\infty)$.*

*(II) For all $\varepsilon > 0$, $\int_0^{\varepsilon} \mu(t)dt > 0$.*

Other consequences of the main theorem are the following results for mappings satisfying contractive conditions of integral type.

**Corollary 2.9.** *We replaced the contractive condition (1) of Theorem 2.2 by the following condition: There exists a $\mu \in \Lambda$ such that*

$$\int_0^{\psi(G(fx,gy,hz))} \mu(t)dt \leq \int_0^{\psi(M(x,y,z))} \mu(t)dt - \int_0^{\varphi(M(x,y,z))} \mu(t)dt. \qquad (62)$$

*Then, f, g, h, R, S, and T have a coincidence point, if the other conditions of Theorem 2.2 be satisfied.*

**Proof 2.10.** *Consider the function $\Gamma(x) = \int_0^x \mu(t)dt$. Then (62) becomes*

$$\Gamma\left(\psi(G(fx,gy,hz))\right) \leq \Gamma(\psi(M(x,y,z))) - \Gamma(\varphi(M(x,y,z))).$$

*Taking $\psi_1 = \Gamma o\psi$ and $\varphi_1 = \Gamma o\varphi$ and applying Theorem 2.2, we obtain the proof (it is easy to verify that $\psi_1$ and $\varphi_1$ are altering distance functions).*

Similar to [43], let $N$ be a fixed positive integer. Let $\{\mu_i\}_{1\leq i\leq N}$ be a family of $N$ functions which belong to $\Lambda$. For all $t \geq 0$, we define

$$\begin{aligned}I_1(t) &= \int_0^t \mu_1(s)ds, \\ I_2(t) &= \int_0^{I_1 t} \mu_2(s)ds = \int_0^{\int_0^t \mu_1(s)ds} \mu_2(s)ds, \\ I_3(t) &= \int_0^{I_2 t} \mu_3(s)ds = \int_0^{\int_0^{\int_0^t \mu_1(s)ds} \mu_2(s)ds} \mu_3(s)ds, \cdots, \\ I_N(t) &= \int_0^{I_{(N-1)}t} \mu_N(s)ds.\end{aligned}$$

We have the following result.

**Corollary 2.11.** *We replaced the inequality (1) of Theorem 2.2 by the following condition:*

$$I_N\left(\psi\left(G(fx,gy,hz)\right)\right) \leq I_N(\psi(M(x,y,z))) - I_N(\varphi(M(x,y,z))). \qquad (63)$$

*Then, f, g, h, R, S, and T have a coincidence point, if the other conditions of Theorem 2.2 be satisfied.*

**Proof 2.12.** *We consider that $\hat{\Psi} = I_N o\psi$ and $\hat{\Phi} = I_N o\varphi$.*

**Competing interests**
The authors declare that they have no competing interests.

**Authors' contributions**
VP, AR, and JRR have worked together on each section of the paper such as the literature review, results and examples. All authors read and approved the final manuscript.

**Acknowledgments**
The authors thank the referees for the extremely careful reading that contributed to the improvement of the manuscript.

**Author details**
[1]Department of Mathematics, Gilan-E-Gharb Branch, Islamic Azad University, Gilan-E-Gharb 67871-54699, Iran. [2]Department of Mathematics, Faculty of Science, Imam Khomeini International University, P.O. Box 34149-16818, Qazvin, Iran. [3]Department of Mathematics, Qaemshahr Branch, Islamic Azad University, Qaemshahr 35953-47631, Iran.

Received: 28 January 2013 Accepted: 3 March 2013
Published: 8 April 2013

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doi:10.1186/2251-7456-7-18
**Cite this article as:** Parvaneh *et al.*: **Common fixed points of six mappings in partially ordered *G*-metric spaces.** *Mathematical Sciences* 2013 **7**:18.