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Fixed point theorems for mappings with common limit range property satisfying generalized (ψ, φ)-weak contractive conditions

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Abstract

In this paper, we prove some common fixed point theorems for weakly compatible mappings in metric spaces satisfying generalized (ψ, φ)-contractive conditions under the common limit range property. We present a fixed point theorem for four finite families of self-mappings which can be utilized to derive common fixed point theorems involving any number of finite mappings. Our results improve and extend the corresponding results of Radenović et al. (Bull. Iranian Math. Soc. 38(3):625–645, 2012). We also furnish some illustrative examples to support our main results.

Keywords: Metric space, Weakly compatible mappings, Property (E.A), Generalized weak contraction, Common limit range property, Fixed point

Introduction and preliminaries

The famous Banach Contraction Principle which is also referred as the Banach fixed point theorem continues to be a very popular and powerful tool in solving existence problems in pure and applied sciences which include biology, medicine, physics, and computer science. It evidently plays a crucial role in nonlinear analysis. This theorem states that if (X, d) is a complete metric space and $T : X \to X$ is a contraction mapping, i.e.,

$$d(Tx, Ty) \leq kd(x, y),$$

for all $x, y \in X$, and k is a non-negative real number such that $k < 1$, then T has a unique fixed point in X. Moreover, this fixed point can be explicitly obtained as a limit of repeated iteration of the mapping, initiating at any point of the underlying space. Obviously, every contraction is a continuous function but not conversely. Many mathematicians (e.g., [1-6]) proved several fixed point theorems to explore some new contraction-type mappings in order to generalize the classical Banach Contraction Principle.

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The concept of weak contraction was introduced by Alber and Guerre-Delabriere [7] in 1997, wherein the authors introduced the following notion for mappings defined on a Hilbert space X.

Consider the following set of real functions $\Phi = \{\varphi : [0, +\infty) \to [0, +\infty) : \varphi$ is lower semi-continuous and $\varphi^{-1}(\{0\}) = \{0\}\}$.

A mapping $T : X \to X$ is called a φ-weak contraction if there exists a function $\varphi \in \Phi$ such that

$$d(Tx, Ty) \leq d(x, y) - \varphi(d(x, y)),$$

for all $x, y \in X$.

Alber and Guerre-Delabriere [7] also showed that each φ-weak contraction on a Hilbert space has a unique fixed point. Thereafter, Rhoades [8] showed that the results contained in [7] are also valid for any Banach space. In particular, he generalized the Banach Contraction Principle which follows in case one chooses $\varphi(t) = (1 - k)t$.

Zhang and Song [9] proved a common fixed point theorem for two mappings using φ-weak contraction. This result was extended by Dorić [10] and Dutta and Choudhury [11] to a pair of (ψ, φ)-weak contractive mappings. However, the main fixed point theorem for a self-mapping satisfying (ψ, φ)-weak contractive condition contained in Dutta and Choudhury [11] runs as follows:

Let us consider the following set of real functions: $\Psi = \{\psi : [0,+\infty) \to [0,+\infty) : \psi$ is continuous non-decreasing and $\psi^{-1}(\{0\}) = \{0\}\}$.

Theorem 1.1. *Let (X,d) be a complete metric space and let $T : X \to X$ be a self-mapping satisfying*

$$\psi(d(Tx,Ty)) \leq \psi(d(x,y)) - \varphi(d(x,y)),$$

for some $\psi \in \Psi$ and $\varphi \in \Phi$ and all $x,y \in X$. Then, T has a unique fixed point in X.

In recent years, many researchers utilized (ψ,φ)-weak contractive conditions to prove a number of metrical fixed point theorems (e.g., [12-19]). In an important paper, Jachymski [20] showed that some of the results involving two functions $\psi \in \Psi$ and $\varphi \in \Phi$ can be reduced to one function $\varphi^{'} \in \Phi$. Popescu [21] proved a fixed point theorem in a complete metric space and showed that the conditions on functions ψ and φ can be weakened. His result improved the corresponding results of Dutta and Choudhury [11] and Dorić [10].

A common fixed point result generally involves conditions on commutativity, continuity, and contraction along with a suitable condition on the containment of range of one mapping into the range of the other. Hence, one is always required to improve one or more of these conditions in order to prove a new common fixed point theorem.

It can be observed that in the case of two mappings $A,S : X \to X$, one can consider the following classes of mappings for the existence and uniqueness of common fixed points:

$$d(Ax,Ay) \leq F(m(x,y)), \tag{1.1}$$

where F is some function and $m(x,y)$ is the maximum of one of the sets:

$$\begin{aligned} M^5_{A,S}(x,y) &= \{d(Sx,Sy), d(Sx,Ax), d(Sy,Ay), \\ &\qquad d(Sx,Ay), d(Sy,Ax)\}, \\ M^4_{A,S}(x,y) &= \Big\{d(Sx,Sy), d(Sx,Ax), d(Sy,Ay), \\ &\qquad \frac{1}{2}(d(Sx,Ay)+d(Sy,Ax))\Big\}, \\ M^3_{A,S}(x,y) &= \Big\{d(Sx,Sy), \frac{1}{2}(d(Sx,Ax)+d(Sy,Ay)), \\ &\qquad \frac{1}{2}(d(Sx,Ay)+d(Sy,Ax))\Big\}. \end{aligned}$$

