Grudsky and Tarkhanov *Mathematical Sciences* 2013, **7**:14
http://www.iaumath.com/content/7/1/14

Mathematical Sciences
a SpringerOpen Journal

ORIGINAL RESEARCH Open Access

Conformal reduction of boundary problems for harmonic functions in a plane domain with strong singularities on the boundary

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Abstract

We consider the Dirichlet, Neumann and Zaremba problems for harmonic functions in a bounded plane domain with nonsmooth boundary.

Purpose: We wish to construct explicit formulas for solutions of these problems when the boundary curve belongs to one of the following three classes: sectorial curves, logarithmic spirals and spirals of power type.

Methods: To study the problem, we apply the familiar Vekua-Muskhelishvili method which consists in the use of conformal mapping of the unit disk onto the domain to pull back the problem to a boundary problem for harmonic functions in the disk. This in turn later reduces to a Toeplitz operator equation on the unit circle with symbol-bearing discontinuities of the second kind.

Results: We develop a constructive invertibility theory for Toeplitz operators and thus derive solvability conditions as well as explicit formulas for solutions.

Conclusions: Our results raise Fredholm theory for boundary value problems in domains with singularities which are not necessarily rectifiable.

Keywords: Toeplitz operators, Boundary value problems, Singularities

Introduction

Elliptic partial differential equations are known to appear in many applied areas of mathematical physics, to name a few, mechanics of solid media, diffraction theory, hydrodynamics, gravity theory and quantum field theory.

In this paper, we focus on boundary value problems for the Laplace equation in plane domains bounded by nonsmooth curves $\mathcal{C}$. We are primarily interested in domains in which boundaries have a finite number of singular points of the oscillating type. By this, we mean that the curve may be parametrised in a neighbourhood of a singular point z_0 by $z(r) = z_0 + r \exp(\imath \varphi(r))$ for $r \in (0, r_0]$, where r is the distance between z and z_0 and where $\varphi(r)$ is a real-valued function which tends to infinity as $r \to 0$ or is bounded while its derivative is unbounded at $r = 0$. Furthermore, $\varphi(r)$ and $\varphi'(r)$ are allowed to tend to infinity rather quickly, and our study encompasses domains with a non-rectifiable boundary as well.

There is a huge literature devoted to boundary value problems for elliptic equations in domains with nonsmooth boundary (*cf.* [1-3] and the references given there). In most papers, one treats piecewise smooth curves with corner points or cusps (*cf.* [4-9]). One paper [10] is of particular importance for it gives a characterisation of Fredholm boundary value problems in domains with weakly oscillating cuspidal edges on the boundary.

There have been essentially fewer works dealing with more complicated curves $\mathcal{C}$. They mostly focus on qualitative properties, such as existence, uniqueness and stability of solutions, with respect to small perturbations (see for instance [11]). The present paper deals not only with qualitative investigations of boundary value problems in domains whose boundaries strongly oscillate at singular

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points, but also with constructive solutions of such problems. Hence, it sheds some new light on the operator calculus which lies behind the problems.

Our results gain in interest if we realize that the general theory of elliptic boundary value problems in domains with singular points on the boundary has made no essential progress since Kondratiev wrote his seminal paper [12]. The Fredholm property is proven to be equivalent to the invertibility of certain operator-valued symbols, where the problem is as immense as the original one. In order to get rid of operator-valued symbols, one has to carefully analyse the classical problems of potential theory.

Background on Toeplitz operators

Statement of the problem

We restrict ourselves to the Dirichlet and Neumann problems for the following Laplace equation:

$$\Delta u := (\partial/\partial x)^2 u + (\partial/\partial y)^2 u = 0, \tag{1}$$

in a simply connected domain $\mathcal{D}$ with boundary $\mathcal{C}$ in the plane of variables $(x, y) \in \mathbb{R}^2$. The boundary data are as follows:

$$u = u_0 \tag{2}$$

and

$$(\partial/\partial \nu)u = u_1, \tag{3}$$

on $\mathcal{C}$, respectively, where $(\partial/\partial \nu)u$ means the derivative of u in the unit outward the normal vector to $\mathcal{C}$.

We also treat the so-called Zaremba problem (see [13]) where the solution to (1) is required to satisfy the mixed conditions on $\mathcal{C}$, which contain both (2) and (3). More precisely, let $\mathcal{S}$ be a non-empty arc on $\mathcal{C}$, and then the mixed condition in question reads as follows:

$$\begin{aligned} u &= u_0 \ \text{ on } \mathcal{S}, \\ (\partial/\partial \nu)u &= u_1 \ \text{ on } \mathcal{C} \setminus \mathcal{S}, \end{aligned} \tag{4}$$

where u_0 and u_1 are given functions on $\mathcal{S}$ and $\mathcal{C} \setminus \mathcal{S}$, respectively. Although being model for us, problem (4) proves to be of great importance in mathematical physics.

Instead of the normal derivative on $\mathcal{C} \setminus \mathcal{S}$, one can consider an oblique derivative, which can be tangent to the boundary of $\mathcal{S}$. This gives rise to the Sturm-Liouville problems with boundary conditions having discontinuities of the first or second kind. Moreover, the boundary curve $\mathcal{C}$ itself is allowed to bear singularities at the points of $\partial\mathcal{S}$. One question that is still unanswered is whether the eigen- and root functions of such problems are complete in the space L^2 in the domain (*cf.* [14,15] and references therein). The Sturm-Liouville problems in domains with piecewise smooth boundary are also of great interest in multidimensional case.

General description of the method

Our approach to the study of elliptic problems in domains with non-smooth boundary goes back to at least as far as the study of Khuskivadze and Paatashvili [16,17]. It consists in reducing the problem in $\mathcal{D}$ to a singular integral equation on the unit circle by means of a conformal mapping of the unit disk onto $\mathcal{D}$. The coefficients of the singular integral equation obtained in this way fail in general to be continuous for they are intimately connected with the derivative of boundary values of the conformal mapping. This method was successfully used for solving problems in domains with piecewise smooth boundary, where the singular points are corner points or cusps (see [4-7,18]). In this case, the coefficients of the mentioned singular integral equation have discontinuities of the first kind. Since the theory of such equations is well elaborated, one has succeeded in constructing a sufficiently complete theory of boundary value problems for a number of elliptic equations in domains with piecewise smooth boundary. Note that by now, the theory of singular integral equations (or, in other terms, the theory of Toeplitz operators) with oscillating coefficients is well elaborated, too. In particular, in a previous study [19-21], a constructive theory of normal solvability (left and right invertibility) is elaborated in the case of coefficients with rather strong discontinuities (see also the monograph [22]).

The present paper deals with main boundary value problems for the Laplace equation for three classes of boundary curves $\mathcal{C}$, namely sectorial curves, logarithmic and power spirals. A sectorial curve is a plane curve $\mathcal{C}$, such that the angle at which the tangent of $\mathcal{C}$ intersects the real axis is a bounded function in a punctured neighbourhood of any vorticity point of $\mathcal{C}$ (see Definition 3 for more details). As distinct from corner or cuspidal points, the angle need not possess finite one-sided limits at a singular point and may, in general, undergo discontinuities of the second kind. For example, the arc $z(t) = t + \imath t^2 \sin(1/t)$, where $|t| < \varepsilon$ is a part of the sectorial curve with singular point $z(0) = 0$. The main result for sectorial curves is Theorem 8 which reduces the Dirichlet problem with data at a sectorial curve to a Toeplitz operator with sectorial symbol. The theory of such operators is well understood. A logarithmic spiral is a curve of the form $z(r) = r \exp(\imath \delta \ln r)$, where $r \in (0, r_0)$ and δ are fixed real number. Note that in [23], Fredholm theory was developed for potential-type operators on slowly oscillating curves, a typical example being a logarithmic spiral. In the present paper, we not only elaborate the theory of Fredholm boundary value problems with data on logarithmic spirals, but also construct explicit formulas for solution. Finally, by a power spiral, we mean a curve of the form $z(r) = r \exp(\imath c r^{-1/\delta})$, where $\delta > 0$. Notice that for $\delta < 1$, the curve is not rectifiable. However, our method allows one not only to develop a Fredholm theory for the corresponding boundary value problems, but also to obtain formulas for solutions in a closed form.

Reduction of the Dirichlet problem

The Dirichlet problem is the most frequently encountered elliptic boundary value problem. This is not only because the Dirichlet problem is of great interest in applications in electrostatics, gravity theory, incompressible fluid theory, etc., but also since it is a good model where one tests approaches to other, more complicated, problems.

Let $\mathcal{D}$ be a simply connected, bounded domain in the plane of real variables (x, y). The boundary of $\mathcal{D}$ is a closed Jordan curve which we denote by $\mathcal{C}$. Consider the Dirichlet problems (1) and (2) in $\mathcal{D}$ with data u_0 on $\mathcal{C}$. As usual, we introduce a complex structure in $\mathbb{R}^2$ by $z = x + \imath y$ and pick a conformal mapping $z = \mathfrak{c}(\zeta)$ of the unit disk $\mathbb{D} = \{\zeta \in \mathbb{C} : |\zeta| < 1\}$ onto the domain $\mathcal{D}$, cf. Riemann mapping theorem. Throught the paper, we make a standing assumption on the mappings $z = \mathfrak{c}(\zeta)$ under consideration, namely

$$\mathfrak{c}'(0) > 0. \tag{5}$$

Problems (1) and (2) can then be reformulated as follows:

$$\begin{aligned} \frac{1}{|\mathfrak{c}'(\zeta)|^2}\,\Delta U &= 0 \quad \text{for } |\zeta| < 1, \\ U &= U_0 \ \text{ for } |\zeta| = 1, \end{aligned} \tag{6}$$

where $U(\zeta) := u(\mathfrak{c}(\zeta))$ and $U_0(\zeta) := u_0(\mathfrak{c}(\zeta))$.

For $1 \le p < \infty$, we denote by $H^p(\mathbb{D})$ the Hardy space on the unit disk (for the properties of Hardy spaces and conformal mappings, we refer the reader to the classical book [24]). By conformal mapping $z = \mathfrak{c}(\zeta)$, the space is transported to the so-called Hardy-Smirnov space $E^p(\mathcal{D})$ of functions on $\mathcal{D}$. A holomorphic function f on $\mathcal{D}$ is said to belong to $E^p(\mathcal{D})$ if

$$\sup_{r\in(0,1)} \int_{\mathcal{C}_r} |f(z)|^p\, |dz| < \infty,$$

where $\mathcal{C}_r$ is the push-forward of the circle $|\zeta| = r$ by $z = \mathfrak{c}(\zeta)$. It is easy to see that $f \in E^p(\mathcal{D})$ if and only if

$$\sqrt[p]{\mathfrak{c}'(\zeta)}\, f(\mathfrak{c}(\zeta)) \in H^p(\mathbb{D}). \tag{7}$$

It is then a familiar property of the functions of Hardy class $H^p(\mathbb{D})$ that the function $\sqrt[p]{\mathfrak{c}'(\zeta)}\, f(\mathfrak{c}(\zeta))$ has finite non-tangential limit values almost everywhere on the unit circle $\mathbb{T} = \{\zeta \in \mathbb{C} : |\zeta| = 1\}$.

If $\mathcal{C}$ is a rectifiable curve, then the function $z = \mathfrak{c}(\zeta)$ is continuous on the closed unit disk $\overline{\mathbb{D}}$, absolutely continuous on the unit circle $\mathbb{T}$ and $(\mathfrak{c}(e^{\imath t}))' = \imath e^{\imath t} \mathfrak{c}'(e^{\imath t})$ almost everywhere on $\mathbb{T}$. It follows from (7) that $f(z)$ has finite non-tangential limit values almost everywhere on $\mathcal{C}$, and

$$\lim_{r\to 1-} \int_{\mathcal{C}_r} |f(z)|^p\, |dz| = \int_{\mathcal{C}} |f(z)|^p\, |dz|. \tag{8}$$

If $\mathcal{C}$ fails to be rectifiable and there is a function $f \in E^p(\mathcal{D})$ with nonzero non-tangential limit values almost everywhere on $\mathcal{C}$, then relation (7) yields that the derivative $\mathfrak{c}'(\zeta)$ also possesses finite non-tangential limit values almost everywhere on $\mathbb{T}$. Moreover, for $f \in E^p(\mathcal{D})$, the limit on the left-hand side of (8) exists; hence, we are able to interpret integrals over the boundary curve $\mathcal{C}$ like that on the right-hand side.

We will also study boundary value problems in Hardy-Smirnov spaces with weights $E^p(\mathcal{D}, w)$.

Set

$$w(\zeta) = \prod_{k=1}^{n} \left(1 - \frac{\zeta}{\zeta_k}\right)^{-\mu_k} \tag{9}$$

for $\zeta \in \mathbb{D}$, where $\{\zeta_1, \ldots, \zeta_n\}$ are pairwise different points on the unit circle. Here, $\mu_1, \ldots, \mu_n$ are real numbers in the interval $(-1/q, 1/p)$, p and q being conjugate exponents, i.e. $1/p + 1/q = 1$. The weight functions of form (9) are called power weights. The advantage of using such weight functions lies certainly in the fact that they are holomorphic in $\mathbb{D}$. A holomorphic function f in $\mathcal{D}$ is said to lie in $E^p(\mathcal{D}, w)$ if

$$\sup_{r\in(0,1)} \int_{\mathcal{C}_r} |f(z)|^p\, |w(\mathfrak{c}^{-1}(z))|^p\, |dz| < \infty.$$

It is well known that for each harmonic function $u(x, y)$ in $\mathcal{D}$, there is an analytic function $f(z)$ in $\mathcal{D}$ whose real part is u. We therefore look for a solution u for problems (1) and (2), which have the form $u = \Re f$ with $f \in E^p(\mathcal{D}, w)$. The boundary condition $u = u_0$ is understood in the sense of non-tangential limit values of u almost everywhere on $\mathcal{C}$.

Definition 1. *Given any Dirichlet data u_0 on $\mathcal{C}$ of class $L^p(\mathcal{C}, w)$ in the sense that*

$$\int_{\mathbb{T}} |u_0(\mathfrak{c}(\zeta))|^p\, |w(\zeta)|^p\, |\mathfrak{c}'(\zeta)|\, |d\zeta| < \infty,$$

we shall say that problems (1) *and* (2) *possess a solution in $\Re E^p(\mathcal{D}, w)$ if there is a harmonic function u in $\mathcal{D}$, such that $u = \Re f$ for some $f \in E^p(\mathcal{D}, w)$ and $u = u_0$ on $\mathcal{C}$.*

If $w \equiv 1$ (i.e. all μ_k vanish), then we recover the Hardy-Smirnov spaces $E^p(\mathcal{D})$ and $\Re E^p(\mathcal{D})$, respectively.

We proceed to reduce the Dirichlet problem. By the previously mentioned data, we can look for f of the following form:

$$f(\mathfrak{c}(\zeta)) = \frac{h^+(\zeta)}{w(\zeta)\,\sqrt[p]{\mathfrak{c}'(\zeta)}}$$

for $\zeta \in \mathbb{D}$, where h^+ is an analytic function of Hardy class $H^p(\mathbb{D})$.

By Theorem 4 in [24], the conformal mapping $z = \mathfrak{c}(\zeta)$ is bijective and continuous on the closed unit disk. Hence, the function $U(\zeta) = u(\mathfrak{c}(\zeta))$ has finite non-tangential limit values almost everywhere on $\mathbb{T}$, and in this way, $U(\zeta) = U_0(\zeta)$ is understood on the unit circle $\mathbb{T}$. This enables us to rewrite problem (6) in the following form:

$$\Re\left(\frac{h^+(\zeta)}{w(\zeta)\sqrt[p]{\mathfrak{c}'(\zeta)}}\right) = U_0(\zeta)$$

for $\zeta \in \mathbb{T}$, where h^+ is an analytic function of Hardy class $H^p(\mathbb{D})$. The latter problem can in turn be reformulated as follows:

$$\frac{1}{2}\left(\frac{h^+(\zeta)}{w(\zeta)\sqrt[p]{\mathfrak{c}'(\zeta)}} + \frac{h^-(\zeta)}{\overline{w(\zeta)\sqrt[p]{\mathfrak{c}'(\zeta)}}}\right) = U_0(\zeta)$$

for $\zeta \in \mathbb{T}$, where

$$\begin{aligned} h^-(\zeta) &= \overline{h^+(\zeta)} \\ &= \overline{h^+\left(\frac{\zeta}{|\zeta|^2}\right)} \\ &= \overline{h^+\left(\frac{1}{\bar{\zeta}}\right)} \end{aligned}$$

can be specified within analytic functions of Hardy class H^p in the complement of the closed unit disk.

More precisely,

$$\overline{h^+\left(\frac{1}{\bar{\zeta}}\right)}$$

belongs to the Hardy class H^p in the complement of $\bar{\mathbb{D}}$ up to an additive complex constant if the functions of Hardy class H^p in $\mathbb{C} \setminus \bar{\mathbb{D}}$ are assumed to vanish at infinity.

Recall the definition of Hardy spaces $H^{p\pm}$ on the unit circle. Let $h \in L^p(\mathbb{T})$, where $1 \le p \le \infty$. We parametrise the points of $\mathbb{T}$ by $\zeta = \exp(\imath t)$ with $t \in [0, 2\pi]$. Let

$$h(\zeta) \sim \sum_{j=-\infty}^{\infty} c_j(h)\,\zeta^j$$

be the Fourier series of h, the coefficients being

$$c_j(h) := \frac{1}{2\pi\imath}\int_{\mathbb{T}} h(\zeta)\,\frac{d\zeta}{\zeta^{j+1}}.$$

Then, $h \in H^{p+}$, if $c_j(h) = 0$ for all integers $j < 0$, and $h \in H^{p-}$, if $c_j(h) = 0$ for all integers $j \ge 0$.

The functions of H^{p+} are non-tangential limit values on $\mathbb{T}$ of functions of Hardy class $H^p(\mathbb{D})$. The functions of H^{p-} are non-tangential limit values on $\mathbb{T}$ of functions of Hardy class H^p in $\mathbb{C} \setminus \bar{\mathbb{D}}$, which vanish at the point at infinity. Moreover, for $1 < p < \infty$, the space $L^p(\mathbb{T})$ splits into the topological direct sum $H^{p+} \oplus H^{p-}$, as is well known.

Finally, we transform the Dirichlet problem to the following:

$$a(\zeta)\,h^+(\zeta) + h^-(\zeta) = g(\zeta) \tag{10}$$

for $\zeta \in \mathbb{T}$, where

$$a(\zeta) = \frac{\overline{w(\zeta)}}{w(\zeta)}\sqrt[p]{\frac{\overline{\mathfrak{c}'(\zeta)}}{\mathfrak{c}'(\zeta)}} = \frac{\overline{w(\zeta)}}{w(\zeta)}\exp\left(-\imath\frac{2}{p}\arg\mathfrak{c}'(\zeta)\right)$$

and $g(\zeta) = 2U_0(\zeta)\,\overline{w(\zeta)}\,\overline{\sqrt[p]{\mathfrak{c}'(\zeta)}}$.

It is well known from the theory of conformal mappings that

$$\arg\mathfrak{c}'(\zeta) = \alpha(\mathfrak{c}(\zeta)) - \arg\zeta - \frac{\pi}{2}$$

for $\zeta \in \mathbb{T}$, where $\alpha(\mathfrak{c}(\zeta))$ is the angle at which the tangent of $\mathcal{C}$ at the point $z = \mathfrak{c}(\zeta)$ intersects the real axis. Note that $g \in L^p(\mathbb{T})$.

Now, let

$$(S_{\mathbb{T}}g)(\zeta) := \frac{1}{\pi\imath}\int_{\mathbb{T}}\frac{g(\zeta')}{\zeta'-\zeta}\,d\zeta', \quad \zeta \in \mathbb{T},$$

stand for the singular Cauchy integral. If $1 < p < \infty$, then $S_{\mathbb{T}}$ is a bounded operator in $L^p(\mathbb{T})$, and the operators

$$P^{\pm}_{\mathbb{T}} := \frac{1}{2}\,(I \pm S_{\mathbb{T}})$$

prove to be continuous projections in $L^p(\mathbb{T})$ called analytic projections. They are intimately related with the classical decomposition of $L^p(\mathbb{T})$ into the direct sum of traces on $\mathbb{T}$ of Hardy class H^p functions in $\mathbb{D}$ and $\mathbb{C} \setminus \overline{\mathbb{D}}$, respectively.

More precisely, we get

$$P^{\pm}_{\mathbb{T}}\,L^p(\mathbb{T}) = H^{p\pm},$$

whence $P^{\pm}_{\mathbb{T}}\,H^{p\pm} = H^{p\pm}$ and $P^{\pm}_{\mathbb{T}}\,H^{p\mp} = 0$.

On applying $P^+_{\mathbb{T}}$ to both sides of equality (10) and taking into account that $(P^+_{\mathbb{T}}h^-)(\zeta) = h^-(\infty)$ and $h^-(\infty) = \overline{h^+(0)}$, we get

$$(T(a)h^+)(\zeta) + \overline{h^+(0)} = g^+(\zeta) \tag{11}$$

for $\zeta \in \mathbb{T}$, where $T(a) := P^+_{\mathbb{T}}aP^+_{\mathbb{T}}$ is a Toeplitz operator with symbol a on $L^p(\mathbb{T})$ and $g^+(\zeta) = (P^+_{\mathbb{T}}g)(\zeta)$ for $\zeta \in \mathbb{T}$.

We thus arrived at the following result.

Theorem 1.

(1) *If $u = \Re f$ with $f \in E^p(\mathcal{D}, w)$ is a solution of the Dirichlet problem in $\mathcal{D}$, then $h^+(\zeta) = w(\zeta)\sqrt[p]{\mathfrak{c}'(\zeta)}\,f(\mathfrak{c}(\zeta))$ is a solution of Equation 11.*

(2) *If $h^+ \in H^{p+}$ is a solution of (11) and the kernel of $T(a)$ is zero, then the function $u(\mathfrak{c}(\zeta)) = \Re\,(h^+(\zeta)/w(\zeta)\sqrt[p]{\mathfrak{c}'(\zeta)})$ is a solution of the Dirichlet problem in $\mathcal{D}$.*

Proof. (1) has already been proven; it remains to show (2). Let $h^+ \in H^{p+}$ satisfy (11). This equality transforms to the following:

$$a(\zeta)h^+(\zeta) + h^-(\zeta) = g(\zeta) \tag{12}$$

where $h^-(\zeta) = -(P_{\mathbb{T}}^-(ah^+))(\zeta) + (P_{\mathbb{T}}^- g)(\zeta) + \overline{h^+(0)}$.

Since

$$\begin{aligned} \overline{a(\zeta)} &= \frac{1}{a(\zeta)}, \\ a(\zeta)\,\overline{g(\zeta)} &= g(\zeta), \end{aligned}$$

as is easy to check, we deduce from (12) that $a(\zeta)\,\overline{h^-(\zeta)} + \overline{h^+(\zeta)} = g(\zeta)$.

Applying $P_{\mathbb{T}}^+$ to both sides of this equality yields the following:

$$\left(T(a)\,\overline{h^-}\right)(\zeta) + \overline{h^+(0)} = g^+(\zeta).$$

Comparing this with (11), we get $T(a)(\overline{h^-} - h^+)(\zeta) = 0$ whence $h^-(\zeta) = \overline{h^+(\zeta)}$ for all $\zeta \in \mathbb{T}$, for the kernel of $T(a)$ is zero.