A further possible generalization is to consider four mappings instead of two and ascertain analogous common fixed point theorems. In the case of four mappings $A,B,S,T : X \to X$, the corresponding sets take the form

$$\begin{aligned} M^5_{A,B,S,T}(x,y) &= \{d(Sx,Ty), d(Sx,Ax), d(Ty,By), \\ &\qquad d(Sx,By), d(Ty,Ax)\}, \\ M^4_{A,B,S,T}(x,y) &= \Big\{d(Sx,Ty), d(Sx,Ax), d(Ty,By), \\ &\qquad \frac{1}{2}(d(Sx,By)+d(Ty,Ax))\Big\}, \\ M^3_{A,B,S,T}(x,y) &= \Big\{d(Sx,Ty),\ \frac{1}{2}(d(Sx,Ax)+d(Ty,By)), \\ &\qquad \frac{1}{2}(d(Sx,By)+d(Ty,Ax))\Big\}. \end{aligned}$$

In this case (1.1) is usually replaced by

$$d(Ax,By) \leq F(m(x,y)), \tag{1.2}$$

where $m(x,y)$ is the maximum of one of the M sets.

Similarly, we can define the M sets for six mappings $A,B,H,R,S,T : X \to X$ as

$$\begin{aligned} M^5_{A,B,H,R,S,T}(x,y) &= \{d(SRx,THy), d(SRx,Ax), d(THy,By), \\ &\qquad (SRx,By), d(THy,Ax)\}, \\ M^4_{A,B,H,R,S,T}(x,y) &= \Big\{d(SRx,THy), d(SRx,Ax), d(THy,By), \\ &\qquad \frac{1}{2}(d(SRx,By)+d(THy,Ax))\Big\}, \\ M^3_{A,B,H,R,S,T}(x,y) &= \Big\{d(SRx,THy), \frac{1}{2}(d(SRx,Ax)+d(THy,By)), \\ &\qquad \frac{1}{2}(d(SRx,By)+d(THy,Ax))\Big\}, \end{aligned} \tag{1.3}$$

and the contractive condition is again in the form (1.2). Using different arguments of control functions, Radenović et al. [22] proved some common fixed point results for two and three mappings using (ψ,φ)-weak contractive conditions and improved several known metrical fixed point theorems.

Motivated by these results, we prove some common fixed point theorems for two pairs of weakly compatible mappings with common limit range property satisfying generalized (ψ,φ)-weak contractive conditions. Many known fixed point results are improved, especially the ones proved in [22] and also contained in the references cited therein. We also obtain a fixed point theorem for four finite families of self-mappings. Some related results are also derived besides furnishing illustrative examples.

Definition 1.1. Let A and S be two mappings from a metric space (X,d) into itself. Then, the mappings are said to

1. be commuting if $ASx = SAx$ for all $x \in X$,
2. be compatible [23] if $\lim_{n\to\infty} d(ASx_n, SAx_n) = 0$ for each sequence $\{x_n\}$ in X such that $\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Sx_n$,
3. be non-compatible [5] if there exists a sequence $\{x_n\}$ in X such that $\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Sx_n$ but $\lim_{n\to\infty} d(ASx_n, SAx_n)$ is either nonzero or nonexistent,
4. be weakly compatible [24] if they commute at their coincidence points, that is, $ASx = SAx$ whenever $Ax = Sx$, for some $x \in X$,
5. satisfy the property (E.A) [25] if there exists a sequence $\{x_n\}$ in X such that $\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Sx_n = t$, for some $t \in X$.

For further details, comparisons, and illustrations on systematic spaces, we refer to Singh and Tomar [26] and Murthy [4].

Any pair of compatible as well as non-compatible self-mappings of a metric space (X, d) satisfies the property (E.A), but a pair of mappings satisfying the property (E.A) need not be non-compatible (see Example 1 of [27]).

In 2005, Liu et al. [28] defined the notion of common property (E.A) for hybrid pairs of mappings, which contain the property (E.A).

Definition 1.2. Liu, 2005 [28] Two pairs (A, S) and (B, T) of self-mappings of a metric space (X, d) are said to satisfy the common property (E.A) if two sequences $\{x_n\}$ and $\{y_n\}$ in X exist such that

$$\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Sx_n = \lim_{n\to\infty} By_n = \lim_{n\to\infty} Ty_n = t,$$

for some $t \in X$.

It can be observed that the fixed point results usually require closedness of the underlying subspaces for the existence of common fixed points under the property (E.A) and common property (E.A). In 2011, Sintunavarat and Kumam [29] coined the idea of 'common limit range property' (see also [30]). Most recently, Imdad et al. [31] extended the notion of common limit range property to two pairs of self-mappings which relax the closedness requirements of the underlying subspaces.