Equality (12) can then be rewritten as follows:

$$\frac{1}{2}\left(\frac{h^+(\zeta)}{w(\zeta)\sqrt{\mathfrak{c}'(\zeta)}} + \overline{\frac{h^+(\zeta)}{w(\zeta)\sqrt{\mathfrak{c}'(\zeta)}}}\right) = U_0(\zeta)$$

for $\zeta \in \mathbb{T}$. Since the function $U_0(\zeta) = u_0(\mathfrak{c}(\zeta))$ is real-valued, the last equality just amounts to saying that $\Re F(\zeta) = U_0(\zeta)$ for $\zeta \in \mathbb{T}$, where

$$F(\zeta) = \frac{h^+(\zeta)}{w(\zeta)\sqrt{\mathfrak{c}'(\zeta)}}.$$

The function $\Re F(\zeta)$ is harmonic in $\mathbb{D}$, and it has non-tangential limit values almost everywhere on $\mathbb{T}$, which coincide with $U_0(\zeta)$. Moreover, $f(z) = F(\mathfrak{c}^{-1}(z))$ is of weighted Hardy-Smirnov class $E^p(\mathcal{D}, w)$, and $u(x, y) = \Re f(z)$ is a solution of the Dirichlet problem in $\mathcal{D}$, as desired. □

Corollary 1. *If the operator* $T(a)$ *is invertible on the space* H^{p+} *and*

$$\left(T(a)^{-1}1\right)(\zeta) = w(\zeta)\,\sqrt[p]{\mathfrak{c}'(\zeta)/\mathfrak{c}'(0)}, \tag{13}$$

for $\zeta \in \mathbb{T}$, *then the Dirichlet problem in* $\mathcal{D}$ *has a unique solution of the following form:*

$$u(z) = \Re\left(\frac{1}{w(\zeta)\sqrt[p]{\mathfrak{c}'(\zeta)}}\left((T(a)^{-1}g^+)(\zeta) - \frac{1}{2}(T(a)^{-1}g^+)(0) \times (T(a)^{-1}1)(\zeta)\right)\right)$$

with $z = \mathfrak{c}(\zeta)$, *where* $g^+ = P_{\mathbb{T}}^+\left(2u_0(\mathfrak{c}(\zeta))\,\overline{w(\zeta)}\,\overline{\sqrt[p]{\mathfrak{c}'(\zeta)}}\right)$.

Proof. Applying the operator $T(a)^{-1}$ to (11) yields the following:

$$h^+(\zeta) + \left(T(a)^{-1}\overline{h^+(0)}\right)(\zeta) = \left(T(a)^{-1}g^+\right)(\zeta)$$

for all $\zeta \in \mathbb{T}$. Since both sides of the equality extend to holomorphic functions in the disk, we can set $\zeta = 0$. By (13), we get $\left(T(a)^{-1}1\right)(0) = 1$; hence,

$$2\,\Re\, h^+(0) = \left(T(a)^{-1}g^+\right)(0).$$

We thus conclude that the general solution of (11) has the following form:

$$\begin{aligned} h^+(\zeta) = {} & \left(T(a)^{-1}g^+\right)(\zeta) - \frac{1}{2}\left(T(a)^{-1}g^+\right)(0) \\ & \times (T(a)^{-1}1)(\zeta) + \imath\, c\left(T(a)^{-1}1\right)(\zeta), \end{aligned}$$

where c is an arbitrary real constant. From (13), it follows that

$$\frac{\left(T(a)^{-1}1\right)(\zeta)}{w(\zeta)\sqrt[p]{\mathfrak{c}'(\zeta)}} = \frac{1}{\sqrt{\mathfrak{c}'(0)}}$$

is a real number. Therefore, $u(z) = \Re\left(\frac{h^+(\zeta)}{w(\zeta)\sqrt[p]{\mathfrak{c}'(\zeta)}}\right)$ is actually independent of c, establishing the corollary. □

Remark 1. *Condition (13) is actually fulfilled in all cases to be treated in this work (see Remark 3 below).*

If Equation 11 has many solutions, then we must specify among them those solutions which give rise to solutions of the Dirichlet problem in weighted Hardy-Smirnov spaces.

Factorisation of symbols

The results of this section with detailed explanations, proofs and corresponding references can be found elsewhere [25-27].

Let $L^\infty(\mathbb{T})$ be the space of all essentially bounded functions on the unit circle $\mathbb{T}$, $H^{\infty\pm}$ the Hardy spaces on $\mathbb{T}$ which consist of the restrictions to $\mathbb{T}$ of bounded analytic functions in $\mathbb{D}$ and $\mathbb{C} \setminus \overline{\mathbb{D}}$, respectively, and $C(\mathbb{T})$ the space of all continuous functions on $\mathbb{T}$.

A bounded linear operator A on a Hilbert space H is said to be normally solvable if its range $\operatorname{im} A$ is closed. A normally solvable operator is called Fredholm if its kernel and cokernel are finite dimensional. In this case, the index of A is introduced as follows:

$$\operatorname{ind} A := \alpha(A) - \beta(A),$$

where $\alpha(A) = \dim \ker A$ and $\beta(A) = \dim \operatorname{coker} A$.

The symbol $a(\zeta)$ of a Toeplitz operator $T(a)$ is said to admit a p-factorisation, with $1 < p < \infty$, if it can be represented in the following form:

$$a(\zeta) = a^+(\zeta)\zeta^\kappa a^-(\zeta), \tag{14}$$

where κ is an integer number,

$$\begin{array}{rlrl} a^+ & \in H^{q+}, & a^- & \in H^{p-} \oplus \{c\}, \\ 1/a^+ & \in H^{p+}, & 1/a^- & \in H^{q-} \oplus \{c\}, \end{array} \qquad (15)$$

p and q are conjugate exponents (i.e. $1/p + 1/q = 1$), and $(1/a^+)S_{\mathbb{T}}(1/a^-)$ is a bounded operator on $L^p(\mathbb{T})$.

The functions a^+ and a^- in (14) are determined uniquely up to a constant factor. On putting the additional condition $a^-(\infty) = 1$, one determines the factorisation uniquely.

Remark 2. *As proven in [27], the factorisation of the symbol a with property (15) only is also unique up to a multiplicative constant.*

Theorem 2. *An operator $T(a)$ is Fredholm in the space H^{p+} if and only if the symbol $a(\zeta)$ admits a p-factorisation. If $T(a)$ is Fredholm, then ind $T(a) = -\kappa$.*

Theorem 3. *Let $a \in L^\infty(\mathbb{T})$ and $a(\zeta) \neq 0$ almost everywhere on $\mathbb{T}$; then at least one of the numbers $\alpha(T(a))$ and $\beta(T(a))$ is equal to zero.*

Combining Theorems 2 and 3, we get a criterion of invertibility for Toeplitz operators.

Corollary 2. *An operator $T(a)$ is invertible on H^{p+} if and only if the symbol $a(\zeta)$ admits a p-factorisation with $\kappa = 0$. In this case,*

$$(T(a))^{-1} = (1/a^+)P^+_{\mathbb{T}}(1/a^-).$$

Proof. If $\kappa = 0$, then $\alpha(T(a)) = \beta(T(a))$; thus, both $\alpha(T(a))$ and $\beta(T(a))$ vanish. Hence, it follows that $T(a)$ is invertible on H^{p+}.

We now establish the formula for the inverse operator $(T(a))^{-1}$. Let $f \in H^{p+}$, and then

$$\begin{aligned} \left((1/a^+)P^+_{\mathbb{T}}(1/a^-)\right) T(a)f &= \left((1/a^+)P^+_{\mathbb{T}}(1/a^-)\right) P^+_{\mathbb{T}}(af) \\ &= \left((1/a^+)P^+_{\mathbb{T}}(1/a^-)\right) af \\ &= (1/a^+)P^+_{\mathbb{T}}a^+f \\ &= (1/a^+)a^+f \\ &= f, \end{aligned}$$

and similarly

$$\begin{aligned} T(a)\left((1/a^+)P^+_{\mathbb{T}}(1/a^-)\right)f &= P^+_{\mathbb{T}}a\left((1/a^+)P^+_{\mathbb{T}}(1/a^-)\right)f \\ &= P^+_{\mathbb{T}}a^-P^+_{\mathbb{T}}(1/a^-)f \\ &= P^+_{\mathbb{T}}a^-(1/a^-)f \\ &= f. \end{aligned}$$

Here, we have used the familiar equalities $P^+_{\mathbb{T}}h^-P^+_{\mathbb{T}} = P^+_{\mathbb{T}}h^-$ and $P^+_{\mathbb{T}}h^+P^+_{\mathbb{T}} = h^+P^+_{\mathbb{T}}$ which are valid for all $h^- \in H^{q-} \oplus \{c\}$ and $h^+ \in H^{q+}$. □

Given a non-vanishing function $a \in C(\mathbb{T})$, we denote by $\mathrm{ind}_{a(\mathbb{T})}(0)$ the winding number of the curve $a(\mathbb{T})$ about the origin or the index of the origin with respect to $a(\mathbb{T})$.

Theorem 4. *Suppose $a \in C(\mathbb{T})$, then the operator $T(a)$ is Fredholm in the space H^{p+} if and only if $a(\zeta) \neq 0$ for all $\zeta \in \mathbb{T}$. Under this condition, the index of $T(a)$ is given by the following:*

$$\mathrm{ind}\, T(a) = -\mathrm{ind}_{a(\mathbb{T})}(0).$$

We now introduce the concept of sectoriality which is of crucial importance in this paper.

Definition 2. *A function $a \in L^\infty(\mathbb{T})$ is called p-sectorial if $\operatorname{ess\,inf} |a(\zeta)| > 0$ and if there is a real number φ_0 such that*

$$\sup_{\zeta \in \mathbb{T}} |\arg(\exp(\imath\varphi_0)a(\zeta))| < \frac{\pi}{\max\{p, q\}} \qquad (16)$$

for all $\zeta \in \mathbb{T}$.

A function $a \in L^\infty(\mathbb{T})$ is said to be locally p-sectorial if $\operatorname{ess\,inf} |a(\zeta)| > 0$, and for any $\zeta_0 \in \mathbb{T}$, there is an open arc containing ζ_0, such that (16) is satisfied for all ζ in the arc with some $\varphi_0 \in \mathbb{R}$ depending on ζ_0. Each p-sectorial curve is obviously locally p-sectorial.

Theorem 5.

(1) If $a(\zeta)$ is a p-sectorial symbol, then the operator $T(a)$ is invertible in the space H^{p+}.
(2) If $a(\zeta)$ is a locally p-sectorial symbol, then $T(a)$ is a Fredholm operator in H^{p+}.

Suppose $h \in GH^{\infty+}$, that is, $h \in H^{\infty+}$ and $1/h \in H^{\infty+}$, then the operator $T(h)$ is invertible in H^{p+}. Indeed, it is easy to check that $(T(h))^{-1} = T(1/h)$. Analogously, if $h \in GH^{\infty-}$, then the operator $T(h)$ is invertible in H^{p+} and $(T(h))^{-1} = T(1/h)$.

Theorem 6.

(1) Let $a(\zeta) = h(\zeta)a_0(\zeta)$, where $h \in GH^{\infty\pm}$ and $a_0 \in L^\infty(\mathbb{T})$, and then the operator $T(a)$ is Fredholm in the space H^{p+} if and only if the operator $T(a_0)$ is Fredholm, in which case ind $T(a) =$ ind $T(a_0)$.
(2) Let $a(\zeta) = c(\zeta)a_0(\zeta)$, where $c \in C(\mathbb{T})$ and $a_0 \in L^\infty(\mathbb{T})$, and then $T(a)$ is Fredholm in H^{p+} if and only if $c(\zeta)$ vanishes at no point of $\mathbb{T}$ and $T(a_0)$ is Fredholm, in which case $\mathrm{ind}\, T(a) = \mathrm{ind}\, T(a_0) - \mathrm{ind}_{c(\mathbb{T})}(0)$.

Proof. This is a straightforward consequence of Theorems 2 and 4 and Corollary 2. □

In conclusion, we give a brief summary on Toeplitz operators with symbols having discontinuities of the first kind (the reader is referred to Chapter 5 of [26]). Let PC stand for the space of all piecewise continuous functions on $\mathbb{T}$ which have at most finitely many jumps. Suppose $a(\zeta) \in PC$ and $\zeta_1, \ldots, \zeta_n$ are the points of discontinuity of

a. Given any $\mathfrak{f} \in \mathbb{C}$ and $\zeta_0 \in \mathbb{T}$, we introduce a function $a_{\mathfrak{f},\zeta_0} \in PC$ by the following:

$$a_{\mathfrak{f},\zeta_0}(\zeta) := \exp\left(\imath \mathfrak{f} \arg\left(-\frac{\zeta}{\zeta_0}\right)\right),$$

for $\zeta \in \mathbb{T}$, where $\arg z \in (-\pi, \pi]$. It is easily seen that $a_{\mathfrak{f},\zeta_0}$ has at most one point of discontinuity at $\zeta = \zeta_0$, with jump

$$\begin{aligned} a_{\mathfrak{f},\zeta_0}(e^{\imath 0-}\zeta_0) &= \exp(\imath \mathfrak{f}\pi), \\ a_{\mathfrak{f},\zeta_0}(e^{\imath 0+}\zeta_0) &= \exp(-\imath \mathfrak{f}\pi). \end{aligned}$$

If $a(e^{\imath 0\pm}\zeta_k) \neq 0$ for all $k = 1, \ldots, n$, then there are complex numbers $\mathfrak{f}_k$ with the property that

$$\frac{a(e^{\imath 0-}\zeta_k)}{a(e^{\imath 0+}\zeta_k)} = \exp(2\imath \mathfrak{f}_k \pi),$$

so

$$a(\zeta) = c(\zeta) \prod_{k=1}^{n} a_{\mathfrak{f}_k,\zeta_k}(\zeta), \tag{17}$$

where $c \in C(\mathbb{T})$.

Theorem 7. *Let $a(\zeta) \in PC$, and then the operator $T(a)$ is Fredholm in H^{p+} if and only if the following are met:*

(1) $a(e^{\imath 0\pm}\zeta) \neq 0$ *for all* $\zeta \in \mathbb{T}$,
(2) There are integer numbers κ_k such that $\kappa_k - \frac{1}{q} < \Re \mathfrak{f}_k < \kappa_k + \frac{1}{p}$.

Under conditions (1) and (2), the index of the operator $T(a)$ in H^{p+} is actually given by the following:

$$\operatorname{ind} T(a) = -\left(\operatorname{ind}_{c(\mathbb{T})}(0) + \sum_{k=1}^{n} \kappa_k\right), \tag{18}$$

which is due to (17).

Remark 3. *If the operator $T(a)$ with symbol of (10) is invertible and $a(\zeta)$ admit a p-factorisation $a = a_+ a_-$ with*

$$\begin{aligned} a_+(\zeta) &= \frac{1}{w(\zeta)\sqrt[p]{\mathfrak{c}'(\zeta)}}, \\ a_-(\zeta) &= \overline{w(\zeta)\sqrt[p]{\mathfrak{c}'(\zeta)}}, \end{aligned}$$

then condition (13) holds true.

Indeed, by Corollary 2, we get the following:

$$\begin{aligned} \left(T(a)^{-1}1\right)(\zeta) &= w(\zeta)\sqrt[p]{\mathfrak{c}'(\zeta)} \left(P^+_{\mathbb{T}} \frac{1}{\overline{w\sqrt[p]{\mathfrak{c}'}}}\right)(\zeta) \\ &= w(\zeta)\sqrt[p]{\mathfrak{c}'(\zeta)} \frac{1}{\overline{\sqrt[p]{\mathfrak{c}'(0)}}} \\ &= w(\zeta)\sqrt[p]{\frac{\mathfrak{c}'(\zeta)}{\mathfrak{c}'(0)}}, \end{aligned}$$

as desired.

Conformal reduction of Dirichlet problems

Sectorial curves

In this section, we consider a simply connected domain $\mathcal{D} \Subset \mathbb{R}^2$ whose boundary $\mathcal{C}$ is smooth away from a finite number of points. By this, it is meant that $\mathcal{C}$ is a Jordan curve of the following form:

$$\mathcal{C} = \bigcup_{k=1}^{n} \mathcal{C}_k,$$

where $\mathcal{C}_k = [z_{k-1}, z_k]$ is an arc with initial point z_{k-1} and endpoint z_k which are located after each other in the positive direction on $\mathcal{C}$, and $z_n = z_0$. Moreover, (z_{k-1}, z_k) is smooth for all k.

Definition 3. *The curve $\mathcal{C}$ is called p-sectorial if, for each $k = 1, \ldots, n$, there is a neighbourhood (z_k^-, z_k^+) of z_k on $\mathcal{C}$ and a real number φ_k, such that*

$$\sup_{z \in (z_k^-, z_k^+) \setminus \{z_k\}} |\alpha(z) - \varphi_k| < \begin{cases} \dfrac{\pi}{2}, & \text{if } p \geq 2, \\ \dfrac{\pi}{2}(p-1), & \text{if } 1 < p < 2, \end{cases} \tag{19}$$

where $\alpha(z)$ is the angle at which the tangent of $\mathcal{C}$ at the point z intersects the real axis.

If z_k is a conical point of $\mathcal{C}$, then the angle at which the tangent of $\mathcal{C}$ at z intersects the real axis has jump $j_k < \pi$ when z passes through z_k. It follows that (19) is fulfilled at z_k with a suitable φ_k, if $p \geq 2$, and is fulfilled if, moreover, $j_k < (p-1)\pi$, if $1 < p < 2$. If z_k is a cuspidal point of $\mathcal{C}$, then the angle has jump $j_k = \pi$ when z passes through z_k. Hence, condition (19) is violated, i.e. cuspidal points are prohibited for sectorial curves.

Example 1. *Let $\mathcal{C}$ be a curve parametrised in a neighbourhood of the singular point $z(0) = 0$ by $z(t) = t + \imath y(t)$ with $|t| \leq \varepsilon$, where $y(t)$ is a continuous function on the interval $[-\varepsilon, \varepsilon]$ whose derivative is continuous away from zero in $(-\varepsilon, \varepsilon)$ and bounded in this interval. The angle at which the tangent of $\mathcal{C}$ at the point $z(t)$ then intersects the real axis is $\arg z'(t) = \arg(1 + \imath y'(t))$. Thus, for $p \geq 2$, the curve $z = z(t)$ is a part of a sectorial curve. If $1 < p < 2$, then $z = z(t)$ is a part of a sectorial curve provided that*

$$\sup_{t \in (-\varepsilon, \varepsilon)} |z'(t)| < \tan \frac{\pi}{2}(p-1).$$

In particular, for $p \geq 2$, the curves $z(t) = t + \imath |t|^{1+\varrho} \sin |t|^{\varrho}$ with $\varrho > 0$ and $z(t) = t + \imath t \sin \ln |t|$, the parameter t varying over $[-\varepsilon, \varepsilon]$, are parts of sectorial curves.

Example 2. *Consider the curve $z(t) = t + \imath t \sin t^{-1}$, where $|t| \leq \varepsilon$. Here, we readily get*

$$\arg z'(t) = \arg\left(1 + \imath\left(\sin\frac{1}{t} - \frac{1}{t}\cos\frac{1}{t}\right)\right)$$

whence

$$\sup_{t\in(-\delta,\delta)} |z'(t)| = \frac{\pi}{2}$$

for all $\delta > 0$. Hence, the discontinuity at point $z(0) = 0$ is not of the sectorial type (the curve oscillates rapidly at 0).

Dirichlet data on sectorial curves

Theorem 8. *Suppose $\mathcal{C}$ is p-sectorial for $1 < p < \infty$ and $w(\zeta) \equiv 1$, then the Toeplitz operator $T(a)$ corresponding to the Dirichlet problem is invertible.*

Proof. Recall that the symbol of the Toeplitz operator in question is as follows:

$$a(\zeta) = \exp\left(-\imath \frac{2}{p} \arg \mathfrak{c}'(\zeta)\right),$$

where $\arg \mathfrak{c}'(\zeta) = \alpha(\mathfrak{c}(\zeta)) - \arg \zeta - \frac{\pi}{2}$ for $\zeta \in \mathbb{T}$.

The idea of the proof is to represent the symbol in the form $a(\zeta) = c(\zeta)a_0(\zeta)$, where a_0 is p-sectorial and $c \in C(\mathbb{T})$ is such that $\operatorname{ind}_{c(\mathbb{T})}(0) = 0$. To this end, we first choose a continuous branch of the function $\arg \mathfrak{c}'(\zeta)$ on $\mathbb{T} \setminus \{\zeta_1, \ldots, \zeta_n\}$, where $z_k = \mathfrak{c}(\zeta_k)$ for $k = 1, \ldots, n$. Consider an arc (ζ_1, ζ_1^+) on $\mathbb{T}$ and take the branch of $\arg \mathfrak{c}'(\zeta)$ such that (19) holds for $k = 1$. Hence, it follows that the argument of $a(\zeta)$ satisfies the following:

$$\sup_{\zeta\in(\zeta_1,\zeta_1^+)} \left| -\frac{2}{p} \arg \mathfrak{c}'(\zeta) - \psi_1 \right| < \begin{cases} \frac{\pi}{p}, & \text{if } p \geq 2, \\ \frac{\pi}{q}, & \text{if } 1 < p < 2, \end{cases} \tag{20}$$

where

$$\psi_k = -\frac{2}{p}\varphi_k + \frac{2}{p} \arg \zeta_k + \frac{\pi}{p}$$

for $k = 1, \ldots, n$.

We then extend $\arg \mathfrak{c}'(\zeta)$ to a continuous function on the arc (ζ_1, ζ_2). Note that the right-hand side of (20) can be written as follows:

$$\frac{\pi}{\max\{p, q\}}$$

for all $1 < p < \infty$.

It is easy to see that there is an integer number j_2 with the property that

$$\sup_{\zeta\in(\zeta_2^-,\zeta_2)} \left| -\frac{2}{p} \arg \mathfrak{c}'(\zeta) - (\psi_2 + 2\pi j_2) \right| < \frac{\pi}{\max\{p, q\}}, \tag{21}$$

where ψ_2 is defined previously.

Choose the continuous branch of $\arg \mathfrak{c}'(\zeta)$ on (ζ_2, ζ_2^+), such that (21) is still valid with (ζ_2^-, ζ_2) replaced by (ζ_2, ζ_2^+).

We now extend $\arg \mathfrak{c}'(\zeta)$ to a continuous function on the arc (ζ_2, ζ_3), and so on. On proceeding in this fashion, we get a continuous branch of $\arg \mathfrak{c}'(\zeta)$ on all of (ζ_n, ζ_1) satisfying the following:

$$\sup_{\zeta\in(\zeta_1^-,\zeta_1)} \left| -\frac{2}{p} \arg \mathfrak{c}'(\zeta) - (\psi_1 + 2\pi j_1) \right| < \frac{\pi}{\max\{p, q\}} \tag{22}$$

with some integer j_1.

The task is now to show that $j_1 = 0$, so the inequality (20) actually holds with (ζ_1, ζ_1^+) replaced by $(\zeta_1^-, \zeta_1^+) \setminus \{\zeta_1\}$. For this purpose, we link any two points z_k^- and z_k^+ together by a smooth curve $\mathcal{A}_k$, such that

1. $\tilde{\mathcal{C}} = \left((z_1^+, z_2^-) \cup \ldots \cup (z_n^+, z_1^-)\right) \cup (\mathcal{A}_1 \cup \ldots \cup \mathcal{A}_n)$ is a smooth closed curve which bounds a simply connected domain $\tilde{\mathcal{D}}$.
2. The angle $\tilde{\alpha}(z)$ at which the tangent of $\tilde{\mathcal{C}}$ at the point z intersects the real axis satisfies (19).

Consider a conformal mapping $z = \tilde{\mathfrak{c}}(\zeta)$ of $\mathbb{D}$ onto $\tilde{\mathcal{D}}$. By the very construction, $\tilde{\alpha}(z) = \alpha(z)$ holds for all $z \in (z_1^+, z_2^-) \cup \ldots \cup (z_n^+, z_1^-)$. Suppose (22) is valid with $j_1 \neq 0$, then, in particular,

$$\left| -\frac{2}{p} \arg \tilde{\mathfrak{c}}'(\tilde{\zeta}_1^-) - (\psi_1 + 2\pi j_1) \right| < \frac{\pi}{\max\{p, q\}}$$

where $z_1^- = \tilde{\mathfrak{c}}(\tilde{\zeta}_1^-)$. From this, we deduce that the function $\arg \tilde{\mathfrak{c}}'(\zeta)$ has a nonzero increment (equal to $2\pi j_1$) when the point ζ makes one turn along the unit circle $\mathbb{T}$ starting from the point $\tilde{\zeta}_1^+$ with $z_1^+ = \tilde{\mathfrak{c}}(\tilde{\zeta}_1^+)$. Hence, it follows, by the argument principle, that the function $\tilde{\mathfrak{c}}'$ has zeros in $\mathbb{D}$, which contradicts the conformality of $\tilde{\mathfrak{c}}$. Thus, $j_1 = 0$ in (22).