Definition 1.3. Sintunavarat, 2012 [29] A pair (A, S) of self-mappings of a metric space (X, d) is said to satisfy the common limit range property with respect to S, denoted by (CLR_S), if there exists a sequence $\{x_n\}$ in X such that

$$\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Sx_n = t,$$

where $t \in S(X)$.

Thus, one can infer that a pair (A, S) satisfying the property (E.A) along with the closedness of the subspace $S(X)$ always enjoys the (CLR_S) property with respect to the mapping S (see Examples 2.16–2.17 of [31]).

Definition 1.4. Imdad, 2012 [31] Two pairs (A, S) and (B, T) of self-mappings of a metric space (X, d) are said to satisfy the common limit range property with respect to mappings S and T, denoted by (CLR_{ST}) if two sequences $\{x_n\}$ and $\{y_n\}$ in X exist such that

$$\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Sx_n = \lim_{n\to\infty} By_n = \lim_{n\to\infty} Ty_n = t,$$

where $t \in S(X) \cap T(X)$.

Definition 1.5. Imdad, 2009 [32] Two families of self-mappings $\{A_i\}_{i=1}^m$ and $\{S_k\}_{k=1}^n$ are said to be pairwise commuting if

1. $A_iA_j = A_jA_i$ for all $i, j \in \{1, 2, \ldots, m\}$,
2. $S_kS_l = S_lS_k$ for all $k, l \in \{1, 2, \ldots, n\}$,
3. $A_iS_k = S_kA_i$ for all $i \in \{1, 2, \ldots, m\}$ and $k \in \{1, 2, \ldots, n\}$.

Main results

Now, we state and prove our main results for four mappings employing the common limit range property in metric spaces. Firstly, we prove the following lemma.

Lemma 2.1. *Let A, B, S and T be self-mappings of a metric space (X, d). Suppose that*

1. *the pair (A, S) satisfies the (CLR_S) property $\big($resp. (B, T) satisfies the (CLR_T) property$\big)$,*
2. *$A(X) \subset T(X)$ $\big($resp. $B(X) \subset S(X)\big)$,*
3. *$T(X)$ $\big($resp. $S(X)\big)$ is a closed subset of X,*
4. *$\{By_n\}$ converges for every sequence $\{y_n\}$ in X whenever $\{Ty_n\}$ converges (resp. $\{Ax_n\}$ converges for every sequence $\{x_n\}$ in X whenever $\{Sx_n\}$ converges),*
5. *there exist $\varphi \in \Phi$ and $\psi \in \Psi$ such that*

$$\psi(d(Ax, By)) \leq \psi(m(x, y)) - \varphi(m(x, y)), \qquad (2.1)$$

for all $x, y \in X$, where

$$m(x, y) = \max M^5_{A,B,S,T}(x, y).$$

Then, the pairs (A, S) and (B, T) share the (CLR_{ST}) property.

Proof. Since the pair (A, S) satisfies the (CLR_S) property, a sequence $\{x_n\}$ in X exists such that

$$\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Sx_n = t,$$

where $t \in S(X)$. By Lemma 2.1 item (2), $A(X) \subset T(X)$, and for each sequence $\{x_n\}$, there exists a sequence $\{y_n\}$ in

X such that $Ax_n = Ty_n$. Therefore, due to the closedness of $T(X)$,

$$\lim_{n\to\infty} Ty_n = \lim_{n\to\infty} Ax_n = t,$$

so that $t \in T(X)$ and in all $t \in S(X) \cap T(X)$. Thus, we have $Ax_n \to t$, $Sx_n \to t$ and $Ty_n \to t$ as $n \to \infty$. By Lemma 2.1 item (4), the sequence $\{By_n\}$ converges, and in all, we need to show that $By_n \to t$ as $n \to \infty$. Let, on the contrary that $By_n \to z(\neq t)$ as $n \to \infty$. On using inequality (2.1) with $x = x_n, y = y_n$, we have

$$\psi(d(Ax_n, By_n)) \leq \psi(m(x_n, y_n)) - \varphi(m(x_n, y_n)), \quad (2.2)$$

where

$$m(x_n, y_n) = \max\{d(Sx_n, Ty_n), d(Sx_n, Ax_n), d(Ty_n, By_n), d(Sx_n, By_n), d(Ty_n, Ax_n)\}.$$

Taking the limit as $n \to \infty$ in (2.2), we get

$$\begin{aligned}\lim_{n\to\infty} \psi(d(Ax_n, By_n)) &\leq \lim_{n\to\infty} \psi(m(x_n, y_n)) - \lim_{n\to\infty} \varphi(m(x_n, y_n)),\\ \lim_{n\to\infty} \psi(d(t,z)) &\leq \psi\left(\lim_{n\to\infty} m(x_n, y_n)\right), - \varphi\left(\lim_{n\to\infty} m(x_n, y_n)\right),\end{aligned} \quad (2.3)$$

where

$$\begin{aligned}\lim_{n\to\infty} m(x_n, y_n) &= \max\{d(t,t), d(t,t), d(t,z), d(t,z), d(t,t)\}\\ &= \max\{0, 0, d(t,z), d(t,z), 0\} = d(t,z).\end{aligned}$$

From (2.3), we obtain

$$\psi(d(t,z)) \leq \psi(d(t,z)) - \varphi(d(t,z)),$$

so that $d(t,z) = 0$, i.e., $t = z$ which is a contradiction. Hence, $By_n \to t$ which shows that the pairs (A, S) and (B, T) share the (CLR_{ST}) property. This concludes the proof. □

Remark 2.1. In general, the converse of Lemma 2.1 is not true (see Example 3.5 of [31]).