We have thus chosen a continuous branch of the function $\arg \mathfrak{c}'(\zeta)$ on the set $\mathbb{T} \setminus \{\zeta_1, \ldots, \zeta_n\}$, satisfying

$$\sup_{\zeta\in(\zeta_k^-,\zeta_k^+)\setminus\{\zeta_k\}} \left| -\frac{2}{p} \arg \mathfrak{c}'(\zeta) - (\psi_k + 2\pi j_k) \right| < \frac{\pi}{\max\{p, q\}} \tag{23}$$

for all $k = 1, \ldots, n$, where j_k is integer and $j_1 = 0$. This allows one to construct the desired factorisation of $a(\zeta)$.

We first define $c(\zeta)$ away from the arcs $\left(\zeta_k^-, \zeta_k^+\right)$ which encompass singular points ζ_k of $\mathfrak{c}'(\zeta)$. Namely, we set

$$c(\zeta) := \exp\left(-\imath \frac{2}{p} \arg \mathfrak{c}'(\zeta)\right)$$

for $\zeta \in \mathbb{T} \setminus \bigcup_{k=1}^{n} (\zeta_k^-, \zeta_k^+)$.

To define $c(\zeta)$ in any arc (ζ_k^-, ζ_k^+) with $k = 1, \ldots, n$, we pick an $\varepsilon_k > 0$ small enough so that $\arg \zeta_k^- + \varepsilon_k <$

$\arg \zeta_k < \arg \zeta_k^+ - \varepsilon_k$. The symbol $c(\zeta)$ is then defined by the following:

$$c(\zeta) := \exp\left(-\imath \frac{2}{p} \frac{(\arg \zeta_k^- + \varepsilon_k - \arg \zeta) \arg \mathfrak{c}'(\zeta_k^-) + (\arg \zeta - \arg \zeta_k^-) \tilde{\varphi}_k}{\varepsilon_k}\right),$$

if $\zeta \in (\zeta_k^-, e^{\imath \varepsilon_k} \zeta_k^-]$,

$$c(\zeta) := \exp\left(-\imath \frac{2}{p} \tilde{\varphi}_k\right),$$

if $\zeta \in (e^{\imath \varepsilon_k} \zeta_k^-, e^{-\imath \varepsilon_k} \zeta_k^+)$, and

$$c(\zeta) := \exp\left(-\imath \frac{2}{p} \frac{(\arg \zeta_k^+ - \arg \zeta) \tilde{\varphi}_k + (\arg \zeta - \arg \zeta_k^+ + \varepsilon_k) \arg \mathfrak{c}'(\zeta_k^+)}{\varepsilon_k}\right),$$

if $\zeta \in [e^{-\imath \varepsilon_k} \zeta_k^+, \zeta_k^+)$. Here, $\tilde{\varphi}_k = -\frac{p}{2}(\psi_k + 2\pi j_k)$.

Obviously, $c(\zeta)$ is a non-vanishing continuous function of $\zeta \in \mathbb{T}$.

From (23) it follows that $\operatorname{ind}_{c(\mathbb{T})}(0) = 0$. Put

$$a_0(\zeta) := \frac{a(\zeta)}{c(\zeta)}$$

for $\zeta \in \mathbb{T}$, and then

$$\arg a_0(\zeta) = 0$$

for all $\zeta \in \mathbb{T} \setminus \bigcup_{k=1}^{n} (\zeta_k^-, \zeta_k^+)$. Moreover, if the numbers $\varepsilon_1, \ldots, \varepsilon_n$ are small enough, then

$$\sup_{\zeta \in (\zeta_k^-, \zeta_k^+)} |\arg a_0(\zeta)| \le \frac{\pi}{\max\{p, q\}}$$

for all $k = 1, \ldots, n$. Hence, $a_0(\zeta)$ is a p-sectorial symbol, which yields the desired factorisation.

By Theorem 5(1), we conclude that the Toeplitz operator $T(a_0)$ is invertible in the space H^{p+}. Moreover, Theorem 6(2) shows that $T(a)$ is Fredholm of index zero. Finally, Theorem 3 implies that the operator $T(a)$ is actually invertible, as desired. □

Corollary 1 gives the solution of the Dirichlet problem in $\mathcal{D}$ via the inverse operator $T(a)^{-1}$. If $a(\zeta)$ admits a p-factorisation, then Corollary 2 yields an explicit formula for $T(a)^{-1}$. In case the boundary of $\mathcal{D}$ is a sectorial curve, it is possible to construct a p-factorisation of $a(\zeta)$ with the help of conformal mapping $z = \mathfrak{c}(\zeta)$.

Theorem 9. *Let $\mathcal{C}$ be a p-sectorial curve; then the Dirichlet problem has a unique solution $u = \Re f$ with $f \in E^p(\mathcal{D})$, and this solution is of the following form:*

$$u(z) = \Re \int_{\mathbb{T}} \frac{1}{2\pi \imath} \frac{\zeta + \mathfrak{c}^{-1}(z)}{\zeta - \mathfrak{c}^{-1}(z)} u_0(\mathfrak{c}(\zeta)) \frac{d\zeta}{\zeta}$$

for $z \in \mathcal{D}$.

Proof. According to Theorems 8 and 2, a p-factorisation of the symbol of Toeplitz operator corresponding to the Dirichlet problem in a domain with p-sectorial boundary, if there is any, looks like $a(\zeta) = a^+(\zeta) a^-(\zeta)$. We begin with the following representation:

$$a(\zeta) = \left(\frac{\overline{\mathfrak{c}'(\zeta)}}{\mathfrak{c}'(\zeta)}\right)^{1/p}$$

for $\zeta \in \mathbb{T}$, *cf.* (10).

In the case of p-sectorial curves, the angle $\alpha(z)$ is bounded, so the curve $\mathcal{C}$ is rectifiable. By a well-known result (see for instance [24]), the derivative $\mathfrak{c}'(\zeta)$ belongs to H^{1+}, whence $\sqrt[p]{\mathfrak{c}'(\zeta)} \in H^{p+}$ and $\overline{\sqrt[p]{\mathfrak{c}'(\zeta)}} \in H^{p-} \oplus \{c\}$. Comparing this with $a(\zeta) = a^+(\zeta) a^-(\zeta)$ we get the following:

$$\sqrt[p]{\mathfrak{c}'(\zeta)} a^+(\zeta) = \overline{\sqrt[p]{\mathfrak{c}'(\zeta)}} (1/a^-(\zeta)).$$

By (15), the left-hand side of this equality belongs to H^{1+}; the right-hand side, to $H^{1-} \oplus \{c\}$. Hence, it follows that

$$\begin{aligned} \sqrt[p]{\mathfrak{c}'(\zeta)} a^+(\zeta) &= c, \\ \overline{\sqrt[p]{\mathfrak{c}'(\zeta)}} (1/a^-(\zeta)) &= c \end{aligned}$$

where c is a complex constant. The factorisation $a(\zeta) = a^+(\zeta) a^-(\zeta)$ with

$$\begin{aligned} a^+(\zeta) &= c \left(1/\sqrt[p]{\mathfrak{c}'(\zeta)}\right), \\ a^-(\zeta)) &= \tfrac{1}{c} \overline{\sqrt[p]{\mathfrak{c}'(\zeta)}} \end{aligned}$$

satisfies (15), and $(T(a))^{-1} = \sqrt[p]{\mathfrak{c}'(\zeta)} P_{\mathbb{T}}^+ (1/\overline{\sqrt[p]{\mathfrak{c}'(\zeta)}})$, which is due to Corollary 2. This establishes the theorem when combined with the formula of Corollary 1. We fill in details.

We first observe that, according to Remark 3, condition (13) is fulfilled. Hence, we may use the formula of Corollary 1. Set

$$N(\zeta) := \left(T(a)^{-1} g^+\right)(\zeta) - \frac{1}{2} \left(T(a)^{-1} g^+\right)(0) (T(a)^{-1} 1)(\zeta)$$

for $\zeta \in \mathbb{T}$. An easy computation shows that

$$\begin{aligned} N(\zeta) &= \sqrt[p]{\mathfrak{c}'(\zeta)} P_{\mathbb{T}}^+ \left(2u_0(\mathfrak{c}(\zeta))\right)(\zeta) \\ &\quad - \sqrt[p]{\mathfrak{c}'(0)} P_{\mathbb{T}}^+ \left(u_0(\mathfrak{c}(\zeta))\right)(0) \sqrt[p]{\mathfrak{c}'(\zeta)} (1/\overline{\sqrt[p]{\mathfrak{c}'(0)}}) \\ &= \sqrt[p]{\mathfrak{c}'(\zeta)} P_{\mathbb{T}}^+ \left(2u_0(\mathfrak{c}(\zeta))\right)(\zeta) \\ &\quad - P_{\mathbb{T}}^+ \left(u_0(\mathfrak{c}(\zeta))\right)(0) \sqrt[p]{\mathfrak{c}'(\zeta)} \end{aligned}$$

holds for almost all $\zeta \in \mathbb{T}$. On writing the projection $P_{\mathbb{T}}^+$ as the Cauchy integral, we get the following:

$$\begin{aligned} N(\zeta) &= \frac{\sqrt[p]{\mathfrak{c}'(\zeta)}}{2\pi \imath} \int_{\mathbb{T}} \left(\frac{2u_0(\mathfrak{c}(\zeta'))}{\zeta' - \zeta} - \frac{u_0(\mathfrak{c}(\zeta'))}{\zeta'}\right) d\zeta' \\ &= \sqrt[p]{\mathfrak{c}'(\zeta)} \int_{\mathbb{T}} \frac{1}{2\pi \imath} \frac{\zeta' + \zeta}{\zeta' - \zeta} u_0(\mathfrak{c}(\zeta')) \frac{d\zeta'}{\zeta'} \end{aligned}$$

for all $\zeta \in \mathbb{D}$. Since

$$u(z) = \Re \frac{N(\mathfrak{c}^{-1}(z))}{\sqrt[p]{\mathfrak{c}'(\mathfrak{c}^{-1}(z))}},$$

the proof is complete. □

Dirichlet data on logarithmic spirals

Let Σ_a be a horizontal half-strip of the form $\Sigma_a = \{\mathfrak{z} \in \mathbb{C} : \Re\mathfrak{z} > 0,\ \Im\mathfrak{z} \in (0,a)\}$, with a being a positive number. Consider the mapping

$$z = \ell_\varphi(\mathfrak{z}) := \exp\left(-e^{\imath\varphi}\mathfrak{z}\right)$$

of the half-strip into the complex plane $\mathbb{C}_z$, where $\varphi \in \left(-\frac{\pi}{2}, 0\right) \cup \left(0, \frac{\pi}{2}\right)$.

A direct computation shows that the mapping $z = \ell_\varphi(\mathfrak{z})$ is conformal if and only if $a < 2\pi\ \cos\varphi$.

For $\upsilon \in [0,a]$, set $\mathcal{R}_\upsilon = \{\mathfrak{z} \in \mathbb{C} : \Re\mathfrak{z} \geq 0,\ \Im\mathfrak{z} = \upsilon\}$, then the curve $\mathcal{S}_{\upsilon,\varphi} := \ell_\varphi(\mathcal{R}_\upsilon)$ is a spiral. Indeed, if $r = |z|$ and $\vartheta = \arg z$, then any point $z \in \mathcal{S}_{\upsilon,\varphi}$ is characterised by the following:

$$\begin{cases} r = \exp(-\Re\mathfrak{z}\ \cos\varphi + \upsilon\sin\varphi), \\ \vartheta = -\Re\mathfrak{z}\ \sin\varphi - \upsilon\cos\varphi. \end{cases}$$

Hence, it follows that $\mathcal{S}_{\upsilon,\varphi}$ can be described by the following equation:

$$r = \left(\exp\frac{\upsilon}{\sin\varphi}\right)\exp\left(\vartheta\ \cot\varphi\right), \tag{24}$$

where ϑ runs over $(-\upsilon\cos\varphi, +\infty)$, if $\varphi \in (-\pi/2, 0)$, and over $(-\infty, -\upsilon\cos\varphi)$, if $\varphi \in (0, \pi/2)$.

Denote by $\mathcal{D}_{a,\varphi}$ the image of Σ_a by ℓ_φ. This is a domain in the z-plane whose boundary is the following composite curve:

$$\mathcal{C}_{a,\varphi} := \mathcal{S}_{0,\varphi} \cup \mathcal{S}_{a,\varphi} \cup b_{a,\varphi},$$

where $\mathcal{S}_{0,\varphi}$ and $\mathcal{S}_{a,\varphi}$ are given by (24) and the arc $b_{a,\varphi}$ by $z = \exp(-e^{\imath\varphi}\imath\upsilon)$ with $\upsilon \in [0,a]$. Thus, $z = \ell_\varphi(\mathfrak{z})$ is a conformal mapping of Σ_a onto $\mathcal{D}_{a,\varphi}$ which transforms the boundary of Σ_a onto $\mathcal{C}_{a,\varphi}$.

It is easily seen that $\mathcal{C}_{a,\varphi}$ is a rectifiable curve. Indeed, the arc length of $\mathcal{S}_{\upsilon,\varphi}$ can be evaluated by the following formula:

$$L = \int_{\vartheta_1}^{\vartheta_2} \sqrt{(r(\vartheta))^2 + (r'(\vartheta))^2}\, d\vartheta.$$

We assume for definiteness that $\varphi \in (-\pi/2, 0)$, and then

$$\begin{aligned} L(\mathcal{S}_{\varphi,a}) &= \int_{-a\cos\varphi}^{\infty} r(\vartheta)\sqrt{1 + (\cot\varphi)^2}\, d\vartheta \\ &= \frac{\exp\frac{a}{\sin\varphi}}{|\sin\varphi|}\int_{-a\cos\varphi}^{\infty} \exp\left(\vartheta\ \cot\varphi\right) d\vartheta \\ &= \frac{\exp\frac{a}{\sin\varphi}}{\cos\varphi}\ \exp\left(-a\cos\varphi\ \cot\varphi\right) \end{aligned}$$

is finite, as desired.

Our next objective is to find a conformal mapping of $\mathbb{D}$ onto $\mathcal{D}_{a,\varphi}$. To this end, we compose three well-known conformal mappings:

1. $\eta = \frac{1}{\imath}\frac{\zeta+1}{\zeta-1}$ maps the unit disk $\mathbb{D}$ conformally onto the upper half-plane $\mathbb{H} := \{\eta \in \mathbb{C} : \Im\eta > 0\}$.
2. $\theta = \eta + \sqrt{\eta^2 - 1}$ maps $\mathbb{H}$ conformally onto the complement in $\mathbb{H}$ of the closed unit disk $\mathbb{H} \setminus \overline{\mathbb{D}}$, the branch of $\sqrt{\eta^2 - 1}$ being chosen according to the condition $\Im\sqrt{\eta^2 - 1} \geq 0$.
3. $\mathfrak{z} = \frac{a}{\pi}\ \ln\theta$ maps $\mathbb{H} \setminus \overline{\mathbb{D}}$ conformally onto the strip Σ_a.

In this way, we arrive at the conformal mapping $z = \mathfrak{c}_{a,\varphi}(\zeta)$ of $\mathbb{D}$ onto $\mathcal{D}_{a,\varphi}$ given by the following:

$$\mathfrak{c}_{a,\varphi}(\zeta) = \exp\left(-e^{\imath\varphi}\,\frac{a}{\pi}\,\ln\left(\eta + \sqrt{\eta^2 - 1}\right)\right). \tag{25}$$

With $\eta = \eta(\zeta)$, we get the following:

$$\begin{aligned} \mathfrak{c}'_{a,\varphi}(\zeta) &= -e^{\imath\varphi}\,\frac{a}{\pi}\,\frac{1 + \frac{\eta}{\sqrt{\eta^2-1}}}{\eta + \sqrt{\eta^2 - 1}}\,\eta'(\zeta)\,\mathfrak{c}_{a,\varphi}(\zeta) \\ &= e^{\imath\varphi}\,\frac{a}{\pi}\frac{2}{\imath}\,\frac{1}{\sqrt{\eta^2 - 1}}\,\frac{1}{(\zeta - 1)^2}\,\mathfrak{c}_{a,\varphi}(\zeta) \end{aligned}$$

for all $\zeta \in \mathbb{D}$. Note that $\eta(\imath) = -1$, $\eta(-\imath) = 1$ and

$$\arg\frac{1}{(\zeta - 1)^2} = \pi - \arg\zeta$$

for all $\zeta \in \mathbb{T}$ different from 1, where the equality is understood modulo entire multiples of 2π.

Since $\eta(\zeta)$ runs over $(-\infty, -1]$, if $\zeta \in (1, \imath]$, over $[-1, 1]$, if $\zeta \in [\imath, -\imath]$, and over $[1, \infty]$, if $\zeta \in [-\imath, 1)$, it follows that

$$\arg\mathfrak{c}'_{a,\varphi}(\zeta) = \begin{cases} \varphi - \frac{\pi}{2} - \arg\zeta - a\cos\varphi - \frac{a}{\pi}\sin\varphi\ \ln|\eta + \sqrt{\eta^2-1}|, & \text{if } \zeta \in (1, \imath), \\ \varphi - \arg\zeta - \frac{a}{\pi}\cos\varphi\ \arccos\eta, & \text{if } \zeta \in [\imath, -\imath], \\ \varphi + \frac{\pi}{2} - \arg\zeta - \frac{a}{\pi}\sin\varphi\ \ln(\eta + \sqrt{\eta^2-1}), & \text{if } \zeta \in (-\imath, 1). \end{cases} \tag{26}$$

The second equality is due to the fact that $|\eta(\zeta)+\sqrt{(\eta(\zeta))^2-1}|=1$ holds for all $\zeta\in[\imath,-\imath]$.

Having disposed of this preliminary step, we introduce the Toeplitz operator $T(a_{a,\varphi})$ with the following symbol:

$$a_{a,\varphi}(\zeta)=\exp\left(-\imath\frac{2}{p}\arg\mathfrak{c}'_{a,\varphi}(\zeta)\right),$$

cf. (10). This operator is responsible for the solvability of the Dirichlet problem in the space $\Re E^p(\mathcal{D}_{a,\varphi})$.

Theorem 10. *The operator $T(a_{a,\varphi})$ in H^{p+} with $1<p<\infty$ is Fredholm if and only if*

$$p\neq\frac{a}{\pi}\cos\varphi.$$

If $p>\frac{a}{\pi}\cos\varphi$, then $T(a_{a,\varphi})$ is invertible in the space H^{p+}. If $p<\frac{a}{\pi}\cos\varphi$, then $\operatorname{ind}T(a_{a,\varphi})=1$.

Proof. The following function is introduced:

$$h_{a,\varphi}(\zeta):=\exp\left(-\imath\frac{2}{p}\left(\varphi-\frac{a}{\pi}\sin\varphi\,\ln(\eta+\sqrt{\eta^2-1})\right)\right),$$

which obviously belongs to $GH^{\infty+}$. Indeed,

$$|h_{a,\varphi}(\zeta)|=\begin{cases}\exp\left(-\frac{2}{p}a\sin\varphi\right), & \text{if } \zeta\in(1,\imath),\\ 1, & \text{if } \zeta\in(-\imath,1),\end{cases}$$

and $h_{a,\varphi}(\zeta)$ is continuous at each point of the arc $[\imath,-\imath]\subset\mathbb{T}$. Let us consider the following quotient:

$$a_0(\zeta)=\frac{a_{a,\varphi}(\zeta)}{h_{a,\varphi}(\zeta)}$$

for $\zeta\in\mathbb{T}$.

An easy computation shows that

$$\arg a_0(\zeta)=\begin{cases}\frac{2}{p}\left(\frac{\pi}{2}+\arg\zeta+a\cos\varphi\right), & \text{if } \zeta\in(1,\imath),\\ \frac{2}{p}\left(\arg\zeta+\frac{a}{\pi}\cos\varphi\,\arccos\eta\right), & \text{if } \zeta\in[\imath,-\imath],\\ \frac{2}{p}\left(-\frac{\pi}{2}+\arg\zeta\right), & \text{if } \zeta\in(-\imath,1).\end{cases}$$

Hence, $a_0(\zeta)$ is a PC function with discontinuity points $\{1,\imath,-\imath\}$. One verifies readily that

$$\begin{aligned}\arg a_0(e^{\imath 0-}\,1)&=\frac{2}{p}\frac{3}{2}\pi,\\ \arg a_0(e^{\imath 0+}\,1)&=\frac{2}{p}\left(\frac{\pi}{2}+a\cos\varphi\right),\\ \arg a_0(e^{\imath 0-}\,\imath)&=\frac{2}{p}(\pi+a\cos\varphi),\\ \arg a_0(e^{\imath 0+}\,\imath)&=\frac{2}{p}\left(\frac{\pi}{2}+a\cos\varphi\right),\\ \arg a_0(e^{\imath 0+}(-\imath))&=\frac{2}{p}\frac{3}{2}\pi,\\ \arg a_0(e^{\imath 0+}(-\imath))&=\frac{2}{p}\pi.\end{aligned}$$

We thus conclude that $a_0(\zeta)$ possesses a representation (17) with $\zeta_1=1$, $\zeta_2=\imath$, $\zeta_3=-\imath$ and

$$\mathfrak{f}_1=\frac{1}{p}\left(1-\frac{a}{\pi}\cos\varphi\right),\quad \mathfrak{f}_2=\mathfrak{f}_3=\frac{1}{2}\frac{1}{p}.$$

Observe that

$$-\frac{1}{q}<\mathfrak{f}_2=\mathfrak{f}_3<\frac{1}{p}$$

for all $p\in(1,\infty)$. Therefore, we may apply Theorem 7 with $\kappa_2=\kappa_3=0$. Since we always have

$$\mathfrak{f}_1<\frac{1}{p},$$

the following cases may occur:

1. If
$$-\frac{1}{q}<\mathfrak{f}_1<\frac{1}{p},$$
then $\kappa_1=0$. By Theorem 7, the operator $T(a_0)$ is Fredholm, so the operator $T(a_{a,\varphi})$ is Fredholm, too, which is due to Theorem 6(1). The inequality $-1/q<\mathfrak{f}_1$ can be rewritten in the following form:
$$1-p<1-\frac{a}{\pi}\cos\varphi,$$
which just amounts to $p>\frac{a}{\pi}\cos\varphi$.
2. If
$$-1-\frac{1}{q}<\mathfrak{f}_1<-1+\frac{1}{p}=-\frac{1}{q},$$
then $\kappa_1=-1$. Since $a<2\pi\cos\varphi$, the left inequality is automatically fulfilled, so the entire inequality reduces to the following:
$$\mathfrak{f}_1<-1+\frac{1}{p},$$
i.e. $p<\frac{a}{\pi}\cos\varphi$.
3. Obviously,
$$p=\frac{a}{\pi}\cos\varphi$$
if and only if $\mathfrak{f}_1=-1/q$. In this case, there is no entire number κ_1 with the property that
$$\kappa_1-\frac{1}{q}<\mathfrak{f}_1<\kappa_1+\frac{1}{p}.$$

By Theorem 7, $T(a_0)$ is not Fredholm. From $a_0=\frac{a_{a,\varphi}}{h_{a,\varphi}}$, it follows that $T(a_{a,\varphi})$ is not Fredholm.

We now suppose that $T(a_0)$ is Fredholm. For the symbol a_0, we get then a representation (17) with $\zeta_1=1$, $\zeta_2=\imath$ and $\zeta_3=-\imath$ and $\mathfrak{f}_k$ specified previously. More precisely,

$$a_0(\zeta)=c(\zeta)\prod_{k=1}^{3}a_{\mathfrak{f}_k,\zeta_k}(\zeta),$$

where $c \in C(\mathbb{T})$. Moreover, the winding number of the cycle $c(\mathbb{T})$ with respect to the origin vanishes. Hence, Theorem 7 applies to the operator $T(a_0)$; in particular, the index of $T(a_0)$ is evaluated by formula (18). To complete the proof, it suffices to use Theorem 6. □

Theorem 10 allows one to construct explicit formulas for solutions of the Dirichlet problem in domains bounded by logarithmic spirals.