Theorem 2.1. *Let A, B, S and T be self-mappings of a metric space (X, d) satisfying the inequality (2.1) of Lemma 2.1. If the pairs (A, S) and (B, T) satisfy the (CLR_{ST}) property, then (A, S) and (B, T) have a coincidence point each. Moreover, A, B, S and T have a unique common fixed point provided both the pairs (A, S) and (B, T) are weakly compatible.*

Proof. If the pairs (A, S) and (B, T) enjoy the (CLR_{ST}) property, then two sequences $\{x_n\}$ and $\{y_n\}$ in X exist such that

$$\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Sx_n = \lim_{n\to\infty} By_n = \lim_{n\to\infty} Ty_n = t,$$

where $t \in S(X) \cap T(X)$. Since $t \in S(X)$, a point $u \in X$ exists such that $Su = t$. We assert that $Au = Su$. Using inequality (2.1) with $x = u, y = y_n$, we get

$$\psi(d(Au, By_n)) \leq \psi(m(u, y_n)) - \varphi(m(u, y_n)), \quad (2.4)$$

where

$$m(u, y_n) = \max\{d(Su, Ty_n), d(Su, Au), d(Ty_n, By_n), d(Su, By_n), d(Ty_n, Au)\}.$$

Taking the limit as $n \to \infty$ in (2.4), we get

$$\begin{aligned}\lim_{n\to\infty} \psi(d(Au, By_n)) &\leq \lim_{n\to\infty} \psi(m(u, y_n)) - \lim_{n\to\infty} \varphi(m(u, y_n)),\\ \lim_{n\to\infty} \psi(d(Au, t)) &\leq \psi\left(\lim_{n\to\infty} m(u, y_n)\right) - \varphi\left(\lim_{n\to\infty} m(u, y_n)\right)\end{aligned} \quad (2.5)$$

where

$$\begin{aligned}\lim_{n\to\infty} m(u, y_n) &= \max\{d(t,t), d(t, Au), d(t,t), d(t,t), d(t, Au)\}\\ &= \max\{0, d(t, Au), 0, 0, d(t, Au)\}\\ &= d(Au, t),\end{aligned}$$

which in turn yields

$$\psi(d(Au, t)) \leq \psi(d(Au, t)) - \varphi(d(Au, t)),$$

so that $\varphi(d(Au, t)) = 0$, i.e., $d(Au, t) = 0$. Hence $Au = Su = t$. Therefore, u is a coincidence point of the pair (A, S).

As $t \in T(X)$, there exists a point $v \in X$ such that $Tv = t$. We show that $Bv = Tv$. Using inequality (2.1) with $x = u$, $y = v$, we get

$$\psi(d(t, Bv)) = \psi(d(Au, Bv)) \leq \psi(m(u, v)) - \varphi(m(u, v)), \quad (2.6)$$

where

$$\begin{aligned}m(u, v) &= \max\{d(Su, Tv), d(Su, Au), d(Tv, Bv), d(Su, Bv), d(Tv, Au)\}\\ &= \max\{d(t,t), d(t,t), d(t, Bv), d(t, Bv), d(t,t)\}\\ &= \max\{0, 0, d(t, Bv), d(t, Bv), 0\} = d(t, Bv),\end{aligned}$$

which in turn yields

$$\psi(d(t, Bv)) \leq \psi(d(t, Bv)) - \varphi(d(t, Bv)),$$

so that $\varphi(d(t, Bv)) = 0$, i.e., $d(t, Bv) = 0$. Hence, $Bv = Tv = t$, which shows that v is a coincidence point of the pair (B, T).

Since the pair (A, S) is weakly compatible, and $Au = Su$; therefore, $At = ASu = SAu = St$. Now, we assert that t is a common fixed point of the pair (A, S). Using inequality (2.1) with $x = t, y = v$, we have

$$\psi(d(At,t)) = \psi(d(At,Bv)) \le \psi(m(t,v)) - \varphi(m(t,v)), \tag{2.7}$$

where

$$\begin{aligned} m(t,v) &= \max\{d(St,Tv), d(St,At), d(Tv,Bv), d(St,Bv), \\ &\qquad d(Tv,At)\} \\ &= \max\{d(At,t), d(At,At), d(t,t), d(At,t), d(t,At)\} \\ &= \max\{d(At,t), 0, 0, d(At,t), d(t,At)\} = d(At,t), \end{aligned}$$

which in turn yields

$$\psi(d(At,t)) \le \psi(d(At,t)) - \varphi(d(At,t)),$$

so that $\varphi(d(At,t)) = 0$, i.e., $At = t = St$, which shows that t is a common fixed point of the pair (A, S).