Theorem 11. *Let $z = \mathfrak{c}_{a,\varphi}(\zeta)$ be the conformal mapping of $\mathbb{D}$ onto $\mathcal{D}_{a,\varphi}$ given by (25).*

(1) *If $p > \frac{a}{\pi} \cos\varphi$, then the Dirichlet problems (1) and (2) have a unique solution in $\Re E^p(\mathcal{D}_{a,\varphi})$ given by the following:*

$$u(z) = \Re \int_{\mathbb{T}} \frac{1}{2\pi\imath} \frac{\zeta + \mathfrak{c}^{-1}_{a,\varphi}(z)}{\zeta - \mathfrak{c}^{-1}_{a,\varphi}(z)} u_0(\mathfrak{c}_{a,\varphi}(\zeta)) \frac{d\zeta}{\zeta}$$

for $z \in \mathcal{D}_{a,\varphi}$.

(2) *If $p < \frac{a}{\pi} \cos\varphi$, then the Dirichlet problems (1) and (2) have infinitely many solutions in $\Re E^p(\mathcal{D}_{a,\varphi})$ given by the following:*

$$u(z) = \Re \left(\frac{c + \bar{c}\, \mathfrak{c}^{-1}_{a,\varphi}(z)}{1 - \mathfrak{c}^{-1}_{a,\varphi}(z)} - \frac{\mathfrak{c}^{-1}_{a,\varphi}(z)}{1 - \mathfrak{c}^{-1}_{a,\varphi}(z)} \times \int_{\mathbb{T}} \frac{1}{\pi\imath} \frac{\zeta - 1}{\zeta - \mathfrak{c}^{-1}_{a,\varphi}(z)} u_0(\mathfrak{c}_{a,\varphi}(\zeta)) \frac{d\zeta}{\zeta} \right)$$

for $z \in \mathcal{D}_{a,\varphi}$, where c is an arbitrary complex constant.

Proof. Let $p > \frac{a}{\pi} \cos\varphi$. By Theorem 10, the operator $T(a_{a,\varphi})$ is invertible in the space H^{p+}. Furthermore, Theorem 2 ensures the existence of p-factorisation of the following form:

$$a_{a,\varphi}(\zeta) = a^+(\zeta) a^-(\zeta),$$

where $a^{\pm}(\zeta)$ bear properties (15). On the other hand, we have the following:

$$a_{a,\varphi}(\zeta) = \frac{\sqrt[p]{\overline{\mathfrak{c}'_{a,\varphi}(\zeta)}}}{\sqrt[p]{\mathfrak{c}'_{a,\varphi}(\zeta)}}, \tag{27}$$

cf. (10).

Rewrite $\mathfrak{c}'_{a,\varphi}(\zeta)$ in the following form:

$$\mathfrak{c}'_{a,\varphi}(\zeta) = e^{\imath\varphi} \frac{a}{\pi} \frac{\sqrt{2}}{\sqrt{\zeta^2 + 1}} \frac{1}{1 - \zeta} \mathfrak{c}_{a,\varphi}(\zeta)$$

for $\zeta \in \mathbb{T}$. Close to either of the points 1, $\imath$ and $-\imath$, one derives easily the following asymptotic relations:

$$\begin{aligned} |\mathfrak{c}'_{a,\varphi}(\zeta)| &\sim C\,|\zeta - 1|^{-1+\frac{a}{\pi}\cos\varphi} && \text{as } \zeta \to e^{\imath 0\pm}, \\ |\mathfrak{c}'_{a,\varphi}(\zeta)| &\sim C\,|\zeta - \imath|^{-\frac{1}{2}} && \text{as } \zeta \to e^{\imath 0\pm}\imath, \\ |\mathfrak{c}'_{a,\varphi}(\zeta)| &\sim C\,|\zeta + \imath|^{-\frac{1}{2}} && \text{as } \zeta \to e^{\imath 0\pm}(-\imath). \end{aligned} \tag{28}$$

On taking into account the estimates $a < 2\pi\cos\varphi$ and $p > (a/\pi)\cos\varphi$, one sees that the factorisation of $a_{a,\varphi}$ given by (27) satisfies (15). By Remark 2, equality (27) is actually a p-factorisation of $a_{a,\varphi}$ with $\kappa = 0$.

Analysis similar to that in the proof of Theorem 9 completes now the proof of part (1).

Suppose $p < \frac{a}{\pi}\cos\varphi$. By Theorem 10, the p-factorisation of the symbol $a_{a,\varphi}$ is of the following form:

$$a_{a,\varphi}(\zeta) = a^+(\zeta)\, \zeta^{-1}\, a^-(\zeta),$$

where $a^{\pm}(\zeta)$ satisfy (15). Therefore, (27) fails to be a p-factorisation of this symbol. We correct it in the following manner. Set

$$\tilde{a}^+(\zeta) = \frac{1 - \zeta}{\sqrt[p]{\mathfrak{c}'_{a,\varphi}(\zeta)}},$$

$$\tilde{a}^-(\zeta) = \frac{\sqrt[p]{\overline{\mathfrak{c}'_{a,\varphi}(\zeta)}}}{1 - \frac{1}{\zeta}},$$

then

$$a_{a,\varphi}(\zeta) = -\tilde{a}^+(\zeta)\, \zeta^{-1}\, \tilde{a}^-(\zeta)$$

is a p-factorisation of $a_{a,\varphi}$. To show this, it suffices to establish (15) in a neighbourhood of the point $\zeta = 1$.

From (28), it follows that

$$\begin{aligned} |\left(\tilde{a}^+(\zeta)\right)^{\pm 1}| &\sim |\zeta - 1|^{\pm\epsilon} \text{ as } \zeta \to 1, \\ |\left(\tilde{a}^-(\zeta)\right)^{\pm 1}| &\sim |\zeta - 1|^{\mp\epsilon} \text{ as } \zeta \to 1, \end{aligned}$$

where $\epsilon = \frac{1}{p}\left(1 - \frac{a}{\pi}\cos\varphi\right) + 1$. Since $a < 2\pi\cos\varphi$, we get the following:

$$q\epsilon > \frac{q}{p}(1 - 2) + q = q\left(1 - \frac{1}{p}\right) = 1,$$

whence $\tilde{a}^+ \in H^{q+}$. On the other hand, from $p < \frac{a}{\pi}\cos\varphi$ we deduce that

$$-p\epsilon = -\left(1 - \frac{a}{\pi}\cos\varphi\right) - p > -(1 - p) - p = -1,$$

whence $(\tilde{a}^+)^{-1} \in H^{p+}$. Similarly, we obtain the following:

$$\begin{aligned} \tilde{a}^- &\in H^{p-} \oplus \{c\}, \\ (\tilde{a}^-)^{-1} &\in H^{q-} \oplus \{c\}. \end{aligned}$$

Consider Equation 11:

$$T(a_{a,\varphi})(h^+)(\zeta) + \overline{h^+(0)} = g^+(\zeta),$$

for $\zeta \in \mathbb{T}$, *cf.* (11). Find all solutions of this equation and choose among them those solutions which give rise to solutions of our Dirichlet problem. In the case under study, the operator $T(a_{a,\varphi})$ has a right inverse of the following form:

$$(T(a_{a,\varphi}))_r^{-1} = -\zeta \, \frac{1}{\tilde{a}^+} P_{\mathbb{T}}^+ \frac{1}{\tilde{a}^-};$$

see Theorem 2.1 in [22] and elsewhere. Moreover, the general solution of (11) proves to be as follows:

$$\begin{aligned} h^+(\zeta) &= -\,(T(a_{a,\varphi}))_r^{-1} \left(h^+(0) - g^+\right)(\zeta) + c\, \frac{1}{\tilde{a}^+(\zeta)} \\ &= \overline{h^+(0)}\, \frac{\zeta}{\tilde{a}^+(\zeta)} P_{\mathbb{T}}^+ \left(\frac{1}{\tilde{a}^-}\right)(\zeta) \\ &\quad - \frac{\zeta}{\tilde{a}^+(\zeta)} P_{\mathbb{T}}^+ \left(\frac{g^+}{\tilde{a}^-}\right)(\zeta) + \frac{c}{\tilde{a}^+(\zeta)} \\ &= \overline{h^+(0)}\, \frac{\zeta}{\tilde{a}^+(\zeta)} \frac{1}{\sqrt[p]{\mathfrak{c}'_{a,\varphi}(0)}} \\ &\quad - \frac{\zeta}{\tilde{a}^+(\zeta)} P_{\mathbb{T}}^+ \left(\frac{g^+}{\tilde{a}^-}\right)(\zeta) + \frac{c}{\tilde{a}^+(\zeta)}, \end{aligned} \tag{29}$$

where c is an arbitrary complex constant. Put $\zeta = 0$ in (29). Since the right inverse $(T(a_{a,\varphi}))_r^{-1}$ maps H^{p+} to functions which vanish at the origin, we immediately get the following:

$$h^+(0) = c\, \sqrt[p]{\mathfrak{c}'_{a,\varphi}(0)}.$$

Thus, (29) yields the following:

$$\begin{aligned} h^+(\zeta) &= \frac{\bar{c}\,\zeta + c}{\tilde{a}^+(\zeta)} - \frac{\zeta}{\tilde{a}^+(\zeta)} P_{\mathbb{T}}^+ \left(\frac{g^+}{\tilde{a}^-}\right)(\zeta) \\ &= \left(\frac{c + \bar{c}\,\zeta}{1-\zeta} - \frac{\zeta}{1-\zeta}\, P_{\mathbb{T}}^+ \left(\frac{g^+}{\tilde{a}^-}\right)(\zeta)\right) \sqrt[p]{\mathfrak{c}'_{a,\varphi}(\zeta)}. \end{aligned}$$

Since

$$P_{\mathbb{T}}^+ \left(\frac{g^+}{\tilde{a}^-}\right)(\zeta) = P_{\mathbb{T}}^+ \left(\frac{g}{\tilde{a}^-}\right)(\zeta) = P_{\mathbb{T}}^+ \left(\frac{\zeta - 1}{\zeta}\, 2U_0\right)(\zeta)$$

is the limit value for almost all $\zeta \in \mathbb{T}$ of the Cauchy integral

$$\int_{\mathbb{T}} \frac{1}{2\pi\imath} \frac{1}{\zeta' - \zeta} \frac{\zeta' - 1}{\zeta'}\, 2u_0(\mathfrak{c}(\zeta'))\, d\zeta',$$

it follows that

$$\begin{aligned} u(\mathfrak{c}(\zeta)) &= \Re\, \frac{h^+(\zeta)}{\sqrt[p]{\mathfrak{c}'_{a,\varphi}(\zeta)}} \\ &= \Re \left(\frac{c + \bar{c}\,\zeta}{1-\zeta} - \frac{\zeta}{1-\zeta} \int_{\mathbb{T}} \frac{1}{\pi\imath} \frac{\zeta' - 1}{\zeta' - \zeta}\, u_0(\mathfrak{c}(\zeta'))\, \frac{d\zeta'}{\zeta'}\right) \end{aligned} \tag{30}$$

for all $\zeta \in \mathbb{D}$.

Now, let $\zeta \in \mathbb{T}$. A trivial verification shows that

$$\Re\, \frac{c + \bar{c}\,\zeta}{1-\zeta} = 0$$

for any complex constant c, provided that $\zeta \neq 1$. Moreover, since the function U_0 is real-valued, we obtain the following:

$$\Re \left(\frac{\zeta}{\zeta - 1}\, P_{\mathbb{T}}^+ \left(\frac{\zeta - 1}{\zeta}\, 2U_0\right)(\zeta)\right) = U_0(\zeta).$$

Hence, it follows that (30) is a general solution of Dirichlet problems (1) and (2), as desired. □

The inverse mapping of $\mathfrak{c}_{a,\varphi} : \mathbb{D} \to \mathcal{D}_{a,\varphi}$ is given explicitly by the following:

$$\mathfrak{c}_{a,\varphi}^{-1}(z) = \frac{\cosh(c \ln z) - \imath}{\cosh(c \ln z) + \imath},$$

where $c = \left(\frac{a}{\pi} e^{\imath\varphi}\right)^{-1}$.

The condition $a < 2\pi \cos\varphi$ implies $\frac{a}{\pi}\cos\varphi < 2\cos^2\varphi$ whence $p > \frac{a}{\pi}\cos\varphi$ for all $p \geq 2\cos^2\varphi$. In particular, the condition $p > \frac{a}{\pi}\cos\varphi$ is satisfied if $p \geq 2$ or if $|\varphi| \geq \frac{\pi}{4}$.

Remark 4. *If $p = (a/\pi)\cos\varphi$, then the operator $T(a_{a,\varphi})$ is not Fredholm. One can show that it has zero null space and dense range in this case. What is still lacking is an explicit description of the range.*

Dirichlet data on spirals of power type

In this section, we consider the Dirichlet problem in Hardy-Smirnov spaces with weights $E^p(\mathcal{D}, w)$.

Let $a > 0$ and $\gamma > 0$. Consider the domain $\mathcal{D}_{a,\gamma}$ in the plane of complex variable θ that is bounded by the following curves:

$$\begin{aligned} \mathcal{S}_{0,\gamma} &= \{\theta \in \mathbb{C} : \Re\theta \geq 1,\ \Im\theta = 0\}, \\ \mathcal{S}_{a,\gamma} &= \{\theta \in \mathbb{C} : \theta = (\Re\theta + \imath a)^\gamma,\ \Re\theta \geq 1\}, \\ b_{a,\gamma} &= \{\theta \in \mathbb{C} : \theta = (1 + \imath \Im\theta)^\gamma,\ \Im\theta \in (0, a)\}. \end{aligned}$$

The boundary of $\mathcal{D}_{a,\gamma}$ is thus the composite curve $\mathcal{C}_{a,\gamma} = \mathcal{S}_{0,\gamma} \cup \mathcal{S}_{a,\gamma} \cup b_{a,\gamma}$, with each arc being smooth.

Given any $\delta > 0$, we define the following:

$$z = h_\delta(\theta) := \frac{\exp(\imath\theta)}{\theta^\delta},$$

for $\theta \in \mathcal{D}_{a,\gamma}$. This function maps $\mathcal{D}_{a,\gamma}$ onto a domain $\mathcal{D}_{a,\gamma,\delta}$.

Consider the curve $h_\delta(\mathcal{S}_{0,\gamma})$ in the z-plane. Introduce the polar coordinates $r = |z|$ and $\vartheta = \arg z$, and then the parametric representation of the curve just amounts to the following:

$$\begin{cases} r = (\Re\theta)^{-\delta}, \\ \vartheta = \Re\theta, \end{cases}$$

so the equation of $h_\delta(\mathcal{S}_{0,\gamma})$ reduces to $r = \vartheta^{-\delta}$ with $\vartheta \geq 1$. In this way, we obtain what will be referred to as the power spiral.

Note that the curve $h_\delta(\mathcal{S}_{0,\gamma})$ is rectifiable if and only if $\delta > 1$. Indeed, the integral

$$L(h_\delta(\mathcal{S}_{0,\gamma})) = \int_1^\infty \sqrt{(r(\vartheta))^2 + (r'(\vartheta))^2}\, d\vartheta = \int_1^\infty \vartheta^{-\delta}\sqrt{1+\frac{\delta^2}{\vartheta^2}}\, d\vartheta$$

is finite if and only if $\delta > 1$.

Theorem 12. *Assume $\gamma \in (0, 1/2]$ and $\delta > 0$. If $a > 0$ is small enough, then $h_\delta : \mathcal{D}_{a,\gamma} \to \mathcal{D}_{a,\gamma,\delta}$ is a conformal mapping.*

Proof. It is sufficient to show that the curve $h_\delta(\mathcal{C}_{a,\gamma})$ has no self-intersections. The arcs $h_\delta(\mathcal{S}_{0,\gamma})$, $h_\delta(\mathcal{S}_{a,\gamma})$ and $h_\delta(b_{a,\gamma})$ have no self-intersections, which is easy to check. Our next goal is to show that the arcs $h_\delta(\mathcal{S}_{0,\gamma})$ and $h_\delta(\mathcal{S}_{a,\gamma})$ do not meet each other.

Suppose

$$z_1 \in h_\delta(\mathcal{S}_{0,\gamma}), \quad z_2 \in h_\delta(\mathcal{S}_{a,\gamma}),$$

then $z_1 = h_\delta(\mathfrak{z}_1)$ and $z_2 = h_\delta(\mathfrak{z}_2^\gamma)$ for some $\mathfrak{z}_1 \in [1,\infty)$ und $\mathfrak{z}_2 \in [1,\infty) + \imath a$. If $z_1 = z_2$, then a trivial verification shows that

$$\frac{\exp(\imath \mathfrak{z}_1)}{\mathfrak{z}_1^\delta} = \frac{\exp\left(-|\mathfrak{z}_2|^\gamma \sin(\gamma \arg \mathfrak{z}_2)\right)\exp(\imath |\mathfrak{z}_2|^\gamma \cos(\gamma \arg \mathfrak{z}_2))}{|\mathfrak{z}_2|^{\gamma\delta}\exp(\imath\gamma\delta \arg \mathfrak{z}_2)},$$

where $|\mathfrak{z}_2| = \sqrt{(\Re \mathfrak{z}_2)^2 + a^2}$ and $\arg \mathfrak{z}_2 = \arctan \frac{a}{\Re \mathfrak{z}_2}$. The last equality is equivalent to the couple of real equalities:

$$\frac{1}{\mathfrak{z}_1^\delta} = \frac{\exp\left(-|\mathfrak{z}_2|^\gamma \sin(\gamma \arg \mathfrak{z}_2)\right)}{|\mathfrak{z}_2|^{\gamma\delta}},$$
$$\mathfrak{z}_1 = |\mathfrak{z}_2|^\gamma \cos(\gamma \arg \mathfrak{z}_2) - \gamma\delta \arg \mathfrak{z}_2 + 2\pi j,$$

with j being an integer number. In this way, we arrive at an equation relative to the real part of $\mathfrak{z}_2$, namely

$$|\mathfrak{z}_2|^\gamma \cos(\gamma \arg \mathfrak{z}_2) - \gamma\delta \arg \mathfrak{z}_2 + 2\pi j = |\mathfrak{z}_2|^\gamma \exp\left(\frac{1}{\delta}|\mathfrak{z}_2|^\gamma \sin(\gamma \arg \mathfrak{z}_2)\right). \tag{31}$$

If $a \to 0$, then

$$|\mathfrak{z}_2|^\gamma = (\Re \mathfrak{z}_2)^\gamma \left(1 + O\left(\frac{a^2}{(\Re \mathfrak{z}_2)^2}\right)\right),$$
$$\arg \mathfrak{z}_2 = \frac{a}{\Re \mathfrak{z}_2} + O\left(\frac{a^3}{(\Re \mathfrak{z}_2)^3}\right),$$
$$\sin(\gamma \arg \mathfrak{z}_2) = \gamma \frac{a}{\Re \mathfrak{z}_2} + O\left(\frac{a^3}{(\Re \mathfrak{z}_2)^3}\right),$$
$$\cos(\gamma \arg \mathfrak{z}_2) = 1 + O\left(\frac{a^2}{(\Re \mathfrak{z}_2)^2}\right).$$

On substituting these asymptotic formulas into (31), we get the following:

$$(\Re \mathfrak{z}_2)^\gamma \left(1 + O\left(\frac{a^2}{(\Re \mathfrak{z}_2)^2}\right)\right) - \gamma\delta \frac{a}{\Re \mathfrak{z}_2} + O\left(\frac{a^3}{(\Re \mathfrak{z}_2)^3}\right) + 2\pi j$$
$$= (\Re \mathfrak{z}_2)^\gamma \left(1 + O\left(\frac{a^2}{(\Re \mathfrak{z}_2)^2}\right)\right)\exp\left(\frac{1}{\delta}(\Re \mathfrak{z}_2)^\gamma \left(1 + O\left(\frac{a^2}{(\Re \mathfrak{z}_2)^2}\right)\right) \times \left(\gamma \frac{a}{\Re \mathfrak{z}_2} + O\left(\frac{a^3}{(\Re \mathfrak{z}_2)^3}\right)\right)\right)$$

or

$$(\Re \mathfrak{z}_2)^\gamma \left(1 + O\left(\frac{a^2}{(\Re \mathfrak{z}_2)^2}\right)\right) - \gamma\delta \frac{a}{\Re \mathfrak{z}_2} + O\left(\frac{a^3}{(\Re \mathfrak{z}_2)^3}\right) + 2\pi j$$
$$= (\Re \mathfrak{z}_2)^\gamma \left(1 + O\left(\frac{a^2}{(\Re \mathfrak{z}_2)^2}\right)\right) \times \left(1 + \frac{\gamma}{\delta}\frac{a}{(\Re \mathfrak{z}_2)^{1-\gamma}} + O\left(\frac{a^2}{(\Re \mathfrak{z}_2)^{2-2\gamma}}\right)\right).$$

This equation is in turn equivalent to the following:

$$-\gamma\delta \frac{a}{\Re \mathfrak{z}_2} + O\left(\frac{a^2}{(\Re \mathfrak{z}_2)^{2-\gamma}}\right) + 2\pi j = \frac{\gamma}{\delta}\frac{a}{(\Re \mathfrak{z}_2)^{1-2\gamma}} + O\left(\frac{a^2}{(\Re \mathfrak{z}_2)^{2-3\gamma}}\right),$$

i.e.

$$\frac{\gamma}{\delta}\frac{a}{(\Re \mathfrak{z}_2)^{1-2\gamma}} + \delta\gamma \frac{a}{\Re \mathfrak{z}_2} = 2\pi j + O\left(\frac{a^2}{(\Re \mathfrak{z}_2)^{2-3\gamma}}\right).$$

If a is small enough, then

$$0 < \frac{\gamma}{\delta} a \left(\frac{(\Re \mathfrak{z}_2)^{2\gamma} + \delta^2}{\Re \mathfrak{z}_2} + O\left(\frac{a^2}{(\Re \mathfrak{z}_2)^{2-3\gamma}}\right)\right) < 2\pi$$

for any $\Re \mathfrak{z}_2 \in [1,\infty)$. Hence, it follows that (31) has no solutions for any j, so the equation $z_1 = z_2$ is not possible, as desired. □

We now construct a conformal mapping of the unit disk onto $\mathcal{D}_{a,\gamma,\delta}$. To this end, we compose four well-known conformal mappings:

1. $\eta = \frac{1}{\imath}\frac{\zeta+1}{\zeta-1}$ maps the unit disk $\mathbb{D}$ conformally onto the upper half-plane $\mathbb{H} := \{\eta \in \mathbb{C} : \Im \eta > 0\}$.
2. $\mathfrak{z} = \frac{a}{\pi}\ln\left(\eta + \sqrt{\eta^2 - 1}\right)$ maps $\mathbb{H}$ conformally onto the strip Σ_a.
3. $\mathfrak{z} \mapsto \mathfrak{z} + 1$ translates the strip Σ_a horizontally.
4. $\theta = \mathfrak{z}^\gamma$ maps $\Sigma_a + 1$ conformally onto the domain $\mathcal{D}_{a,\gamma}$.

In this way, we obtain the conformal mapping $z = \mathfrak{c}_{a,\gamma,\delta}(\zeta)$ of $\mathbb{D}$ onto $\mathcal{D}_{a,\gamma,\delta}$ given by the following:

$$\mathfrak{c}_{a,\gamma,\delta}(\zeta) = \frac{\exp \imath \left(\frac{a}{\pi}\ln\left(\eta + \sqrt{\eta^2-1}\right) + 1\right)^\gamma}{\left(\frac{a}{\pi}\ln\left(\eta + \sqrt{\eta^2-1}\right) + 1\right)^{\gamma\delta}} \tag{32}$$

with $\eta = \eta(\zeta)$. Recall that the continuous branches of the multi-valued functions under consideration are chosen in such a manner that $\ln\theta = \ln|\theta| + \imath \arg\theta$ with $\arg\theta \in [0, 2\pi)$, $\Im\sqrt{\eta^2 - 1} \geq 0$, and more generally, $\mathfrak{z}^\gamma = |\mathfrak{z}|^\gamma \exp(\imath\gamma \arg\mathfrak{z})$ with $\arg\mathfrak{z} \in [0, 2\pi)$.