Also, the pair (B, T) is weakly compatible, and $Bv = Tv$; therefore, $Bt = BTw = TBw = Tt$. Suppose that $Bt \ne t$. On using inequality (2.1) with $x = u, y = t$, we have

$$\psi(d(t,Bt)) = \psi(d(Au,Bt)) \le \psi(m(u,t)) - \varphi(m(u,t)), \tag{2.8}$$

where

$$\begin{aligned} m(u,t) &= \max\{d(Su,Tt), d(Su,Au), d(Tt,Bt), d(Su,Bt), \\ &\qquad d(Tt,Au)\} \\ &= \max\{d(t,Bt), d(t,t), d(Bt,Bt), d(t,Bt), d(Bt,t)\} \\ &= \max\{d(t,Bt), 0, 0, d(t,Bt), d(Bt,t)\} = d(t,Bt), \end{aligned}$$

which in turn yields

$$\psi(d(t,Bt)) \le \psi(d(t,Bt)) - \varphi(d(t,Bt)),$$

so that $\varphi(d(t,Bt)) = 0$, i.e., $d(t,Bt) = 0$. Therefore, $Bt = t = Tt$ which shows that t is a common fixed point of the pair (B, T) and in all t is a common fixed point of both the pairs (A, S) and (B, T).

To prove the uniqueness of common fixed point, let on contrary that there is another common fixed point $t' \in X$ such that $At' = Bt' = St' = Tt' = t'$. Using inequality (2.1) with $x = t, y = t'$, we have

$$\psi(d(t,t')) = \psi(d(At,Bt')) \le \psi(m(t,t')) - \varphi(m(t,t')), \tag{2.9}$$

where

$$\begin{aligned} m(t,t') &= \max\{d(St,Tt'), d(St,At), d(Tt',Bt'), d(St,Bt'), \\ &\qquad d(Tt',At)\} \\ &= \max\{d(t,t'), d(t,t), d(t',t'), d(t,t'), d(t',t)\} \\ &= \max\{d(t,t'), 0, 0, d(t,t'), d(t',t)\} = d(t,t'), \end{aligned}$$

which in turn yields

$$\psi(d(t,t')) \le \psi(d(t,t')) - \varphi(d(t,t')),$$

so that $\varphi(d(t,t')) = 0$, i.e., $t = t'$. Hence, t is a unique common fixed point of the mappings $A, B, S,$ and T. This concludes the proof. □

Remark 2.2. Theorem 2.1 improves the relevant results of Radenović et al. [22] as the requirements on the closedness and containment among the ranges of the involved mappings are not needed.

Now, we furnish an illustrative example which demonstrates the validity of the hypotheses and degree of generality of our main result over comparable ones from the existing literature.

Example 2.1. Consider $X = [2, 11)$ equipped with the usual metric. Define the self mappings A, B, S and T by

$$Ax = \begin{cases} 2, & \text{if } x \in \{2\} \cup (5,11), \\ 8, & \text{if } x \in (2,5]; \end{cases} \qquad Bx = \begin{cases} 2, & \text{if } x \in \{2\} \cup (5,11), \\ 4, & \text{if } x \in (2,5]; \end{cases}$$

$$Sx = \begin{cases} 2, & \text{if } x = 2, \\ 9, & \text{if } x \in (2,5], \\ \frac{x+1}{3}, & \text{if } x \in (5,11); \end{cases} \qquad Tx = \begin{cases} 2, & \text{if } x = 2, \\ 6, & \text{if } x \in (2,5], \\ x-3, & \text{if } x \in (5,11). \end{cases}$$

Consider two sequences $\{x_n\} = \left\{5 + \frac{1}{n}\right\}_{n\in\mathbb{N}}$, $\{y_n\} = \{2\}$ $\left(\text{or } \{x_n\} = \{2\}, \{y_n\} = \left\{5 + \frac{1}{n}\right\}_{n\in\mathbb{N}}\right)$. The pairs (A, S) and (B, T) satisfy the (CLR_{ST}) property:

$$\begin{aligned} \lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Sx_n &= \lim_{n\to\infty} By_n \\ &= \lim_{n\to\infty} Ty_n = 2 \in S(X) \cap T(X). \end{aligned}$$

Also, $A(X) = \{2, 8\} \nsubseteq [2, 8) = T(X)$ and $B(X) = \{2, 4\} \nsubseteq [2, 4) \cup \{9\} = S(X)$.

Take $\psi \in \Psi$ and $\varphi \in \Phi$ given by

$$\psi(t) = 2t, \qquad \varphi(t) = \frac{2}{7}t.$$

In order to check the contractive condition (2.1), consider the following nine cases: 1° $x = y = 2$; 2° $x = 2$, $y \in (2, 5]$; 3° $x = 2$, $y \in (5, 11)$; 4° $x \in (2, 5]$, $y = 2$; 5° $x, y \in (2, 5]$; 6° $x \in (2, 5]$, $y \in (5, 11)$; 7° $x \in (5, 11)$, $y = 2$; 8° $x \in (5, 11), y \in (2, 5]$; 9° $x, y \in (5, 11)$.