Introduce the function $\mathfrak{z}(\zeta) = \frac{a}{\pi} \ln\left(\eta + \sqrt{\eta^2 - 1}\right) + 1$ of $\zeta \in \mathbb{D}$. A direct computation yields the following:

$$\mathfrak{c}'_{a,\gamma,\delta}(\zeta) = \exp \imath\,(\mathfrak{z}(\zeta))^\gamma \, \frac{\imath\gamma\,(\mathfrak{z}(\zeta))^\gamma - \gamma\delta}{(\mathfrak{z}(\zeta))^{\gamma\delta+1}}\, \mathfrak{z}'(\zeta),$$

where $\mathfrak{z}'(\zeta) = \frac{a}{\pi} \frac{1}{\sqrt{1+\zeta^2}} \frac{\sqrt{2}}{1-\zeta}$. We rewrite this as follows:

$$\mathfrak{c}'_{a,\gamma,\delta}(\zeta) = \sqrt{2}\imath\gamma\, \frac{a}{\pi}\,(1 + \imath\delta\,(\mathfrak{z}(\zeta))^{-\gamma}) \times \frac{(\mathfrak{z}(\zeta))^{\gamma-\gamma\delta-1}}{\sqrt{1+\zeta^2}\,(1-\zeta)}\, \exp\imath\,(\mathfrak{z}(\zeta))^\gamma. \tag{33}$$

For the analysis of the Dirichlet problem in $\mathcal{D}_{a,\gamma,\delta}$, we employ the Hardy-Smirnov spaces with weight $w(\zeta) = (1-\zeta)^{-\mu}$, where $-1/q < \mu < 1/p$.

Theorem 13. *Let $\gamma \in (0, 1/2]$, $\delta > 0$ and let $\mathfrak{c}_{a,\gamma,\delta}(\zeta)$ be the conformal mapping of $\mathbb{D}$ onto $\mathcal{D}_{a,\gamma,\delta}$ given by (32). The Toeplitz operator $T(a_{a,\gamma,\delta})$ is then Fredholm if and only if $\mu \neq 0$. Moreover,*

(1) *If $-\frac{1}{q} < \mu < 0$, then the operator $T(a_{a,\gamma,\delta})$ is invertible.*
(2) *If $0 < \mu < \frac{1}{p}$, then the index of $T(a_{a,\gamma,\delta})$ is equal to -1.*

Proof. Consider the Toeplitz operator $T(a_{a,\gamma,\delta})$ with the following symbol:

$$\begin{aligned} a_{a,\gamma,\delta}(\zeta) &= \exp\left(-\imath\frac{2}{p}\arg\mathfrak{c}'_{a,\gamma,\delta}(\zeta)\right)\frac{\overline{w(\zeta)}}{w(\zeta)} \\ &= \exp\left(-\imath\frac{2}{p}\arg\mathfrak{c}'_{a,\gamma,\delta}(\zeta)\right)\frac{(1-1/\zeta)^{-\mu}}{(1-\zeta)^{-\mu}} \\ &= \exp\left(-\imath\frac{2}{p}\arg\mathfrak{c}'_{a,\gamma,\delta}(\zeta)\right)(-\zeta)^{\mu} \end{aligned}$$

cf. (10). Set

$$h_{a,\gamma,\delta}(\zeta) = \exp\left(-\imath\frac{2}{p}(\mathfrak{z}(\zeta))^\gamma\right)$$

for $\zeta \in \mathbb{T}$. It is easily seen that $(h_{a,\gamma,\delta}(\zeta))^{\pm 1} \in H^{\infty+}$, so we introduce the following function:

$$a_0(\zeta) := \frac{a_{a,\gamma,\delta}(\zeta)}{h_{a,\gamma,\delta}(\zeta)}$$

of $\zeta \in \mathbb{T}$.

Note that

$$\lim_{\zeta\to e^{\imath 0\pm}1} \arg\,(\mathfrak{z}(\zeta))^\gamma = 0$$

for all real γ. Hence, it follows that $a_0(\zeta) \in PC$ with discontinuities at the points 1 and $\pm\imath$.

On taking into account that

$$\arg\frac{1}{1-\zeta} = \frac{\pi}{2} - \frac{1}{2}\arg\zeta$$

with $\arg\zeta \in (0, 2\pi)$, we readily deduce that

$$\begin{aligned} \lim_{\zeta\to e^{\imath 0-}1}\arg a_0(\zeta) - \lim_{\zeta\to e^{\imath 0+}1}\arg a_0(\zeta) &= 2\pi\left(\tfrac{1}{p}+\mu\right), \\ \lim_{\zeta\to e^{\imath 0-}\imath}\arg a_0(\zeta) - \lim_{\zeta\to e^{\imath 0+}\imath}\arg a_0(\zeta) &= \tfrac{\pi}{p}, \\ \lim_{\zeta\to e^{\imath 0-}(-\imath)}\arg a_0(\zeta) - \lim_{\zeta\to e^{\imath 0+}(-\imath)}\arg a_0(\zeta) &= \tfrac{\pi}{p}. \end{aligned}$$

Hence, representation (17) holds for the symbol $a_0(\zeta)$ with $\zeta_1 = 1$, $\zeta_2 = \imath$, $\zeta_3 = -\imath$, and

$$\mathfrak{f}_1 = \frac{1}{p} + \mu, \quad \mathfrak{f}_2 = \mathfrak{f}_3 = \frac{1}{2}\frac{1}{p},$$

and a continuous function $c(\zeta)$, such that the winding number of the curve $c(\mathbb{T})$ about the origin vanishes. Thus, according to Theorem 7, the operator $T(a_{a,\gamma,\delta})$ is Fredholm if and only if $\mu \neq 0$.

Suppose that $-1/q < \mu < 0$, then all the quantities $\mathfrak{f}_1$, $\mathfrak{f}_2$ and $\mathfrak{f}_3$ belong to the following interval:

$$\left(-\frac{1}{q}, \frac{1}{p}\right).$$

In this case, Theorem 7 applies with $\kappa_1 = \kappa_2 = \kappa_3 = 0$. Combining Theorems 7 and 3 yields the invertibility of $T(a_0)$ in the space H^{p+}. By Theorem 6(1), the operator $T(a_{a,\gamma,\delta})$ is invertible in H^{p+}, too, for $a_{a,\gamma,\delta} = h_{a,\gamma,\delta}a_0$. According to Theorem 2, the symbol $a_{a,\gamma,\delta}$ admits a p-factorisation with $\kappa = 0$. Hence, the operator $T(a_{a,\gamma,\delta})$ is invertible.

Now, let $\mu \in \left(0, \frac{1}{p}\right)$, then

$$-\frac{1}{q} + 1 < \mathfrak{f}_1 < \frac{1}{p} + 1, \quad -\frac{1}{q} < \mathfrak{f}_2 = \mathfrak{f}_3 < \frac{1}{p}$$

for all $p \in (1, \infty)$. In this case, Theorem 7 applies with $\kappa_1 = 1$ and $\kappa_2 = \kappa_3 = 0$, according to which the operator $T(a_0)$ in H^{p+} is Fredholm and its index just amounts to -1. By Theorem 2, the symbol $a_{a,\gamma,\delta}$ admits a p-factorisation with $\kappa = 1$ whence ind $T(a_{a,\gamma,\delta}) = -1$. □

The advantage of our method lies in the fact that we construct explicitly the p-factorisations in question.

Theorem 14. *Let $\gamma \in (0, 1/2]$, $\delta > 0$ and let $z = \mathfrak{c}_{a,\gamma,\delta}(\zeta)$ be the conformal mapping of $\mathbb{D}$ onto $\mathcal{D}_{a,\gamma,\delta}$ given by (32).*

(1) *If $-\frac{1}{q} < \mu < 0$, then the Dirichlet problems (1) and (2) have a unique solution in $\Re E^p(\mathcal{D}_{a,\gamma,\delta}, w)$ given by the following:*

$$u(z) = \Re\int_{\mathbb{T}} \frac{1}{2\pi\imath}\frac{\zeta + \mathfrak{c}^{-1}_{a,\gamma,\delta}(z)}{\zeta - \mathfrak{c}^{-1}_{a,\gamma,\delta}(z)}\, u_0(\mathfrak{c}_{a,\gamma,\delta}(\zeta))\,\frac{d\zeta}{\zeta}$$

for $z \in \mathcal{D}_{a,\gamma,\delta}$.

(2) *If $0 < \mu < \frac{1}{p}$, then the Dirichlet problems (1) and (2) have a unique solution in $\Re E^p(\mathcal{D}_{a,\gamma,\delta}, w)$ given by the following:*

$$u(z) = \Re \int_{\mathbb{T}} \frac{1}{\pi\imath} \frac{\mathfrak{c}_{a,\gamma,\delta}^{-1}(z) - 1}{\zeta - 1} \frac{1}{\zeta - \mathfrak{c}_{a,\gamma,\delta}^{-1}(z)} u_0(\mathfrak{c}_{a,\gamma,\delta}(\zeta))\, d\zeta$$

for $z \in \mathcal{D}_{a,\gamma,\delta}$.

Proof. Suppose

$$-\frac{1}{q} < \mu < 0.$$

By Theorem 13, the symbol $a_{a,\gamma,\delta}(\zeta)$ admits a p-factorisation with $\kappa = 0$. Reasoning similar to that in the proof of Theorem 11 shows that this factorisation has the form $a_{a,\gamma,\delta} = a^+a^-$ with

$$a^+(\zeta) = \frac{1}{w(\zeta)} \frac{1}{\sqrt[p]{\mathfrak{c}'_{a,\gamma,\delta}(\zeta)}},$$
$$a^-(\zeta) = \overline{w(\zeta)}\, \overline{\sqrt[p]{\mathfrak{c}'_{a,\gamma,\delta}(\zeta)}}.$$

Corollary 2 now implies

$$\begin{aligned}\left(T(a_{a,\gamma,\delta})^{-1}g^+\right)(\zeta) &= \frac{1}{a^+(\zeta)} \left(P_{\mathbb{T}}^+ \frac{1}{a^-} g^+\right)(\zeta) \\ &= \frac{1}{a^+(\zeta)} \left(P_{\mathbb{T}}^+ \frac{1}{a^-} g\right)(\zeta).\end{aligned}$$

Our next task is to prove that condition (13) is fulfilled. To this end, we observe that

$$\begin{aligned}\left(T(a_{a,\gamma,\delta})^{-1}1\right)(\zeta) &= w(\zeta)\sqrt[p]{\mathfrak{c}'_{a,\gamma,\delta}(\zeta)} \left(P_{\mathbb{T}}^+ \frac{1}{\overline{w}} \frac{1}{\overline{\sqrt[p]{\mathfrak{c}'_{a,\gamma,\delta}}}}\right)(\zeta) \\ &= w(\zeta) \sqrt[p]{\frac{\mathfrak{c}'_{a,\gamma,\delta}(\zeta)}{\mathfrak{c}'_{a,\gamma,\delta}(0)}},\end{aligned}$$

as desired.

Thus, we can use the representation of Corollary 1. In this way we immediately obtain the following:

$$\begin{aligned}u(z) &= \Re \left(\left(P_{\mathbb{T}}^+ 2U_0\right)(\zeta) - \sqrt[p]{\mathfrak{c}'_{a,\gamma,\delta}(0)} \left(P_{\mathbb{T}}^+ U_0\right)(0) \right. \\ &\quad \left. \times \left(P_{\mathbb{T}}^+ \frac{1}{\overline{w}} \frac{1}{\overline{\sqrt[p]{\mathfrak{c}'_{a,\gamma,\delta}}}}\right)(\zeta)\right) \\ &= \Re \left(\left(P_{\mathbb{T}}^+ 2U_0\right)(\zeta) - \left(P_{\mathbb{T}}^+ U_0\right)(0) \right) \\ &= \Re \int_{\mathbb{T}} \frac{1}{2\pi\imath} \frac{\zeta' + \zeta}{\zeta' - \zeta} U_0(\zeta') \frac{d\zeta'}{\zeta'}\end{aligned}$$

with $z = \mathfrak{c}_{a,\gamma,\delta}(\zeta)$. This establishes the formula of part (1).

We now assume that $0 < \mu < \frac{1}{p}$. Put

$$\tilde{a}^+(\zeta) = \frac{1}{1-\zeta} \frac{1}{w(\zeta)} \frac{1}{\sqrt[p]{\mathfrak{c}'_{a,\gamma,\delta}(\zeta)}},$$
$$\tilde{a}^-(\zeta) = \left(1 - \frac{1}{\zeta}\right) \overline{w(\zeta)}\, \overline{\sqrt[p]{\mathfrak{c}'_{a,\gamma,\delta}(\zeta)}},$$

then

$$a_{a,\gamma,\delta}(\zeta) = -\tilde{a}^+(\zeta)\, \zeta\, \tilde{a}^-(\zeta) \tag{34}$$

is a p-factorisation of $a_{a,\gamma,\delta}$. Show that condition (15) holds. For this purpose, it suffices to consider the behaviour of the factors in a neighbourhood of the point $\zeta = 1$.

From (33), it follows that

$$\begin{aligned}&|\left(\tilde{a}^+(\zeta)\right)^{\pm 1}| \sim |\zeta - 1|^{\pm(1/p+\mu-1)} \\ &\left(\ln|\zeta - 1|^{-1}\right)^{\mp(\gamma-\gamma\delta-1)/p} \text{ as} \zeta \to 1, \\ &|\left(\tilde{a}^-(\zeta)\right)^{\pm 1}| \sim |\zeta - 1|^{\mp(1/p+\mu-1)} \\ &\left(\ln|\zeta - 1|^{-1}\right)^{\pm(\gamma-\gamma\delta-1)/p} \text{ as} \zeta \to 1.\end{aligned}$$

Since μ belongs to the interval $\left(0, \frac{1}{p}\right)$, we have the following:

$$q\left(\frac{1}{p} + \mu - 1\right) > -1,$$
$$-p\left(\frac{1}{p} + \mu - 1\right) > -1,$$

whence $\tilde{a}^+ \in H^{q+}$ and $(\tilde{a}^+)^{-1} \in H^{p+}$. Analogously, we get the following:

$$\begin{aligned}\tilde{a}^- &\in H^{p-} \oplus \{c\}, \\ (\tilde{a}^-)^{-1} &\in H^{q-} \oplus \{c\}.\end{aligned}$$

Consider the folllowing equation:

$$(T(a_{a,\gamma,\delta})h^+)(\zeta) + \overline{h^+(0)} = g^+(\zeta) \tag{35}$$

for $\zeta \in \mathbb{T}$, *cf.* (11). In the case under study, the operator $T(a_{a,\gamma,\delta})$ has no inverse operator defined on all of H^{p+} (but a left inverse). We introduce the auxiliary operator $T(\tilde{a}^+\tilde{a}^-)$. This operator is invertible, so applying its inverse to both sides of the equation, we rewrite it equivalently as follows:

$$\begin{aligned}&\frac{1}{\tilde{a}^+} P_{\mathbb{T}}^+ \frac{1}{\tilde{a}^-} P_{\mathbb{T}}^+ \left(-\tilde{a}^+\zeta\tilde{a}^- h^+\right) + (T(\tilde{a}^+\tilde{a}^-))^{-1}\overline{h^+(0)} \\ &= (T(\tilde{a}^+\tilde{a}^-))^{-1} g^+,\end{aligned}$$

that is

$$\begin{aligned}&-\zeta h^+(\zeta) + \overline{h^+(0)}\left(T(\tilde{a}^+\tilde{a}^-)^{-1}1\right)(\zeta) \\ &= \frac{1}{\tilde{a}^+(\zeta)} P_{\mathbb{T}}^+ \left(\frac{1}{\tilde{a}^-} g\right)(\zeta)\end{aligned} \tag{36}$$

for $\zeta \in \mathbb{T}$.

Assume that (36) possesses a solution. On putting $\zeta = 0$ and taking into account that

$$(T(\tilde{a}^+\tilde{a}^-)^{-1}1)(0) = 1,$$

we get the following:

$$\begin{aligned}\overline{h^+(0)} &= \frac{1}{\tilde{a}^+(0)} P^+_{\mathbb{T}} \left(\frac{1}{\tilde{a}^-}g\right)(0) \\ &= \sqrt[p]{\mathfrak{c}'_{a,\gamma,\delta}(0)}\, P^+_{\mathbb{T}} \left(\left(1-\frac{1}{\zeta}\right)^{-1} 2U_0\right)(0) \\ &= \sqrt[p]{\mathfrak{c}'_{a,\gamma,\delta}(0)} \int_{\mathbb{T}} \frac{\zeta}{\zeta-1} 2U_0(\zeta) \frac{1}{2\pi\imath}\frac{d\zeta}{\zeta} \\ &= \sqrt[p]{\mathfrak{c}'_{a,\gamma,\delta}(0)} \int_{\mathbb{T}} \frac{1}{\pi\imath}\frac{U_0(\zeta)}{\zeta-1}\, d\zeta. \end{aligned} \tag{37}$$

As previously mentioned, $(\tilde{a}^-)^{-1}$ lies in $H^{q-} \oplus \{c\}$, so $P^+_{\mathbb{T}}(\tilde{a}^-)^{-1}(\zeta)$ is independent of $\zeta \in \mathbb{D}$. An easy computation shows that

$$P^+_{\mathbb{T}}(\tilde{a}^-)^{-1}(\zeta) = \frac{1}{\sqrt[p]{\mathfrak{c}'_{a,\gamma,\delta}(0)}}$$

for all $\zeta \in \mathbb{D}$ whence

$$h^+(\zeta) = \frac{1}{\zeta}\left(\frac{\overline{h^+(0)}}{\tilde{a}^+(\zeta)} \frac{1}{\sqrt[p]{\mathfrak{c}'_{a,\gamma,\delta}(0)}} - \frac{1}{\tilde{a}^+(\zeta)} P^+_{\mathbb{T}}\left(\frac{1}{\tilde{a}^-}g\right)(\zeta)\right),$$

which is due to (36). Combining this equality with (37) gives the following:

$$\begin{aligned} h^+(\zeta) &= \frac{1}{\zeta}\frac{1}{\tilde{a}^+(\zeta)}\left(\int_{\mathbb{T}}\frac{1}{\pi\imath}\frac{1}{\zeta'-1}U_0(\zeta')\,d\zeta' \right.\\ &\qquad \left. - \int_{\mathbb{T}}\frac{1}{2\pi\imath}\frac{1}{\zeta'-\zeta}\frac{\zeta'}{\zeta'-1}2U_0(\zeta')\,d\zeta'\right) \\ &= -\frac{1}{\tilde{a}^+(\zeta)}\int_{\mathbb{T}}\frac{1}{\pi\imath}\frac{1}{\zeta'-\zeta}\frac{U_0(\zeta')}{\zeta'-1}\,d\zeta' \end{aligned} \tag{38}$$

for $\zeta \in \mathbb{D}$.

On the other hand, putting $\zeta = 0$ in (38), we obtain the following:

$$h^+(0) = -\sqrt[p]{\mathfrak{c}'_{a,\gamma,\delta}(0)} \int_{\mathbb{T}} \frac{1}{\pi\imath}\frac{1}{\zeta'}\frac{U_0(\zeta')}{\zeta'-1}\,d\zeta'.$$

Since $U_0(\zeta)$ is real-valued, it follows that

$$\begin{aligned}\overline{h^+(0)} &= \sqrt[p]{\mathfrak{c}'_{a,\gamma,\delta}(0)} \int_{\mathbb{T}} \frac{1}{\pi\imath}\,\zeta'\,\frac{U_0(\zeta')}{(\zeta')^{-1}-1}\,d\frac{1}{\zeta'} \\ &= \sqrt[p]{\mathfrak{c}'_{a,\gamma,\delta}(0)} \int_{\mathbb{T}} \frac{1}{\pi\imath}\frac{U_0(\zeta')}{\zeta'-1}\,d\zeta'\end{aligned}$$

which coincides with (37).

We have thus proven that each solution of (35) has necessarily form (38), i.e. the solution is unique. Moreover, the function $h^+(\zeta)$ given by (38) satisfies Equation 35.

To complete the proof, it suffices to apply part (2) of Theorem 1. Equation 34 shows that the symbol $a_{a,\gamma,\delta}$ admits a p-factorisation with $\kappa = 1$. By Theorem 2, the index of the Toeplitz operator $T(a_{a,\gamma,\delta})$ is equal to -1. Hence, it follows by Theorem 3 that the kernel of $T(a_{a,\gamma,\delta})$ is zero. We thus conclude that Theorem 1 is applicable.

If $\zeta \in \mathbb{D}$, then

$$\frac{h^+(\zeta)}{w(\zeta)\sqrt[p]{\mathfrak{c}'_{a,\gamma,\delta}(\zeta)}} = \int_{\mathbb{T}} \frac{1}{\pi\imath}\frac{\zeta-1}{\zeta'-1}\frac{1}{\zeta'-\zeta}U_0(\zeta')\,d\zeta',$$

showing part (2). □

Remark 5. *In the case of $\mu = 0$, the operator $T(a_{a,\gamma,\delta})$ fails to be Fredholm. This motivates the introduction of spaces with weights.*

Conformal reduction of Neumann problems

The Neumann problem

Let $\mathcal{D}$ be a simply connected, bounded domain in the plane of real variables (x, y). Denote by $\mathcal{C}$ the boundary of $\mathcal{D}$ which is a Jordan curve. Given a function u_1 on $\mathcal{C}$, we consider the problem of finding a harmonic function u in $\mathcal{D}$ whose outward normal derivative on $\mathcal{C}$ coincides with u_1. In this way, we obtain what has been formulated in (1) and (3).

As usual, we give the plane a complex structure by $z = x + \imath y$ and pick a conformal mapping $z = \mathfrak{c}(\zeta)$ of $\mathbb{D}$ onto $\mathcal{D}$ satisfying (5). We continue to use power weight functions $w(\zeta)$ introduced in (9). A function u_1 on $\mathcal{C}$ is said to belong to $L^p(\mathcal{C}, w)$ if

$$\int_{\mathbb{T}} |u_1(\mathfrak{c}(\zeta))|^p\, |w(\zeta)|^p\, |\mathfrak{c}'(\zeta)|\, |d\zeta| < \infty.$$

Definition 4. *Given any $u_1 \in L^p(\mathcal{C}, w)$, the Neumann problem is said to possess a solution of class $\Re E^{1,p}(\mathcal{D}, w)$ if there exists an analytic function f in $\mathcal{D}$, such that $f' \in E^p(\mathcal{D}, w)$ and the harmonic function $u = \Re f$ satisfies $(\partial/\partial\nu)u = u_1$ on $\mathcal{C}$.*

If the curve $\mathcal{C}$ is rectifiable and $f' \in E^p(\mathcal{D}, w)$ then also $f \in E^p(\mathcal{D}, w)$. However, f no longer needs to belong to $E^p(\mathcal{D}, w)$ if $\mathcal{C}$ fails to be rectifiable.

Our next concern will be to reduce the Neumann problem to Toeplitz operator equations on the unit circle. According to [4,16], if $u = \Re f$ with $f' \in E^p(\mathcal{D}, w)$, then

$$\frac{\partial u}{\partial \nu}(z) = \Re\left(e^{\imath\beta(z)}f'(z)\right) \tag{39}$$

for almost all $z \in \mathcal{C}$, where $\beta(z)$ is the angle at which the outward normal of $\mathcal{C}$ at the point z intersects the real axis. Introduce the following function:

$$h^+(\zeta) = w(\zeta)\sqrt[p]{\mathfrak{c}'(\zeta)}\,f'(\mathfrak{c}(\zeta))$$

of $\zeta \in \mathbb{D}$, which is obviously analytic in the unit disk. Moreover, h^+ can be specified within the Hardy space

H^{p+} on the unit circle, as is easy to check. Put $h^-(\zeta) := \overline{h^+(\zeta)}$ for $\zeta \in \mathbb{T}$. Using (39), we rewrite the equation $(\partial/\partial\nu)u = u_1$ equivalently as follows:

$$\frac{1}{2}\left(e^{\imath\beta(\mathfrak{c}(\zeta))}\,\frac{h^+(\zeta)}{w(\zeta)\sqrt[p]{\mathfrak{c}'(\zeta)}} + e^{-\imath\beta(\mathfrak{c}(\zeta))}\,\frac{h^-(\zeta)}{\overline{w(\zeta)\sqrt[p]{\mathfrak{c}'(\zeta)}}}\right) = U_1(\zeta) \tag{40}$$

for $\zeta \in \mathbb{T}$, where $U_1(\zeta) = u_1(\mathfrak{c}(\zeta))$.