In the cases 1°, 3°, 7°, and 9°, we get that $d(Ax, By) = 0$; (2.1) is trivially satisfied. In the cases 2° and 8°, it is $d(Ax, By) = 2$ and $m(x, y) = 4$; so, (2.1) reduces to

$$\psi(2) = 4 \le \frac{48}{7} = \psi(4) - \varphi(4).$$

In the cases 4° and 6°, we get that $d(Ax, By) = 6$ and $m(x, y) = 7$; so, (2.1) reduces to

$$\psi(6) = 12 \le 12 = \psi(7) - \varphi(7).$$

Finally, in the case 5°, we obtain $d(Ax, By) = 4$ and $m(x, y) = 5$, and again we have

$$\psi(4) = 8 \le \frac{60}{7} = \psi(5) - \varphi(5).$$

Hence, all the conditions of Theorem 2.1 are satisfied, and 2 is a unique common fixed point of the pairs (A, S) and (B, T) which also remains a point of coincidence. Here, one may notice that all the involved mappings are discontinuous at their unique common fixed point 2.

However, notice that the subspaces $S(X)$ and $T(X)$ are not closed subspaces of X, and required inclusions among the ranges of the involved maps do not hold. Therefore, the results of Radenović et al. [22] cannot be used in the context of this example which establishes the genuineness of our extension.

In view of Theorem 2.1 and Lemma 2.1, the following corollary is immediate.

Corollary 2.1. *Let A, B, S and T be self-mappings of a metric space (X, d) satisfying all the hypotheses of Lemma 2.1. Then $A, B, S,$ and T have a unique common fixed point, provided that both the pairs (A, S) and (B, T) are weakly compatible.*

Proof. Owing to Lemma 2.1, it follows that the pairs (A, S) and (B, T) enjoy the (CLR_{ST}) property. Hence, all the conditions of Theorem 2.1 are satisfied; A, B, S and T have a unique common fixed point provided that both the pairs (A, S) and (B, T) are weakly compatible. □

Here, it is worth noting that the conclusions in Example 2.1 cannot be obtained using Corollary 2.1 as conditions (2) and (3) of Lemma 2.1 are not fulfilled. In what follows, we present another example which creates a situation wherein a conclusion can be reached using Corollary 2.1.

Example 2.2. In the setting of Example 2.1, replace the self-mappings S and T by the following, retaining the rest:

$$Sx = \begin{cases} 2, & \text{if } x = 2, \\ 5, & \text{if } x \in (2, 5], \\ \frac{x-1}{2}, & \text{if } x \in (5, 11); \end{cases} \qquad Tx = \begin{cases} 2, & \text{if } x = 2, \\ 8, & \text{if } x \in (2, 5], \\ x-3, & \text{if } x \in (5, 11). \end{cases}$$

Then, as in the earlier example, the pairs (A, S) and (B, T) satisfy the (CLR_{ST}) property. Moreover, inequality (2.1) can be be verified as earlier. Also, as earlier define,

$$\psi(t) = 2t, \qquad \varphi(t) = \frac{2}{7}t.$$

Here, $A(X) = \{2, 8\} \subset [2, 8] = T(X)$ and $B(X) = \{2, 4\} \subset [2, 5] = S(X)$ holds. Thus, all the conditions of Corollary 2.1 are satisfied and 2 is a unique common fixed point of the involved mappings $A, B, S,$ and T.

Remark 2.3. The conclusions of Lemma 2.1, Theorem 2.1 and Corollary 2.1 remain true if we choose $m(x, y) = \max M^4_{A,B,S,T}(x, y)$ or $m(x, y) = \max M^3_{A,B,S,T}(x, y)$.

By setting $A, B, S,$ and T suitably, we can deduce corollaries involving two as well as three self-mappings. As a sample, we can deduce the following corollary involving two self-mappings:

Corollary 2.2. *Let A and S be self-mappings of a metric space (X, d). Suppose that*

1. *the pair (A, S) satisfies the (CLR_S) property,*
2. *there exist $\varphi \in \Phi$ and $\psi \in \Psi$ such that*

$$\psi(d(Ax, Ay)) \leq \psi(m(x, y)) - \varphi(m(x, y)),$$

for all $x, y \in X$, where $m(x, y) = \max M^k_{A,S}(x, y)$ and $k \in \{3, 4, 5\}$.

Then, (A, S) has a coincidence point. Moreover, if the pair (A, S) is weakly compatible, then the pair has a unique common fixed point in X.

As an application of Theorem 2.1, we have the following result involving four finite families of self-mappings.

Theorem 2.2. *Let $\{A_i\}_{i=1}^m$, $\{B_j\}_{r=1}^n$, $\{S_k\}_{k=1}^p$, and $\{T_l\}_{l=1}^q$ be four finite families of self-mappings of a metric space (X, d) with $A = A_1A_2 \cdots A_m$, $B = B_1B_2 \cdots B_n$, $S = S_1S_2 \cdots S_p$, and $T = T_1T_2 \cdots T_q$ satisfying the condition (2.1). Suppose that the pairs (A, S) and (B, T) satisfy the (CLR_{ST}) property, then (A, S) and (B, T) have a point of coincidence each.*

Moreover $\{A_i\}_{i=1}^m$, $\{B_j\}_{j=1}^n$, $\{S_k\}_{k=1}^p$, and $\{T_l\}_{l=1}^q$ have a unique common fixed point if the families $(\{A_i\}, \{S_k\})$ and $(\{B_r\}, \{T_h\})$ commute pairwise where $i \in \{1, 2, \ldots, m\}$, $k \in \{1, 2, \ldots, p\}$, $j \in \{1, 2, \ldots, n\}$, and $l \in \{1, 2, \ldots, q\}$.

Proof. The proof of this theorem can be completed on the lines of Theorem 2.2 of Imdad et al. [31]. □

Remark 2.4. A result similar to Theorem 2.2 can be outlined using Corollary 2.1.
Remark 2.5. Theorem 2.2 extends the results of Radenović et al. [22] and Abbas and Dorić [12].

Now, we indicate that Theorem 2.2 can be utilized to derive common fixed point theorems for any finite number of mappings. As a sample, we can derive a common fixed point theorem for six mappings by setting two families of two members, while the rest by two of single members.

Corollary 2.3. *Let $A, B, H, R, S,$ and T be self-mappings of a metric space (X, d). Suppose that*

1. *the pairs (A, SR) and (B, TH) share the $(CLR_{(SR)(TH)})$ property,*

2. *there exist* $\varphi \in \Phi$ *and* $\psi \in \Psi$ *such that*

$$\psi(d(Ax, By)) \leq \psi(m(x, y)) - \varphi(m(x, y)),$$

for all $x, y \in X$, *where*
$m(x, y) = \max M^k_{A,B,H,R,S,T}(x, y)$, *and* $k \in \{3, 4, 5\}$.

Then, (A, SR) *and* (B, TH) *have a coincidence point each. Moreover,* A, B, H, R, S, *and* T *have a unique common fixed point provided* $AS = SA$, $AR = RA$, $SR = RS$, $BT = TB$, $BH = HB$, *and* $TH = HT$.

By choosing $A_1 = A_2 = \cdots = A_m = A$, $B_1 = B_2 = \cdots = B_n = B$, $S_1 = S_2 = \cdots = S_p = S$, and $T_1 = T_2 = \cdots = T_q = T$ in Theorem 2.2, we get the following corollary:

Corollary 2.4. *Let* A, B, S, *and* T *be self-mappings of a metric space* (X, d). *Suppose that*

1. *the pairs* (A^m, S^p) *and* (B^n, T^q) *share the* (CLR_{S^p,T^q}) *property, where* m, n, p, *and* q *are fixed positive integers;*
2. *there exist* $\varphi \in \Phi$ *and* $\psi \in \Psi$ *such that*

$$\psi(d(A^m x, B^n y)) \leq \psi(m(x, y)) - \varphi(m(x, y)),$$

for all $x, y \in X$, *where*
$m(x, y) = \max M^k_{A^m,B^n,S^p,T^q}(x, y)$, *and* $k \in \{3, 4, 5\}$.

Then, A, B, S, *and* T *have a unique common fixed point provided that* $AS = SA$ *and* $BT = TB$.

Remark 2.6. Notice that Corollary 2.4 is a slight but partial generalization of Theorem 2.1 as the commutativity requirements (that is, $AS = SA$ and $BT = TB$) in this corollary are relatively stronger as compared to weak compatibility in Theorem 2.1.
Remark 2.7. Results similar to Corollary 2.4 can be derived from Corollary 2.1.
Remark 2.8. It may be pointed out that the earlier proved results, namely Theorems 2.1 and 2.2 (also Corollaries 2.1–2.4) remain valid in symmetric space (X, d) whenever d is continuous.

Competing interest
The authors declare that they have no competing interests.

Authors' contributions
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References