Note that

$$\beta(z) = \alpha(z) - \frac{\pi}{2}$$

for $z \in \mathcal{C}$, where $\alpha(z)$ is the angle at which the tangent of $\mathcal{C}$ at the point z intersects the real axis. As mentioned,

$$\arg \mathfrak{c}'(\zeta) = \alpha(\mathfrak{c}(\zeta)) - \arg\zeta - \frac{\pi}{2}$$

for $\zeta \in \mathbb{T}$. Hence, it follows that

$$\begin{aligned}\frac{e^{\imath\beta(\mathfrak{c}(\zeta))}}{w(\zeta)\sqrt[p]{\mathfrak{c}'(\zeta)}} &= \frac{e^{\imath\beta(\mathfrak{c}(\zeta))-\imath\frac{1}{p}\arg\mathfrak{c}'(\zeta)}}{w(\zeta)\sqrt[p]{|\mathfrak{c}'(\zeta)|}}\\ &= \frac{e^{\imath\arg\mathfrak{c}'(\zeta)+\imath\arg\zeta-\imath\frac{1}{p}\arg\mathfrak{c}'(\zeta)}}{w(\zeta)\sqrt[p]{|\mathfrak{c}'(\zeta)|}}\\ &= \frac{\zeta\,e^{\imath\frac{1}{q}\arg\mathfrak{c}'(\zeta)}}{w(\zeta)\sqrt[p]{|\mathfrak{c}'(\zeta)|}},\end{aligned}$$

where $\frac{1}{p}+\frac{1}{q}=1$, so (40) just amounts to the following:

$$b(\zeta)\left(\zeta h^+(\zeta)\right) + \frac{1}{\zeta}\,h^-(\zeta) = g(\zeta)$$

for $\zeta \in \mathbb{T}$, with

$$\begin{aligned}b(\zeta) &= e^{\imath\frac{2}{q}\arg\mathfrak{c}'(\zeta)}\,\frac{\overline{w(\zeta)}}{w(\zeta)},\\ g(\zeta) &= 2\,e^{\imath\frac{1}{q}\arg\mathfrak{c}'(\zeta)}\,\overline{w(\zeta)}\,\sqrt[p]{|\mathfrak{c}'(\zeta)|}\,U_1(\zeta).\end{aligned} \tag{41}$$

On applying the projection $P^+_{\mathbb{T}}$ to both sides of the last equality, we derive the Toeplitz equation:

$$T(b)\left(\zeta h^+\right)(\zeta) = g^+(\zeta) \tag{42}$$

on $\mathbb{T}$, where $g^+ = P^+_{\mathbb{T}}g$. We have used the equality $P^+_{\mathbb{T}}\left(\zeta^{-1}h^-\right) = 0$, which is easily verified.

Suppose now that $h^+(\zeta)$ is a solution of Equation 42 considered in the space H^{p+} and that the pair $h^+(\zeta)$ and $h^-(\zeta) = \overline{h^+(\zeta)}$ satisfies equation 40, put

$$f'(\mathfrak{c}(\zeta)) = \frac{h^+(\zeta)}{w(\zeta)\,\sqrt[p]{\mathfrak{c}'(\zeta)}},$$

then

$$\begin{aligned}u(z) &= \Re\int_{z_0}^{z}\frac{h^+(\mathfrak{c}^{-1}(z'))}{w(\mathfrak{c}^{-1}(z'))\sqrt[p]{\mathfrak{c}'(\mathfrak{c}^{-1}(z'))}}dz'\\ &= \Re\int_{z_0}^{z}h^+(\mathfrak{c}^{-1}(z'))\,\frac{\sqrt[p]{(\mathfrak{c}^{-1})'(z')}}{w(\mathfrak{c}^{-1}(z'))}dz'\end{aligned} \tag{43}$$

for all $z \in \mathcal{D}$, where z_0 is an arbitrary fixed point of $\mathcal{D}$ and the integral is over any path in $\mathcal{D}$ connecting z_0 and z. It is easily seen that solution u is determined uniquely up to a real constant.

Theorem 15. *Let $u_1 \in L^p(\mathcal{C}, w)$.*

1. *If $u = \Re f$ is a solution of the Neumann problem in $\Re E^{1,p}(\mathcal{D}, w)$, then the function $h^+(\zeta) = w(\zeta)\sqrt[p]{\mathfrak{c}'(\zeta)}f'(\mathfrak{c}(\zeta))$ is a solution of Toeplitz equation (42).*
2. *If h^+ is a unique solution of Toeplitz equation (42) in H^{p+}, then the function u given by (43) is a solution of the Neumann problem in the space $\Re E^{1,p}(\mathcal{D}, w)$.*

Proof. This theorem can be proven in much the same way as Theorem 1. □

If the operator $T(b)$ is invertible in the space H^{p+}, then the Neumann problem possesses a solution in $\Re E^{1,p}(\mathcal{D}, w)$ if and only if $(T(b)^{-1}g^+)(0) = 0$. Indeed, applying $T(b)^{-1}$ to both sides of (42) yields $\zeta h^+(\zeta) = (T(b)^{-1}g^+)(\zeta)$. On substituting $\zeta = 0$, we obtain $(T(b)^{-1}g^+)(0) = 0$. Conversely, if the latter condition is fulfilled, then

$$h^+(\zeta) = \frac{1}{\zeta}\,(T(b)^{-1}g^+)(\zeta)$$

is of Hardy class H^{p+}, so (43) gives us the general solution of the Neumann problem.

Corollary 3. *Suppose the symbol $b(\zeta)$ has a p-factorisation of the form $b = b^+b^-$ with*

$$\begin{aligned}b^+(\zeta) &= \frac{\sqrt[q]{\mathfrak{c}'(\zeta)}}{w(\zeta)},\\ b^-(\zeta) &= \frac{\overline{w(\zeta)}}{\sqrt[q]{\overline{\mathfrak{c}'(\zeta)}}},\end{aligned}$$

then

1. *For the Neumann problem to possess a solution in $\Re E^{1,p}(\mathcal{D}, w)$, it is necessary and sufficient that*

$$\frac{1}{\pi\imath}\int_{\mathbb{T}}u_1(\mathfrak{c}(\zeta))\,|\mathfrak{c}'(\zeta)|\,\frac{d\zeta}{\zeta} = 0.$$

2. *If this condition holds, then any solution of the Neumann problem in the space $\Re E^{1,p}(\mathcal{D}, w)$ has the following form:*

$$u(z) = \Re\int_{z_0}^{z}(\ln\mathfrak{c}^{-1}(z'))'\left(\int_{\mathbb{T}}u_1(\mathfrak{c}(\zeta))\,\frac{1}{\pi\imath}\,\frac{|\mathfrak{c}'(\zeta)|}{\zeta-\mathfrak{c}^{-1}(z')}d\zeta\right)dz'$$

for $z \in \mathcal{D}$, where $z_0 \in \mathcal{D}$ is a fixed point and the integration is over any path connecting z_0 and z.

Proof. Using the p-factorisation $b = b^+b^-$, we get the following:

$$\begin{aligned} h^+(\zeta) &= \frac{1}{\zeta}\left(T(b)^{-1}g^+\right)(\zeta) \\ &= \frac{1}{\zeta}\,\frac{1}{b^+(\zeta)}\,P^+_{\mathbb{T}}\left(\frac{g}{b^-}\right)(\zeta) \\ &= \frac{1}{\zeta}\,\frac{1}{b^+(\zeta)}\,P^+_{\mathbb{T}}\left(2\,|\mathfrak{c}'|\,U_1\right)(\zeta) \\ &= \frac{2}{\zeta}\,\frac{w(\zeta)}{\sqrt[q]{\mathfrak{c}'(\zeta)}}\,P^+_{\mathbb{T}}\left(|\mathfrak{c}'|\,U_1\right)(\zeta), \end{aligned}$$

showing part (1) of the corollary.

Furthermore, on writing the analytic projection $P^+_{\mathbb{T}}$ as the Cauchy integral for $\zeta \in \mathbb{D}$, we deduce the following:

$$\frac{h^+(\zeta)}{w(\zeta)\,\sqrt[p]{\mathfrak{c}'(\zeta)}} = \frac{1}{\zeta}\,\frac{1}{\mathfrak{c}'(\zeta)}\int_{\mathbb{T}}\frac{1}{\pi\imath}\,\frac{1}{\zeta'-\zeta}\,|\mathfrak{c}'(\zeta')|U_1(\zeta')\,d\zeta',$$

which establishes part (2). □

Neumann data on sectorial curves

As in section 'Dirichlet data on sectorial curves', we consider a simply connected domain $\mathcal{D} \Subset \mathbb{R}^2$ whose boundary $\mathcal{C}$ is smooth everywhere except for a finite number of points. More precisely, $\mathcal{C}$ is a Jordan curve of the following form:

$$\mathcal{C} = \bigcup_{k=1}^{n} \mathcal{C}_k,$$

where $\mathcal{C}_k = [z_{k-1}, z_k]$ is an arc with initial point z_{k-1} and endpoint z_k which are located after each other in positive direction on $\mathcal{C}$, and $z_n = z_0$. Moreover, (z_{k-1}, z_k) is smooth for all k.

We first notice that the Neumann problem behaves well for sectorial boundary curves with the sectoriality index different from that for the Dirichlet problem. To make it more precise, we observe that the right-hand side of estimate (19) for the variation of the inclination of the tangent of $\mathcal{C}$ close to a singular point z_k can be written as $\min\{p-1,1\}\,\pi/2$ for all $1 < p < \infty$. This has been referred to as p-sectoriality. The right-hand side of the estimate of variation that we might allow in the case of Neumann problem looks like $\min\{1/(p-1),1\}\,\pi/2 = \min\{q-1,1\}\,\pi/2$, which corresponds to the q-sectoriality.

Definition 5. *The curve $\mathcal{C}$ is called q-sectorial if, for each $k = 1,\dots,n$, there is a neighbourhood (z_k^-, z_k^+) of z_k on $\mathcal{C}$ and a real number φ_k, such that*

$$\sup_{z\in(z_k^-,z_k^+)\setminus\{z_k\}} |\alpha(z)-\varphi_k| < \begin{cases} \frac{\pi}{2}\,\frac{1}{p-1}, & \text{if } p \geq 2, \\ \frac{\pi}{2}, & \text{if } 1 < p < 2, \end{cases}$$

where $\alpha(z)$ is the angle at which the tangent of $\mathcal{C}$ at the point z intersects the real axis.

If z_k is a conical point of $\mathcal{C}$, then the angle at which the tangent of $\mathcal{C}$ at z intersects the real axis has jump $j_k < \pi$ when z passes through z_k. Hence, the estimate is fulfilled at z_k with a suitable φ_k, if $1 < p < 2$, and is fulfilled if moreover $j_k < 1/(p-1)\,\pi$, if $p \geq 2$.

Theorem 16. *Let the boundary of $\mathcal{D}$ be a q-sectorial curve, and the Neumann problem has then a solution in $\Re E^{1,p}(\mathcal{D})$ if and only if the condition of Corollary 3(1) is satisfied. In this case, any solution of the problem has the form of Corollary 3(2).*

Proof. With the reasoning similar to that in the proof of Theorem 8, it shows that $b = cb_0$ where b_0 is p-sectorial and $c \in C(\mathbb{T})$ is such that $\mathrm{ind}_{c(\mathbb{T})}(0) = 0$. Combining Theorems 5, 6 and 3, we conclude that the operator $T(b)$ is invertible and so the symbol b admits a p-factorisation of the form $b = b^+b^-$.

On the other hand, (41) yields the following:

$$b(\zeta) = \frac{\sqrt[q]{\mathfrak{c}'(\zeta)}}{\sqrt[q]{\overline{\mathfrak{c}'(\zeta)}}}, \tag{44}$$

whence

$$\frac{\sqrt[q]{\mathfrak{c}'(\zeta)}}{b^+(\zeta)} = \sqrt[q]{\overline{\mathfrak{c}'(\zeta)}}\,b^-(\zeta).$$

Since $\mathfrak{c}' \in H^{1+}$ and $1/b^+ \in H^{p+}$, the left-hand side of this equality belongs to H^{1+} (see for instance [24]). In the same manner, one sees that the right-hand side belongs to $H^{1-} \oplus \{c\}$. Therefore,

$$\begin{aligned} b^+(\zeta) &= c\,\sqrt[q]{\mathfrak{c}'(\zeta)}, \\ b^-(\zeta) &= \frac{1}{c}\,\frac{1}{\sqrt[q]{\overline{\mathfrak{c}'(\zeta)}}}, \end{aligned}$$

where c is a complex constant. Thus, equality (44) is actually a p-factorisation, so Corollary 3 applies to complete the proof. □

Neumann data on logarithmic spirals

Let $\mathcal{D}_{a,\varphi}$ be a simply connected domain bounded by two logarithmic spirals and an auxiliary curve as described in the 'Dirichlet data on logarithmic spirals' section. According to formula (41), the Toeplitz operator corresponding to the Neumann problem in the space $\Re E^{1,p}(\mathcal{D}_{a,\varphi})$ has the following form:

$$b_{a,\varphi}(\zeta) = \exp\left(\imath\,\frac{2}{q}\,\arg \mathfrak{c}'_{a,\varphi}(\zeta)\right)$$

with the argument of $\mathfrak{c}'_{a,\varphi}(\zeta)$ evaluated in (26).

Since $b_{a,\varphi}(\zeta) = (a_{a,\varphi}(\zeta))^{-\frac{p}{q}}$, the symbol $a_{a,\varphi}$ being given before Theorem 10, we obtain the following:

$$b_{a,\varphi}(\zeta) = (h_{a,\varphi}(\zeta))^{-\frac{p}{q}}\,b_0(\zeta)$$

with

$$(h_{a,\varphi}(\zeta))^{-\frac{p}{q}} := \exp\left(\imath\,\frac{2}{q}\left(\varphi - \frac{a}{\pi}\sin\varphi\,\ln(\eta + \sqrt{\eta^2-1})\right)\right)$$

belonging to $GH^{\infty+}$ and $b_0(\zeta) = (a_0(\zeta))^{-\frac{p}{q}}$ belonging to PC. Moreover, $b_0(\zeta)$ has representation (17) with $\zeta_1 = 1$, $\zeta_2 = \imath$ and $\zeta_3 = -\imath$,

$$\mathfrak{f}_1 = -\frac{1}{q}\left(1 - \frac{a}{\pi}\cos\varphi\right), \quad \mathfrak{f}_2 = \mathfrak{f}_3 = -\frac{1}{2}\frac{1}{q},$$

and a continuous function $c(\zeta)$ which is different from zero on $\mathbb{T}$ and satisfies $\mathrm{ind}_{c(\mathbb{T})}(0) = 0$.

We are now in a position to employ Theorem 15 and Corollary 3 for studying the Neumann problem in $\mathcal{D}_{a,\varphi}$.

Theorem 17. *As previously defined, the operator $T(b_{a,\varphi})$ is Fredholm in the space H^{p+} if and only if*

$$\frac{a}{\pi}\cos\varphi \neq q.$$

Proof. Since $a < 2\pi\cos\varphi$, it follows that

$$\mathfrak{f}_1 < \frac{1}{q}\left(2\cos^2\varphi - 1\right) = \frac{1}{q}\cos(2\varphi) \le \frac{1}{q}$$

whence

$$-\frac{1}{q} < \mathfrak{f}_1 < 1 + \frac{1}{p}, \quad -\frac{1}{q} < \mathfrak{f}_2 = \mathfrak{f}_3 < \frac{1}{p}.$$

By Theorem 7, the operator $T(b_{a,\varphi})$ is Fredholm if and only if $\mathfrak{f}_1 \neq 1/p$, as desired. □

Theorem 18. *Assume that*

$$q > \frac{a}{\pi}\cos\varphi, \tag{45}$$

then

(1) *The operator $T(b_{a,\varphi})$ is invertible in H^{p+}, and the symbol has a p-factorisation $b_{a,\varphi} = b^+b^-$ with*

$$b^+(\zeta) = \sqrt[q]{\mathfrak{c}'_{a,\varphi}(\zeta)}, \qquad b^-(\zeta) = \frac{1}{\sqrt[q]{\mathfrak{c}'_{a,\varphi}(\zeta)}}.$$

(2) *For the Neumann problem to possess a solution in $\Re E^{1,p}(\mathcal{D}_{a,\varphi})$, it is necessary and sufficient that*

$$\frac{1}{\pi\imath}\int_{\mathbb{T}} u_1(\mathfrak{c}_{a,\varphi}(\zeta))\,|\mathfrak{c}'_{a,\varphi}(\zeta)|\,\frac{d\zeta}{\zeta} = 0.$$

(3) *Under this condition, any solution of the Neumann problem in $\Re E^{1,p}(\mathcal{D}_{a,\varphi})$ has the following form:*

$$u(z) = \Re\int_{z_0}^{z}(\ln\mathfrak{c}^{-1}_{a,\varphi}(z'))'\left(\int_{\mathbb{T}} u_1(\mathfrak{c}_{a,\varphi}(\zeta))\,\frac{1}{\pi\imath}\,\frac{|\mathfrak{c}'_{a,\varphi}(\zeta)|}{\zeta - \mathfrak{c}^{-1}_{a,\varphi}(z')}d\zeta\right)dz'$$

for $z \in \mathcal{D}_{a,\varphi}$, where $z_0 \in \mathcal{D}_{a,\varphi}$ is an arbitrary fixed point and the integration is over any curve in $\mathcal{D}_{a,\varphi}$ connecting z_0 and z.

Proof. Inequality (45) just amounts to saying that $\mathfrak{f}_1 < 1/p$. In this case, Theorem 7 applies immediately to conclude that the operator $T(b_{a,\varphi})$ is invertible in the space H^{p+}. The p-factorisation is now derived analogously to that in the proof of Theorem 10. The remaining part of the theorem follows readily from Corollary 3. □

Theorem 19. *If*

$$q < \frac{a}{\pi}\cos\varphi, \tag{46}$$

then $T(b_{a,\varphi})$ is a Fredholm operator in H^{p+} of index -1, and the symbol has a p-factorisation $b_{a,\varphi} = -b^+\zeta b^-$ with

$$b^+(\zeta) = \frac{1}{1-\zeta}\sqrt[q]{\mathfrak{c}'_{a,\varphi}(\zeta)}, \qquad b^-(\zeta) = \left(1 - \frac{1}{\zeta}\right)\frac{1}{\sqrt[q]{\mathfrak{c}'_{a,\varphi}(\zeta)}}.$$

Proof. Note that the assumption is equivalent to the following:

$$\frac{1}{p} < \mathfrak{f}_1 < 1 + \frac{1}{p}.$$

Hence, from Theorem 7, it follows that the operator $T(b_{a,\varphi})$ in H^{p+} is Fredholm and its index equals -1. The p-factorisation is established similarly to that in the proof of Theorem 10. □

We now turn to the Neumann problem provided that (46) is satisfied. In our case, Equation 42 looks like the following:

$$-T(b^+\zeta b^-)\left(\zeta h^+\right)(\zeta) = g^+(\zeta) \tag{47}$$

for $\zeta \in \mathbb{T}$. The operator $T(b^+b^-)$ is easily seen to be invertible, and its inverse is given by the following:

$$(T(b^+b^-))^{-1} = \frac{1}{b^+}P^+_{\mathbb{T}}\frac{1}{b^-}.$$

Applying the inverse to both sides of (47) leads us to the following:

$$\begin{aligned}-\zeta^2 h^+(\zeta) &= \frac{1}{b^+(\zeta)}P^+_{\mathbb{T}}\left(\frac{g^+}{b^-}\right)(\zeta)\\ &= \frac{1}{b^+(\zeta)}P^+_{\mathbb{T}}\left(\frac{g}{b^-}\right)(\zeta)\\ &= \frac{1-\zeta}{\sqrt[q]{\mathfrak{c}'_{a,\varphi}(\zeta)}}P^+_{\mathbb{T}}\left(\frac{\zeta}{\zeta-1}|\mathfrak{c}'_{a,\varphi}|\,2U_1\right).\end{aligned}$$

Thus, $h^+ \in H^{p+}$ if and only if the right-hand side of the last equality has a second-order zero at the point $\zeta = 0$, and in this case, the solution is unique. Since $(1-\zeta)/\sqrt[q]{\mathfrak{c}'_{a,\varphi}(\zeta)}$ does not vanish at the origin, we get the following:

Corollary 4. *Let (46) be satisfied, then the condition*

$$\left(\frac{\partial}{\partial\zeta}\right)^j\int_{\mathbb{T}}\frac{1}{\pi\imath}\frac{1}{\zeta'-\zeta}\,\frac{\zeta'}{\zeta'-1}\,|\mathfrak{c}'_{a,\varphi}(\zeta')|\,U_1(\zeta')d\zeta'\Big|_{\zeta=0} = 0,$$

for $j = 0, 1$, is necessary and sufficient in order that the Neumann problem might have a solution in $\Re E^{1,p}(\mathcal{D}_{a,\varphi})$. Under this condition, all solutions in $\Re E^{1,p}(\mathcal{D}_{a,\varphi})$ have the following form:

$$u(z) = \Re \int_{z_0}^{z} (\ln \mathfrak{c}_{a,\varphi}^{-1}(z'))' \frac{\mathfrak{c}_{a,\varphi}^{-1}(z') - 1}{\mathfrak{c}_{a,\varphi}^{-1}(z')} \times \left(\int_{\mathbb{T}} \frac{1}{\pi \imath} \frac{1}{\zeta - \mathfrak{c}_{a,\varphi}^{-1}(z')} \frac{\zeta}{\zeta - 1} |\mathfrak{c}'_{a,\varphi}(\zeta)| \, U_1(\zeta) \, d\zeta \right) dz'$$

for $z \in \mathcal{D}_{a,\varphi}$, where $z_0 \in \mathcal{D}_{a,\varphi}$ is fixed and the outer integral is over any path in $\mathcal{D}_{a,\varphi}$ connecting z_0 and z.

Proof. It remains to establish the formula for solutions. The latter follows readily from the following:

$$h^+(\zeta) = -\frac{1}{\zeta^2} \frac{1-\zeta}{\sqrt[q]{\mathfrak{c}'_{a,\varphi}(\zeta)}} \times \int_{\mathbb{T}} \frac{1}{2\pi \imath} \frac{1}{\zeta' - \zeta} \left(\frac{\zeta'}{\zeta' - 1} |\mathfrak{c}'_{a,\varphi}(\zeta')| \, 2U_1(\zeta') \right) d\zeta'$$

for all $\zeta \in \mathbb{D}$. □

Neumann data on spirals of power type

Let $\mathcal{D}_{a,\gamma,\delta}$ be a simply connected domain bounded by two power-like spirals and an auxiliary curve as described in the 'Dirichlet data on spirals of power type' section. According to formula (41), the Toeplitz operator corresponding to the Neumann problem in $\Re E^{1,p}(\mathcal{D}_{a,\gamma,\delta}, w)$ has the following form:

$$\begin{aligned} b_{a,\gamma,\delta}(\zeta) &= \exp\left(\imath \frac{2}{q} \arg \mathfrak{c}'_{a,\gamma,\delta}(\zeta) \right) \frac{\overline{w(\zeta)}}{w(\zeta)} \\ &= \exp\left(\imath \frac{2}{q} \arg \mathfrak{c}'_{a,\gamma,\delta}(\zeta) \right) (-\zeta)^{\mu}, \end{aligned}$$

where $w(\zeta) = (1-\zeta)^{-\mu}$.

Arguments similar to those in the proof of Theorem 13 show that the symbol $b_{a,\gamma,\delta}$ factorises as follows:

$$b_{a,\gamma,\delta}(\zeta) = h_{a,\gamma,\delta}(\zeta) \, b_0(\zeta)$$

with

$$h_{a,\gamma,\delta}(\zeta) = \exp\left(-\imath \frac{2}{p} (\mathfrak{z}(\zeta))^{\gamma} \right)$$

belonging to $GH^{\infty +}$ and $b_0 \in PC$ having discontinuities at the points 1 and $\pm\imath$. Moreover, $b_0(\zeta)$ admits a representation (17) with $\zeta_1 = 1$, $\zeta_2 = \imath$ and $\zeta_3 = -\imath$,

$$\mathfrak{f}_1 = -\frac{1}{q} + \mu, \quad \mathfrak{f}_2 = \mathfrak{f}_3 = -\frac{1}{2} \frac{1}{q}.$$

and a continuous function $c(\zeta)$ which does not vanish on the circle $\mathbb{T}$ and satisfies $\operatorname{ind}_{c(\mathbb{T})}(0) = 0$. Thus, we may apply Theorem 15 and Corollary 3 to develop the Neumann problem in $\mathcal{D}_{a,\gamma,\delta}$. The exposition is much the same as that for data on logarithmic spirals at the end of the 'Neumann data on logarithmic spirals' section.