1. Ali, J, Imdad, M: An implicit function implies several contraction conditions. Sarajevo J. Math. **4**(17), 269–285 (2008)
2. Ćirić, L j, B: Generalized contractions and fixed point theorems. Publ. Inst. Math. (Beograd) (N.S.) **12**(26), 19–26 (1971)
3. Ćirić, L j, B, Razani, A, Radenović, S, Ume, J S: Common fixed point theorems for families of weakly compatible maps. Comput. Math. Appl. **55**(11), 2533–2543 (2008)
4. Murthy, PP: Important tools and possible applications of metric fixed point theory. Proceedings of the Third World Congress of Nonlinear Analysts, Part 5 (Catania,2000). Nonlinear Anal. **47**(5), 3479–3490 (2001)
5. Pant, RP: Noncompatible mappings and common fixed points. Soochow J. Math. **26**(1), 29–35 (2000)
6. Pant, RP: Discontinuity and fixed points. J. Math. Anal. Appl. **240**(1), 280–283 (1999)
7. Alber, Y I, Guerre-Delabriere, S: Principles of weakly contractive maps in Hilbert spaces. In: *I. Gohberg, Yu. Lyubich, (eds.) New Results in Operator Theory and its Applications*, pp. 7–22. Birkhauser, Switzerland, (1997)
8. Rhoades, BE: Some theorems on weakly contractive maps. Nonlinear Anal. **47**, 2683–2693 (2001)
9. Zhang, Q, Song, Y: Fixed point theory for generalized φ-weak contractions. Appl. Math. Lett. **22**, 75–78 (2009)
10. Dorić, D: Common fixed point for generalized (ψ, φ)-weak contractions. Appl. Math. Lett. **22**, 1896–1900 (2009)
11. Dutta, PN, Choudhury, BS: A generalization of contraction principle in metric spaces. Fixed Point, Theory Appl. **2008**, 1–8 (2008). doi:10.1155/2008/406368
12. Abbas, M, Dorić, D: Common fixed point theorem for four mappings satisfying generalized weak contractive conditions. Filomat. **24**(2), 1–10 (2010)
13. Abbas, M, Khan, M A: Common fixed point theorem of two mappings satisfying a generalized weak contractive condition. Intern. J. Math. Math. Sci. **2009**, 1–9 (2009). doi:10.1155/2009/131068
14. Beg, I, Abbas, M: Coincidence point and invariant approximation for mappings satisfying generalized weak contractive condition. Fixed Point, Theory Appl. **2006**, 1–7 (2006). doi:10.1155/FPTA/2006/74503
15. Berinde, V: Approximating fixed points of weak φ-contractions. Fixed Point, Theory. **4**, 131–142 (2003)
16. Choudhury, BS, Konor, P, Rhoades, BE, Metiya, N: Fixed point theorems for generalized weakly contractive mapping. Nonlinear Anal. **74**, 2116–2126 (2011)
17. Ding, H-Sh, Kadelburg, Z, Karapinar, E, Radenović, S: Common fixed points of weak contractions in cone metric spaces. Abstract Appl, Anal. **2012**, 1–18. doi:10.1155/2012/793862
18. Radenović, S, Kadelburg, Z: Generalized weak contractions in partially ordered metric spaces. Comput. Math. Appl. **60**, 1776–1783 (2010)
19. Razani, A, Yazidi, M: Two common fixed point theorems for compatible mappings. Int. J. Nonlinear Anal. Appl. **2**(2), 7–18 (2011)
20. Jachymski, J: Equivalent conditions for generalized contractions on (ordered) metric spaces. Nonlinear Anal. **74**, 768–774 (2011)
21. Popescu, O: Fixed points for (ψ, ϕ)-weak contractions. Appl. Math. Lett. **24**, 1–4 (2011)
22. Radenović, S, Kadelburg, Z, Jandrlić, D, Jandrlić, A: Some results on weakly contractive maps. Bull Iranian Math. Soc. **38**(3), 625–645 (2012)
23. Jungck, G: Compatible mappings and common fixed points. Internat. J. Math. Math. Sci. **9**(4), 771–779 (1986)
24. Jungck, G, Rhoades, BE: Fixed points for set valued functions without continuity. Indian J. Pure, Appl. Math. **29**(3), 227–238 (1998)
25. Aamri, M, Moutawakil, D El: Some new common fixed point theorems under strict contractive conditions. J. Math. Anal. Appl. **270**(1), 181–188 (2002)

26. Singh, SL, Tomar, A: Weaker forms of commuting maps and existence of fixed points. J. Korean Soc. Math. Edu. Ser. B: Pure, Appl. Math. **10**(3), 145–161 (2003)
27. Fang, JX, Gao, Y: Common fixed point theorems under strict contractive conditions in Menger spaces. Nonlinear Anal. **70**(1), 184–193 (2009)
28. Liu, Y, Wu, J, Li, Z: Common fixed points of single-valued and multivalued maps. Int. J. Math. Math. Sci. **19**, 3045–3055 (2005)
29. Sintunavarat, W, Kumam, P: Common fixed point theorems for a pair of weakly compatible mappings in fuzzy metric spaces. J. Appl. Math. **2011**, 1–14 (2011)
30. Sintunavarat, W, Kumam, P: Common fixed points for R-weakly commuting mappings in fuzzy metric spaces. Ann. Univ. Ferrara Ser. VII Sci. Math. **58**(2), 389–406 (2012)
31. Imdad, M, Pant, BD, Chauhan, S: Fixed point theorems in Menger spaces using the (CLR_{ST}) property and applications. J. Nonlinear Anal. Optim. **3**(2), 225–237 (2012)
32. Imdad, M, Ali, J, Tanveer, M: Coincidence and common fixed point theorems for nonlinear contractions in Menger PM spaces. Chaos. Soliton. Fract. **42**(5), 3121–3129 (2009)

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