Theorem 20. *As defined previously, the operator $T(b_{a,\gamma,\delta})$ is Fredholm in the space H^{p+} if and only if $\mu \neq 0$.*

Proof. Since $\mu \in \left(-\frac{1}{q}, \frac{1}{p}\right)$, we get the following:

$$-\frac{2}{q} < \mathfrak{f}_1 < 1 - \frac{2}{q}, \quad -\frac{1}{q} < \mathfrak{f}_2 = \mathfrak{f}_3 < \frac{1}{p}.$$

By Theorem 7, the operator $T(b_{a,\gamma,\delta})$ is Fredholm if and only if $\mathfrak{f}_1 \neq -1/q$, which is equivalent to $\mu \neq 0$. □

Theorem 21. *Let*

$$\mu \in \left(0, \frac{1}{p}\right), \tag{48}$$

then

(1) The operators $T(b_{a,\gamma,\delta})$ is invertible in the space H^{p+}, and its symbol has a p-factorisation $b_{a,\gamma,\delta} = b^+ b^-$ with

$$\begin{aligned} b^+(\zeta) &= \frac{\sqrt[q]{\mathfrak{c}'_{a,\gamma,\delta}(\zeta)}}{w(\zeta)}, \\ b^-(\zeta) &= \frac{\overline{w(\zeta)}}{\overline{\sqrt[q]{\mathfrak{c}'_{a,\gamma,\delta}(\zeta)}}}. \end{aligned}$$

(2) In order that the Neumann problem might have a solution in $\Re E^{1,p}(\mathcal{D}_{a,\gamma,\delta}, w)$, it is necessary and sufficient that

$$\frac{1}{\pi \imath} \int_{\mathbb{T}} u_1(\mathfrak{c}_{a,\gamma,\delta}(\zeta)) \, |\mathfrak{c}'_{a,\gamma,\delta}(\zeta)| \, \frac{d\zeta}{\zeta} = 0.$$

(3) Under this condition, any solution of the Neumann problem in $\Re E^{1,p}(\mathcal{D}_{a,\gamma,\delta}, w)$ is of the following form:

$$u(z) = \Re \int_{z_0}^{z} (\ln \mathfrak{c}_{a,\gamma,\delta}^{-1}(z'))' \times \left(\int_{\mathbb{T}} u_1(\mathfrak{c}_{a,\gamma,\delta}(\zeta)) \frac{1}{\pi \imath} \frac{|\mathfrak{c}'_{a,\gamma,\delta}(\zeta)|}{\zeta - \mathfrak{c}_{a,\gamma,\delta}^{-1}(z')} d\zeta \right) dz'$$

for $z \in \mathcal{D}_{a,\gamma,\delta}$, where $z_0 \in \mathcal{D}_{a,\gamma,\delta}$ is an arbitrary fixed point and the integration is over any curve in $\mathcal{D}_{a,\gamma,\delta}$ connecting z_0 and z.

Proof. If $\mu \in \left(0, \frac{1}{p}\right)$, then

$$-\frac{1}{q} < \mathfrak{f}_1 < -\frac{1}{q} + \frac{1}{p} < \frac{1}{p}.$$

In this case, Theorem 7 implies readily that the operator $T(b_{a,\gamma,\delta})$ is invertible in the space H^{p+}. The p-factorisation is now derived analogously to that in the proof of Theorem 10. The remaining part of the theorem follows immediately from Corollary 3. □

Theorem 22. *If*

$$\mu \in \left(-\frac{1}{q}, 0\right), \tag{49}$$

then $T(b_{a,\gamma,\delta})$ *is a Fredholm operator in* H^{p+} *of index 1, and the symbol has a* p*-factorisation* $b_{a,\varphi} = -b^{+}\zeta^{-1}b^{-}$ *with*

$$b^{+}(\zeta) = (1-\zeta)\,\frac{\sqrt[q]{\mathfrak{c}'_{a,\gamma,\delta}(\zeta)}}{w(\zeta)},$$
$$b^{-}(\zeta) = \left(1-\frac{1}{\zeta}\right)^{-1}\frac{\overline{w(\zeta)}}{\sqrt[q]{\overline{\mathfrak{c}'_{a,\varphi}(\zeta)}}}.$$

Proof. If $\mu \in \left(-\frac{1}{q}, 0\right)$, then

$$-1-\frac{1}{q} < \mathfrak{f}_1 < -\frac{1}{q} = -1+\frac{1}{p}.$$

Hence, from Theorem 7 it follows that the operator $T(b_{a,\gamma,\delta})$ in H^{p+} is Fredholm and its index equals 1. By Theorem 2, the symbol $b_{a,\gamma,\delta}$ admits a p-factorisation $b_{a,\gamma,\delta} = -b^{+}\zeta^{-1}b^{-}$. Arguing as in the proof of Theorem 11, we deduce that the factors $b^{\pm}$ have the desired form. □

We now turn to the Neumann problem provided that (49) is satisfied. In our case, Equation 42 looks like the following:

$$-T(b^{+}b^{-})\left(h^{+}\right)(\zeta) = g^{+}(\zeta) \tag{50}$$

for $\zeta \in \mathbb{T}$. The operator $T(b^{+}b^{-})$ is invertible, so Equation 50 has a unique solution in H^{p+} given by the following:

$$\begin{aligned} h^{+}(\zeta) &= -\frac{1}{b^{+}(\zeta)}\,P^{+}_{\mathbb{T}}\left(\frac{g^{+}}{b^{-}}\right)(\zeta) \\ &= -\frac{1}{b^{+}(\zeta)}\,P^{+}_{\mathbb{T}}\left(\frac{g}{b^{-}}\right)(\zeta) \\ &= -\frac{1}{1-\zeta}\,\frac{w(\zeta)}{\sqrt[q]{\mathfrak{c}'_{a,\gamma,\delta}(\zeta)}}\,P^{+}_{\mathbb{T}}\left(\left(1-\frac{1}{\zeta}\right)|\mathfrak{c}'_{a,\gamma,\delta}|\,2U_1\right) \end{aligned}$$

for $\zeta \in \mathbb{T}$.

Corollary 5. *Let (49) be satisfied, then for each* $u_1 \in L^p(\partial\mathcal{D}_{a,\gamma,\delta}, w)$, *the Neumann problem has a solution in* $\Re E^{1,p}(\mathcal{D}_{a,\gamma,\delta}, w)$. *The general solution of this problem has the following form:*

$$u(z) = \Re\int_{z_0}^{z}\frac{(\mathfrak{c}^{-1}_{a,\gamma,\delta})'(z')}{1-\mathfrak{c}^{-1}_{a,\gamma,\delta}(z')} \times\left(\int_{\mathbb{T}}\frac{1}{\pi\imath}\,\frac{1}{\zeta-\mathfrak{c}^{-1}_{a,\varphi}(z')}\,\frac{1-\zeta}{\zeta}\,|\mathfrak{c}'_{a,\gamma,\delta}(\zeta)|\,U_1(\zeta)\,d\zeta\right)dz'$$

for $z \in \mathcal{D}_{a,\gamma,\delta}$, *where* $z_0 \in \mathcal{D}_{a,\gamma,\delta}$ *is fixed and the outer integral is over any path in* $\mathcal{D}_{a,\gamma,\delta}$ *connecting* z_0 *and* z.

Proof. It suffices to use the formula $u = \Re f$, where f is determined from the following equation:

$$f'(\mathfrak{c}_{a,\gamma,\delta}(\zeta)) = \frac{h^{+}(\zeta)}{w(\zeta)\sqrt[p]{\mathfrak{c}'_{a,\gamma,\delta}(\zeta)}}$$

for $\zeta \in \mathbb{D}$. □

Conformal reduction of Zaremba problems

The Zaremba problem

Let $\mathcal{C}$ be a Jordan curve in the plane and $\mathcal{D}$ the bounded domain in $\mathbb{R}^2$ whose boundary is $\mathcal{C}$. We assume that $\mathcal{C}$ is smooth almost everywhere and denote by $\tau = \tau(z)$ the unit tangent vector of $\mathcal{C}$ at a point $z \in \mathcal{C}$. As defined previously, by $\nu = \nu(z)$, it is meant that the unit outward normal vector of $\mathcal{C}$ at z which exists for almost all points $z \in \mathcal{C}$.

Moreover, let $\mathcal{S}$ be a non-empty open arc on $\mathcal{C}$. Given any functions u_0 and u_1 on $\mathcal{S}$ and $\mathcal{C}\setminus\mathcal{S}$, respectively, we consider the problem of finding a harmonic function u in $\mathcal{D}$ such that

$$\begin{aligned} u &= u_0 \ \text{ on } \mathcal{S}, \\ (\partial/\partial\nu)u &= u_1 \ \text{ on } \mathcal{C}\setminus\mathcal{S}, \end{aligned}$$

cf. (4). Zaremba wrote in [13] that it was Wirtinger who pointed out to him the great practical importance of this mixed boundary problem.

Our standing assumption is that u_0 has a derivative along the arc $\mathcal{S}$ almost everywhere on $\mathcal{S}$, that is,

$$u_0'(z) := \frac{\partial}{\partial\tau}u_0(z) = \cos\alpha(z)\,\frac{\partial}{\partial x}u_0(z) + \sin\alpha(z)\,\frac{\partial}{\partial y}u_0(z)$$

for $z \in \mathcal{S}$, where $\alpha(z)$ is the angle at which the tangent of $\mathcal{C}$ at z intersects the real axis. Introduce the following function:

$$u_{0,1}(z) = \begin{cases} \imath\, u_0'(z), & \text{if } z \in \mathcal{S}, \\ u_1(z), & \text{if } z \in \mathcal{C}\setminus\mathcal{S}. \end{cases} \tag{51}$$

Our next goal is to reduce the mixed boundary value problem to a Toeplitz operator equation. To this end, we notice that if $u = \Re f$, where f is an analytic function in $\mathcal{D}$, then the Dirichlet condition on $\mathcal{S}$ can be rewritten equivalently up to a constant function as follows:

$$\Re\left(e^{\imath\alpha(z)}f'(z)\right) = u_0'(z) \tag{52}$$

for almost all $z \in \mathcal{S}$. On the other hand, the Neumann condition on $\mathcal{C}\setminus\mathcal{S}$ just amounts to the following:

$$\Re\left(e^{\imath\beta(z)}f'(z)\right) = u_1(z) \tag{53}$$

for almost all $z \in \mathcal{C}\setminus\mathcal{S}$, where $\beta(z)$ is the angle at which the outward normal of $\mathcal{C}$ at z intersects the real axis (see (39)).

Pick a conformal mapping $z = \mathfrak{c}(\zeta)$ of the unit disk $\mathbb{D}$ onto $\mathcal{D}$, such that $\mathfrak{c}'(0) > 0$. Introduce the following weight function:

$$w(\zeta) = \prod_{k=1}^{n} \left(1 - \frac{\zeta}{\zeta_k}\right)^{-\mu_k}$$

for $\zeta \in \mathbb{D}$, where $\mu_k \in \left(-\frac{1}{q}, \frac{1}{p}\right)$, and define

$$\begin{aligned} h^+(\zeta) &= w(\zeta) \sqrt[p]{\mathfrak{c}'(\zeta)} f'(\mathfrak{c}(\zeta)), \\ h^-(\zeta) &= \overline{h^+(\zeta)}. \end{aligned}$$

On substituting these expressions into (52) and (53), we get a system of two real equations for the unknown complex-valued function h^+ of Hardy class on $\mathbb{T}$, namely

$$\begin{aligned} & e^{\imath\alpha(\mathfrak{c}(\zeta))} \frac{h^+(\zeta)}{w(\zeta)\sqrt[p]{\mathfrak{c}'(\zeta)}} + e^{-\imath\alpha(\mathfrak{c}(\zeta))} \frac{h^-(\zeta)}{\overline{w(\zeta)\sqrt[p]{\mathfrak{c}'(\zeta)}}} \\ &= 2\, u_0'(\mathfrak{c}(\zeta)), \quad \zeta \in \mathfrak{c}^{-1}(\mathcal{S}), \\ & e^{\imath\beta(\mathfrak{c}(\zeta))} \frac{h^+(\zeta)}{w(\zeta)\sqrt[p]{\mathfrak{c}'(\zeta)}} + e^{-\imath\beta(\mathfrak{c}(\zeta))} \frac{h^-(\zeta)}{\overline{w(\zeta)\sqrt[p]{\mathfrak{c}'(\zeta)}}} \\ &= 2\, u_1(\mathfrak{c}(\zeta)), \quad \zeta \in \mathfrak{c}^{-1}(\mathcal{C} \setminus \mathcal{S}). \end{aligned} \tag{54}$$

In the section 'The Neumann problem', we have proven that

$$\frac{e^{\imath\beta(\mathfrak{c}(\zeta))}}{w(\zeta)\sqrt[p]{\mathfrak{c}'(\zeta)}} = \frac{\zeta\, e^{\imath \frac{1}{q} \arg \mathfrak{c}'(\zeta)}}{w(\zeta)\sqrt[p]{|\mathfrak{c}'(\zeta)|}},$$

where $\frac{1}{p} + \frac{1}{q} = 1$. On taking into account that $\alpha(z) = \beta(z) + \frac{\pi}{2}$ for $z \in \mathcal{C}$, we can rewrite (54) in the following form:

$$\begin{aligned} \imath\, b(\zeta) \left(\zeta h^+(\zeta)\right) - \imath\, \zeta^{-1} h^-(\zeta) &= g_1(\zeta), \quad \zeta \in \mathfrak{c}^{-1}(\mathcal{S}), \\ b(\zeta) \left(\zeta h^+(\zeta)\right) + \zeta^{-1} h^-(\zeta) &= g_2(\zeta), \quad \zeta \in \mathfrak{c}^{-1}(\mathcal{C} \setminus \mathcal{S}) \end{aligned}$$

where for $\zeta \in \mathbb{T}$, with

$$b(\zeta) = e^{\imath \frac{2}{q} \arg \mathfrak{c}'(\zeta)} \frac{\overline{w(\zeta)}}{w(\zeta)}$$

and

$$\begin{aligned} g_1(\zeta) &= 2\, e^{\imath \frac{1}{q} \arg \mathfrak{c}'(\zeta)}\, \overline{w(\zeta)}\, \sqrt[p]{|\mathfrak{c}'(\zeta)|}\, u_0'(\mathfrak{c}(\zeta)), \\ g_2(\zeta) &= 2\, e^{\imath \frac{1}{q} \arg \mathfrak{c}'(\zeta)}\, \overline{w(\zeta)}\, \sqrt[p]{|\mathfrak{c}'(\zeta)|}\, u_1(\mathfrak{c}(\zeta)). \end{aligned}$$

Multiplying the first equation by $\imath$, we reduce system (54) finally to the following single equation:

$$\sigma_{\mathcal{S}}(\zeta)\, b(\zeta) \left(\zeta h^+(\zeta)\right) + \zeta^{-1} h^-(\zeta) = g(\zeta) \tag{55}$$

on $\mathbb{T}$, where

$$\sigma_{\mathcal{S}}(\zeta) = \begin{cases} -1, & \text{if } \zeta \in \mathfrak{c}^{-1}(\mathcal{S}), \\ 1, & \text{if } \zeta \in \mathfrak{c}^{-1}(\mathcal{C} \setminus \mathcal{S}), \end{cases}$$

and $g(\zeta) = 2\, e^{\imath \frac{1}{q} \arg \mathfrak{c}'(\zeta)}\, \overline{w(\zeta)}\, \sqrt[p]{|\mathfrak{c}'(\zeta)|}\, u_{0,1}(\mathfrak{c}(\zeta))$ (see (51) for the definition of $u_{0,1}$).

Use the following representation of the function $\sigma_{\mathcal{S}}(\zeta)$. Write A for the initial point of $\mathcal{S}$ and E for the end point. Let

$$\begin{aligned} \zeta_1 &= \mathfrak{c}^{-1}(A) =: e^{\imath\varphi_A}, \\ \zeta_2 &= \mathfrak{c}^{-1}(E) =: e^{\imath\varphi_E}, \end{aligned}$$

then

$$\sigma_{\mathcal{S}}(\zeta) = \sigma_{\mathcal{S}}^+(\zeta)\, \sigma_{\mathcal{S}}^-(\zeta) \tag{56}$$

for $\zeta \in \mathbb{T}$ with

$$\begin{aligned} \sigma_{\mathcal{S}}^+(\zeta) &= e^{-\imath \frac{1}{2}\varphi_A} \left(1 - \frac{\zeta}{e^{\imath\varphi_A}}\right)^{-\frac{1}{2}} \left(1 - \frac{\zeta}{e^{\imath\varphi_E}}\right)^{\frac{1}{2}}, \\ \sigma_{\mathcal{S}}^-(\zeta) &= e^{\imath \frac{1}{2}\varphi_E} \left(1 - \frac{e^{\imath\varphi_A}}{\zeta}\right)^{\frac{1}{2}} \left(1 - \frac{e^{\imath\varphi_E}}{\zeta}\right)^{-\frac{1}{2}}. \end{aligned}$$

The functions $\sigma_{\mathcal{S}}^+(\zeta)$ and $\sigma_{\mathcal{S}}^-(\zeta)$ are analytic in the unit disk $\mathbb{D}$ and in the complement of $\mathbb{D}$, respectively, and the branches of these functions are chosen in such a way that

$$\sqrt{1 - \frac{\zeta}{e^{\imath\varphi_{A,E}}}}\,\Big|_{\zeta=0} = 1, \qquad \sqrt{1 - \frac{e^{\imath\varphi_{A,E}}}{\zeta}}\,\Big|_{\zeta=\infty} = 1.$$

Notice that (56) fails to be a p-factorisation of $\sigma_{\mathcal{S}}(\zeta)$, for condition (15) is violated.

We now rewrite Equation 55 in the following form:

$$b(\zeta) \left(\zeta \sigma_{\mathcal{S}}^+(\zeta) h^+(\zeta)\right) + \zeta^{-1} \frac{h^-(\zeta)}{\sigma_{\mathcal{S}}^-(\zeta)} = \frac{g(\zeta)}{\sigma_{\mathcal{S}}^-(\zeta)}.$$

Set

$$\begin{aligned} \tilde{h}^+(\zeta) &= \sigma_{\mathcal{S}}^+(\zeta) h^+(\zeta), \\ \tilde{h}^-(\zeta) &= \tfrac{h^-(\zeta)}{\sigma_{\mathcal{S}}^-(\zeta)}. \end{aligned}$$

Applying to both sides of this equality the analytic projection $P_{\mathbb{T}}^+$ leads us to the following Toeplitz equation:

$$T(b) \left(\zeta \tilde{h}^+\right)(\zeta) = P_{\mathbb{T}}^+ \left(\frac{g}{\sigma_{\mathcal{S}}^-}\right)(\zeta) \tag{57}$$

on $\mathbb{T}$. It is quite natural to look for a solution of this equation in the Hardy space H^{p+}, so we assume that

$$\begin{aligned} \tilde{h}^+ &= \sigma_{\mathcal{S}}^+ w(\zeta) \sqrt[p]{\mathfrak{c}'(\zeta)} f'(\mathfrak{c}(\zeta)) && \in H^{p+}, \\ \tfrac{g}{\sigma_{\mathcal{S}}^-} &= 2\left(\sigma_{\mathcal{S}}^-\right)^{-1} e^{\imath \frac{1}{q} \arg \mathfrak{c}'(\zeta)}\, \overline{w(\zeta)}\, \sqrt[p]{|\mathfrak{c}'(\zeta)|}\, u_{0,1}(\mathfrak{c}(\zeta)) && \in L^p(\mathbb{T}). \end{aligned}$$

These preliminary considerations suggest a functional theoretic setting to treat (1) and (4).

Definition 6. *Given any Zaremba data (u_0, u_1) on $\mathcal{C}$ of class $L^p(\mathcal{C}, w/\sigma_{\mathcal{S}}^-)$ in the sense that*

$$\int_{\mathbb{T}} |u_{0,1}(\mathfrak{c}(\zeta))|^p \left|\frac{w(\zeta)}{\sigma_{\mathcal{S}}^-(\zeta)}\right|^p |\mathfrak{c}'(\zeta)|\, |d\zeta| < \infty,$$

we shall say that the Zaremba problem possesses a solution in $\Re E^{1,p}(\mathcal{D}, \sigma_{\mathcal{S}}^+ w)$ if there is a harmonic function u in $\mathcal{D}$, such that $u = \Re f$ for some $f \in E^{1,p}(\mathcal{D}, \sigma_{\mathcal{S}}^+ w)$ and u satisfies (4) on $\mathcal{C}$.

If $\tilde{h}^+$ is a solution of (57), then

$$f'(\mathfrak{c}(\zeta)) = \frac{\tilde{h}^+(\zeta)}{\sigma_{\mathcal{S}}^+(\zeta)w(\zeta)\sqrt[p]{\mathfrak{c}'(\zeta)}}$$

and

$$u(z) = u_s(z) + c \tag{58}$$

for all $z \in \mathcal{D}$, where

$$u_s(z) = \Re \int_{z_0}^{z} \tilde{h}^+(\mathfrak{c}^{-1}(z')) \frac{\sqrt[p]{(\mathfrak{c}^{-1})'(z')}}{\sigma_{\mathcal{S}}^+(\mathfrak{c}^{-1}(z'))w(\mathfrak{c}^{-1}(z'))} dz',$$

cf. (43), $z_0 = \mathfrak{c}^{-1}(0)$, the integral is over any path in $\mathcal{D}$ connecting the points z_0 and z, and c an arbitrary real constant. Each function u of family (58) satisfies the following conditions:

$$\begin{aligned} (\partial/\partial\tau)u &= u_0' \ \text{ on } \mathcal{S}, \\ (\partial/\partial\nu)u &= u_1 \ \text{ on } \mathcal{C}\setminus\mathcal{S}, \end{aligned}$$

but not necessarily the first condition of (4). The latter is satisfied by the one and only function of family (58). For finding the corresponding constant c, we observe that the function u_0 is continuous almost everywhere on $\mathcal{S}$. Suppose u_s is continuous up to at least one continuity point $z_1 \in \mathcal{S}$ of u_0, then $u_0(z_1) = u_s(z_1) + c$, implying $c = u_0(z_1) - u_s(z_1)$.

Theorem 23. *Let $u_{0,1} \in L^p(\mathcal{C}, w/\sigma_{\mathcal{S}}^-)$.*

1. *If $u = \Re f$ is a solution of the Zaremba problem in $\Re E^{1,p}(\mathcal{D}, \sigma_{\mathcal{S}}^+ w)$, then the function $\tilde{h}^+(\zeta) = \sigma_{\mathcal{S}}^+(\zeta)w(\zeta)\sqrt[p]{\mathfrak{c}'(\zeta)}f'(\mathfrak{c}(\zeta))$ is a solution of the Toeplitz equation (57) in H^{p+}.*
2. *If $\tilde{h}^+$ is a unique solution of the Toeplitz equation (57) in H^{p+}, then the function u given by (58) is a solution of the Zaremba problem in the space $\Re E^{1,p}(\mathcal{D}, \sigma_{\mathcal{S}}^+ w)$.*

Proof. This theorem just summarises the reduction of the Zaremba problem to a Toeplitz operator equation, as stated previously. The proof is analogous to that of Theorem 1. □

If the operator $T(b)$ is invertible in the space H^{p+}, then for the Zaremba problem to possess a solution in $\Re E^{1,p}(\mathcal{D}, \sigma_{\mathcal{S}}^+ w)$, it is necessary and sufficient that the condition

$$T(b)^{-1}\left(P_{\mathbb{T}}^+ \frac{g}{\sigma_{\mathcal{S}}^-}\right)(0) = 0 \tag{59}$$

be fulfilled. Indeed, applying $T(b)^{-1}$ to both sides of (57) yields the following:

$$\zeta\tilde{h}^+(\zeta) = T(b)^{-1}\left(P_{\mathbb{T}}^+ \frac{g}{\sigma_{\mathcal{S}}^-}\right)(\zeta).$$

On substituting $\zeta = 0$, we obtain (59). Conversely, if the latter condition is fulfilled, then

$$\tilde{h}^+(\zeta) = \frac{1}{\zeta}\, T(b)^{-1}\left(P_{\mathbb{T}}^+ \frac{g}{\sigma_{\mathcal{S}}^-}\right)(\zeta)$$

is of Hardy class H^{p+}, so (58) gives us the general solution of the Zaremba problem.

Corollary 6. *Assume that the symbol $b(\zeta)$ has a p-factorisation of the form $b = b^+b^-$ with*

$$\begin{aligned} b^+(\zeta) &= \frac{\sqrt[q]{\mathfrak{c}'(\zeta)}}{w(\zeta)}, \\ b^-(\zeta) &= \frac{\overline{w(\zeta)}}{\sqrt[q]{\overline{\mathfrak{c}'(\zeta)}}}, \end{aligned}$$

then

1. *For the Zaremba problem to possess a solution in $\Re E^{1,p}(\mathcal{D}, \sigma_{\mathcal{S}}^+ w)$, it is necessary and sufficient that*
$$\frac{1}{\pi\imath}\int_{\mathbb{T}} u_{0,1}(\mathfrak{c}(\zeta))\, \frac{|\mathfrak{c}'(\zeta)|}{\sigma_{\mathcal{S}}^-(\zeta)}\, \frac{d\zeta}{\zeta} = 0.$$
2. *Under this condition, the problem has a unique solution in $\Re E^{1,p}(\mathcal{D}, \sigma_{\mathcal{S}}^+ w)$ given by $u = u_s + c$ with*
$$u_s(z) = \Re\int_{z_0}^{z} \frac{(\ln \mathfrak{c}^{-1}(z'))'}{\sigma_{\mathcal{S}}^+(\mathfrak{c}^{-1}(z'))} \times \left(\int_{\mathbb{T}} u_{0,1}(\mathfrak{c}(\zeta))\, \frac{|\mathfrak{c}'(\zeta)|}{\sigma_{\mathcal{S}}^-(\zeta)}\, \frac{1}{\pi\imath}\, \frac{1}{\zeta - \mathfrak{c}^{-1}(z')}\, d\zeta\right) dz'$$
for $z \in \mathcal{D}$, where $z_0 = \mathfrak{c}^{-1}(0)$, the outer integral is over any curve connecting z_0 and z, and $c = u_0(z_1) - u_s(z_1)$.

Proof. Using the p-factorisation $b = b^+b^-$, we get the following:

$$\begin{aligned} \tilde{h}^+(\zeta) &= \frac{1}{\zeta}\, T(b)^{-1}\left(P_{\mathbb{T}}^+ \frac{g}{\sigma_{\mathcal{S}}^-}\right)(\zeta) \\ &= \frac{1}{\zeta}\, \frac{1}{b^+(\zeta)}\, P_{\mathbb{T}}^+\left(\frac{1}{b^-}\, P_{\mathbb{T}}^+ \frac{g}{\sigma_{\mathcal{S}}^-}\right)(\zeta) \\ &= \frac{2}{\zeta}\, \frac{w(\zeta)}{\sqrt[q]{\mathfrak{c}'(\zeta)}}\, P_{\mathbb{T}}^+\left(\frac{|\mathfrak{c}'|}{\sigma_{\mathcal{S}}^-}\, U_{0,1}\right)(\zeta), \end{aligned}$$

where $U_{0,1} := u_{0,1} \circ \mathfrak{c}$. Here, we have used the equality

$$P_{\mathbb{T}}^+ \frac{1}{b^-}\, P_{\mathbb{T}}^+ = P_{\mathbb{T}}^+ \frac{1}{b^-}$$

and the fact that $\sqrt[p]{|\mathfrak{c}'(\zeta)|}\sqrt[q]{|\mathfrak{c}'(\zeta)|} = |\mathfrak{c}'(\zeta)|$. This proves part (1) of the corollary.

Furthermore, on writing the analytic projection $P^+_{\mathbb{T}}$ as the Cauchy integral for $\zeta \in \mathbb{D}$, we deduce the following:

$$\frac{\tilde{h}^+(\zeta)}{\sigma^+_{\mathcal{S}}(\zeta)\, w(\zeta)\, \sqrt[p]{\mathfrak{c}'(\zeta)}} = \frac{1}{\zeta}\, \frac{1}{\sigma^+_{\mathcal{S}}(\zeta)\mathfrak{c}'(\zeta)} \times \int_{\mathbb{T}} \frac{1}{\pi\imath}\, \frac{1}{\zeta'-\zeta}\, \frac{|\mathfrak{c}'(\zeta')|}{\sigma^-_{\mathcal{S}}(\zeta')}\, U_{0,1}(\zeta')\, d\zeta',$$

which establishes part (2). □

Zaremba data on sectorial curves

Corollary 6 readily applies to the Zaremba problem in domains bounded by q-sectorial curves.

Theorem 24. *Let $\mathcal{C}$ be a q-sectorial curve and (u_0, u_1) Zaremba data on $\mathcal{C}$ of class $L^p(\mathcal{C}, 1/\sigma^-_{\mathcal{S}})$, then the Zaremba problem has a solution in $\Re E^{1,p}(\mathcal{D}, \sigma^+_{\mathcal{S}})$ if and only if*

$$\frac{1}{\pi\imath} \int_{\mathbb{T}} u_{0,1}(\mathfrak{c}(\zeta))\, \frac{|\mathfrak{c}'(\zeta)|}{\sigma^-_{\mathcal{S}}(\zeta)}\, \frac{d\zeta}{\zeta} = 0.$$

If the solvability condition is fulfilled, then the Zaremba problem has actually a unique solution in the space $\Re E^{1,p}(\mathcal{D}, \sigma^+_{\mathcal{S}})$. This solution is of the form $u = u_s + c$ with

$$u_s(z) = \Re \int_{z_0}^{z} \frac{(\ln \mathfrak{c}^{-1}(z'))'}{\sigma^+_{\mathcal{S}}(\mathfrak{c}^{-1}(z'))} \times \left(\int_{\mathbb{T}} u_{0,1}(\mathfrak{c}(\zeta))\, \frac{|\mathfrak{c}'(\zeta)|}{\sigma^-_{\mathcal{S}}(\zeta)}\, \frac{1}{\pi\imath}\, \frac{1}{\zeta - \mathfrak{c}^{-1}(z')}\, d\zeta \right) dz'$$

for $z \in \mathcal{D}$, where $z_0 = \mathfrak{c}^{-1}(0)$, the outer integral is over any curve connecting z_0 and z, and $c = u_0(z_1) - u_s(z_1)$, z_1 being an arbitrary point on the smooth part of $\mathcal{S}$.

Zaremba data on logarithmic spirals

Suppose $\mathcal{D}_{a,\varphi}$ is a simply connected domain in the plane bounded by two logarithmic spirals and an auxiliary curve as described in the 'Dirichlet data on logarithmic spirals' section. As defined previously, the symbol of the Toeplitz operator corresponding to the Zaremba problem in $\Re E^{1,p}(\mathcal{D}_{a,\varphi}, \sigma^+_{\mathcal{S}})$ is as follows:

$$b_{a,\varphi}(\zeta) = \exp\left(\imath\, \frac{2}{q} \arg \mathfrak{c}'_{a,\varphi}(\zeta) \right),$$

the argument of $\mathfrak{c}'_{a,\varphi}(\zeta)$ being given by (26).

The arguments given at the beginning of the 'Neumann data on logarithmic spirals' section still hold for $b_{a,\varphi}$. To study the Zaremba problem in the domain $\mathcal{D}_{a,\varphi}$, we employ Theorem 23 and Corollary 6. Our standing assumption on the Zaremba data (u_0, u_1) on the logarithmic spiral is $u_{0,1} \in L^p(\mathcal{C}_{a,\varphi}, 1/\sigma^-_{\mathcal{S}})$.

Theorem 25. *Suppose that*

$$q > \frac{a}{\pi}\, \cos\varphi$$

then

1. *(1) For the Zaremba problem to possess a solution in $\Re E^{1,p}(\mathcal{D}_{a,\varphi}, \sigma^+_{\mathcal{S}})$, it is necessary and sufficient that*

$$\frac{1}{\pi\imath} \int_{\mathbb{T}} u_{0,1}(\mathfrak{c}_{a,\varphi}(\zeta))\, \frac{|\mathfrak{c}'_{a,\varphi}(\zeta)|}{\sigma^-_{\mathcal{S}}(\zeta)}\, \frac{d\zeta}{\zeta} = 0.$$

2. *(2) If this solvability condition is fulfilled, then the Zaremba problem has actually a unique solution in the space $\Re E^{1,p}(\mathcal{D}_{a,\varphi}, \sigma^+_{\mathcal{S}})$. The solution is of the form $u = u_s + c$ with*

$$u_s(z) = \Re \int_{z_0}^{z} \frac{(\ln \mathfrak{c}^{-1}_{a,\varphi}(z'))'}{\sigma^+_{\mathcal{S}}(\mathfrak{c}^{-1}_{a,\varphi}(z'))} \times \left(\int_{\mathbb{T}} u_{0,1}(\mathfrak{c}_{a,\varphi}(\zeta))\, \frac{|\mathfrak{c}'_{a,\varphi}(\zeta)|}{\sigma^-_{\mathcal{S}}(\zeta)}\, \frac{1}{\pi\imath}\, \frac{1}{\zeta - \mathfrak{c}^{-1}_{a,\varphi}(z')}\, d\zeta \right) dz'$$

for $z \in \mathcal{D}_{a,\varphi}$, where $z_0 = \mathfrak{c}^{-1}_{a,\varphi}(0)$, the outer integral is over any path connecting z_0 and z, and $c = u_0(z_1) - u_s(z_1)$, z_1 being an arbitrary point on the smooth part of $\mathcal{S}$.

Proof. Indeed, under the assumption of the theorem, the operator $T(b_{a,\varphi})$ is invertible in the space H^{p+}, and the symbol has a p-factorisation of the form $b_{a,\varphi} = b^+ b^-$ with

$$b^+(\zeta) = \sqrt[q]{\mathfrak{c}'_{a,\varphi}(\zeta)},$$
$$b^-(\zeta) = \frac{1}{\sqrt[q]{\overline{\mathfrak{c}'_{a,\varphi}(\zeta)}}},$$

which is due to Theorem 18. Applying Corollary 6, we get the remaining part of the theorem. □

Theorem 26. *Let*

$$q < \frac{a}{\pi}\, \cos\varphi,$$

then

1. *(1) The Zaremba problem has a solution in the space $\Re E^{1,p}(\mathcal{D}_{a,\varphi}, \sigma^+_{\mathcal{S}})$ if and only if*

$$\left(\frac{\partial}{\partial\zeta} \right)^j \int_{\mathbb{T}} \frac{1}{\pi\imath}\, \frac{1}{\zeta'-\zeta}\, \frac{\zeta'}{\zeta'-1}\, \frac{|\mathfrak{c}'_{a,\varphi}(\zeta')|}{\sigma^-_{\mathcal{S}}(\zeta')}\, U_{0,1}(\zeta')d\zeta' \Big|_{\zeta=0} = 0$$

for $j = 0, 1$.

2. *(2) Under these conditions, the solution is unique and has the following form: $u = u_s + c$ with*

$$u_s(z) = \Re \int_{z_0}^{z} \frac{(\ln \mathfrak{c}^{-1}_{a,\varphi}(z'))'}{\sigma^+_{\mathcal{S}}(\mathfrak{c}^{-1}_{a,\varphi}(z'))}\, \frac{\mathfrak{c}^{-1}_{a,\varphi}(z') - 1}{\mathfrak{c}^{-1}_{a,\varphi}(z')} \times \left(\int_{\mathbb{T}} \frac{1}{\pi\imath}\, \frac{1}{\zeta - \mathfrak{c}^{-1}_{a,\varphi}(z')}\, \frac{\zeta}{\zeta - 1}\, \frac{|\mathfrak{c}'_{a,\varphi}(\zeta)|}{\sigma^-_{\mathcal{S}}(\zeta)}\, U_{0,1}(\zeta)\, d\zeta \right) dz'$$

for $z \in \mathcal{D}_{a,\varphi}$, where $z_0 = \mathfrak{c}^{-1}_{a,\varphi}(0)$, the outer integral is over any path in $\mathcal{D}_{a,\varphi}$ connecting z_0 and z, and $c = u_0(z_1) - u_s(z_1)$, z_1 being an arbitrary point on the smooth part of $\mathcal{S}$.

Proof. Indeed, under the assumption of the theorem, $T(b_{a,\varphi})$ is a Fredholm operator in H^{p+} of index -1, and its symbol has a p-factorisation of the form $b_{a,\varphi} = -b^+ \zeta b^-$ with

$$b^+(\zeta) = \frac{1}{1-\zeta} \sqrt[q]{\mathfrak{c}'_{a,\varphi}(\zeta)},$$
$$b^-(\zeta) = \left(1 - \frac{1}{\zeta}\right) \frac{1}{\overline{\sqrt[q]{\mathfrak{c}'_{a,\varphi}(\zeta)}}}$$

(see Theorem 19), so Equation (57) is as follows:

$$-T(b^+ \zeta b^-) \left(\zeta \tilde{h}^+\right)(\zeta) = P^+_{\mathbb{T}} \left(\frac{g}{\sigma^-_{\mathcal{S}}}\right)(\zeta)$$

for $\zeta \in \mathbb{T}$. The operator $T(b^+ b^-)$ is invertible. Applying the inverse to both sides of the last equality, it yields the following:

$$-\zeta^2 \tilde{h}^+(\zeta) = \frac{1}{b^+(\zeta)} P^+_{\mathbb{T}} \left(\frac{1}{b^-} \frac{g}{\sigma^-_{\mathcal{S}}}\right)(\zeta)$$
$$= \frac{1-\zeta}{\sqrt[q]{\mathfrak{c}'_{a,\varphi}(\zeta)}} P^+_{\mathbb{T}} \left(\frac{\zeta}{\zeta - 1} \frac{|\mathfrak{c}'_{a,\varphi}|}{\sigma^-_{\mathcal{S}}} 2U_{0,1}\right)(\zeta).$$

The function $\tilde{h}^+$ determined from this equality belongs to H^{p+} if and only if the right-hand side of the equality has zero of multiplicity two at the point $\zeta = 0$. Since $(1-\zeta)/\sqrt[q]{\mathfrak{c}'_{a,\varphi}(\zeta)}$ does not vanish at the origin, familiar reasoning completes the proof. □

Remark 6. *We do not consider $q = \frac{a}{\pi} \cos\varphi$, in which case, the operator $T(b_{a,\varphi})$ fails to be Fredholm in H^{p+}.*

Zaremba data on spirals of power type

Consider the Zaremba problem in a simply connected domain $\mathcal{D}_{a,\gamma,\delta}$ bounded by two power-like spirals and an auxiliary curve as described in the 'Dirichlet data on spirals of power type' section. The functional theoretic setting is suggested by the particular method we use for the study and consists of weighted Hardy-Smirnov spaces $\Re E^{1,p}(\mathcal{D}_{a,\gamma,\delta}, \sigma^+_{\mathcal{S}} w)$, where $w(\zeta) = (1-\zeta)^{-\mu}$ with $-1/q < \mu < 1/p$. From what has been shown in section 'The Zaremba problem', it follows that the Toeplitz operator corresponding to the Zaremba problem has the following symbol:

$$b_{a,\gamma,\delta}(\zeta) = \exp\left(\imath \frac{2}{q} \arg \mathfrak{c}'_{a,\gamma,\delta}(\zeta)\right) \frac{\overline{w(\zeta)}}{w(\zeta)}$$
$$= \exp\left(\imath \frac{2}{q} \arg \mathfrak{c}'_{a,\gamma,\delta}(\zeta)\right) (-\zeta)^{\mu},$$

for $\zeta \in \mathbb{T}$.

Our standing assumption on the Zaremba data (u_0, u_1) on power-like spirals is $u_{0,1} \in L^p(\partial \mathcal{D}_{a,\gamma,\delta}, w/\sigma^-_{\mathcal{S}})$.

Theorem 27. *If*

$$\mu \in \left(0, \frac{1}{p}\right),$$

then

(1) *For the Zaremba problem to possess a solution in $\Re E^{1,p}(\mathcal{D}_{a,\gamma,\delta}, \sigma^+_{\mathcal{S}} w)$, it is necessary and sufficient that*

$$\frac{1}{\pi\imath} \int_{\mathbb{T}} U_{0,1}(\zeta) \frac{|\mathfrak{c}'_{a,\gamma,\delta}(\zeta)|}{\sigma^-_{\mathcal{S}}(\zeta)} \frac{d\zeta}{\zeta} = 0.$$

(2) *If this solvability condition is fulfilled, then the Zaremba problem has a unique solution in the space $\Re E^{1,p}(\mathcal{D}_{a,\gamma,\delta}, \sigma^+_{\mathcal{S}} w)$. The solution is of the form $u = u_s + c$ with*

$$u_s(z) = \Re \int_{z_0}^{z} \frac{(\ln \mathfrak{c}^{-1}_{a,\gamma,\delta}(z'))'}{\sigma^+_{\mathcal{S}}(\mathfrak{c}^{-1}_{a,\gamma,\delta}(z'))} \times \left(\int_{\mathbb{T}} \frac{1}{\pi\imath} \frac{1}{\zeta - \mathfrak{c}^{-1}_{a,\gamma,\delta}(z')} \frac{|\mathfrak{c}'_{a,\gamma,\delta}(\zeta)|}{\sigma^-_{\mathcal{S}}(\zeta)} U_{0,1}(\zeta)\, d\zeta\right) dz'$$

for $z \in \mathcal{D}_{a,\gamma,\delta}$, where $z_0 = \mathfrak{c}^{-1}_{a,\gamma,\delta}(0)$, the outer integral is over any path connecting z_0 and z, and $c = u_0(z_1) - u_s(z_1)$, z_1 being an arbitrary point on the smooth part of $\mathcal{S}$.

Proof. Indeed, under the assumption of the theorem, the operator $T(b_{a,\gamma,\delta})$ is invertible in the space H^{p+}, and the symbol has a p-factorisation of the form $b_{a,\gamma,\delta} = b^+ b^-$ with

$$b^+(\zeta) = \frac{\sqrt[q]{\mathfrak{c}'_{a,\gamma,\delta}(\zeta)}}{w(\zeta)},$$
$$b^-(\zeta) = \frac{\overline{w(\zeta)}}{\overline{\sqrt[q]{\mathfrak{c}'_{a,\gamma,\delta}(\zeta)}}},$$

which is due to Theorem 21, and Corollary 6 gives to us all statements of the theorem. □

Theorem 28. *If*

$$\mu \in \left(-\frac{1}{q}, 0\right),$$

then, given any data of class $L^p(\partial \mathcal{D}_{a,\gamma,\delta}, w/\sigma^-_{\mathcal{S}})$, the Zaremba problem possesses a unique solution in the space $\Re E^{1,p}(\mathcal{D}_{a,\varphi}, \sigma^+_{\mathcal{S}} w)$. This solution is given by $u = u_s + c$ with

$$u_s(z) = \Re \int_{z_0}^{z} \frac{1}{\mathfrak{c}^{-1}_{a,\gamma,\delta}(z') - 1} \frac{(\mathfrak{c}^{-1}_{a,\gamma,\delta})'(z')}{\sigma^+_{\mathcal{S}}(\mathfrak{c}^{-1}_{a,\varphi}(z'))} \times \left(\int_{\mathbb{T}} \frac{1}{\pi\imath} \frac{1}{\zeta - \mathfrak{c}^{-1}_{a,\gamma,\delta}(z')} \frac{\zeta - 1}{\zeta} \frac{|\mathfrak{c}'_{a,\gamma,\delta}(\zeta)|}{\sigma^-_{\mathcal{S}}(\zeta)} U_{0,1}(\zeta) d\zeta\right) dz'$$

for $z \in \mathcal{D}_{a,\gamma,\delta}$, where $z_0 = \mathfrak{c}^{-1}_{a,\gamma,\delta}(0)$, the outer integral is over any curve in $\mathcal{D}_{a,\gamma,\delta}$ connecting z_0 and z, and $c =$

$u_0(z_1) - u_s(z_1)$, z_1 *being an arbitrary point on the smooth part of* $\mathcal{S}$.

Proof. By Theorem 22, $T(b_{a,\gamma,\delta})$ is a Fredholm operator in H^{p+} of index 1, and its symbol has a p-factorisation of the form $b_{a,\varphi} = -b^+\zeta^{-1}b^-$ with

$$b^+(\zeta) = (1-\zeta)\,\frac{\sqrt[q]{\mathfrak{c}'_{a,\gamma,\delta}(\zeta)}}{w(\zeta)},$$
$$b^-(\zeta) = \left(1-\frac{1}{\zeta}\right)^{-1}\frac{\overline{w(\zeta)}}{\sqrt[q]{\overline{\mathfrak{c}'_{a,\varphi}(\zeta)}}}.$$

Thus, Equation 57 is as follows:

$$-T(b^+\zeta^{-1}b^-)\left(\zeta\tilde{h}^+\right)(\zeta) = T(b^+b^-)\tilde{h}^+(\zeta)$$
$$= P^+_{\mathbb{T}}\left(\frac{g}{\sigma^-_{\mathcal{S}}}\right)(\zeta)$$

for $\zeta \in \mathbb{T}$. The operator $T(b^+b^-)$ is invertible. Applying the inverse to both sides of the last equality yields the following unique solution:

$$-\tilde{h}^+(\zeta) = \frac{1}{b^+(\zeta)}P^+_{\mathbb{T}}\left(\frac{1}{b^-}\frac{g}{\sigma^-_{\mathcal{S}}}\right)(\zeta)$$
$$= \frac{1}{1-\zeta}\,\frac{w(\zeta)}{\sqrt[q]{\mathfrak{c}'_{a,\gamma,\delta}(\zeta)}}\,P^+_{\mathbb{T}}\left(\frac{\zeta-1}{\zeta}\,\frac{|\mathfrak{c}'_{a,\gamma,\delta}|}{\sigma^-_{\mathcal{S}}}\,2U_{0,1}\right)(\zeta).$$

The function $\tilde{h}^+$ defined by this equality belongs to H^{p+}, as is easy to check. We thus obtain $u = \Re f$, where f is a holomorphic function in the domain $\mathcal{D}_{a,\gamma,\delta}$, satisfying the following:

$$f'(\mathfrak{c}_{a,\gamma,\delta}(\zeta)) = \frac{1}{\zeta-1}\,\frac{1}{\mathfrak{c}'_{a,\gamma,\delta}(\zeta)}\,\frac{1}{\sigma^+_{\mathcal{S}}(\zeta)}$$
$$\times P^+_{\mathbb{T}}\left(\frac{\zeta-1}{\zeta}\,\frac{|\mathfrak{c}'_{a,\gamma,\delta}|}{\sigma^-_{\mathcal{S}}}\,2U_{0,1}\right)(\zeta)$$

for all $\zeta \in \mathbb{D}$. On arguing as in (58), we derive the desired formula for the solution u. □

Remark 7. *We do not consider* $\mu = 0$*, in which case, the operator* $T(b_{a,\gamma,\delta})$ *fails to be Fredholm in* H^{p+}.

Conclusions

We investigated the main boundary value problems for harmonic functions in a simply connected plane domain with strong singularities on the boundary. We developed a Fredholm theory of such problems in weighted function spaces when the boundary curve is of one of the following three classes: sectorial curves, logarithmic spirals and spirals of power type. Moreover, we elaborated a constructive invertibility theory for Toeplitz operators and thus derive explicit solvability conditions as well as formulas for solutions.

Competing interests
Both authors declare that they have no competing interest.

Authors' contributions
Both SG and NT worked in close connection over all range of problems considered in the paper. Both authors read and approved the final manuscript.

Acknowledgments
The authors acknowledge the financial support of the DFG (German Research Society), grant TA 289/8-1, and the CONACYT (National Society of Science and Technology), grant 102800. We are grateful to the referees who pointed out some errors in an earlier version and made helpful comments.

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Received: 7 December 2012 Accepted: 30 January 2013
Published: 3 March 2013

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doi:10.1186/2251-7456-7-14
Cite this article as: Grudsky and Tarkhanov: **Conformal reduction of boundary problems for harmonic functions in a plane domain with strong singularities on the boundary.** *Mathematical Sciences* 2013 **7**:14.