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A variational approach to a singular elliptic system involving critical Sobolev-Hardy exponents and concave-convex nonlinearities

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Abstract

This paper is concerned with a quasilinear elliptic system, which involves the Caffarelli-Kohn-Nirenberg inequality and multiple critical exponents. The existence and multiplicity results of positive solutions are obtained by variational methods.

Keywords: Nehari manifold, Critical Hardy-Sobolev exponent, Elliptic system, Multiple positive solutions, Concave-convex nonlinearities

MSC (2000): 35A15; 35B33; 35J70

Introduction

The aim of this paper is to establish the existence of nontrivial solutions to the following quasilinear elliptic system:

$$\begin{cases} -\Delta_p u - \mu \dfrac{|u|^{p-2}u}{|x|^p} = \dfrac{1}{p^*}F_u(x,u,v) + \lambda \dfrac{|u|^{q-2}u}{|x|^s}, & x \in \Omega, \\ -\Delta_p v - \mu \dfrac{|v|^{p-2}v}{|x|^p} = \dfrac{1}{p^*}F_v(x,u,v) + \theta \dfrac{|v|^{q-2}v}{|x|^s}, & x \in \Omega, \\ u = v = 0, & x \in \partial\Omega, \end{cases} \tag{1}$$

where $0 \in \Omega$ is a bounded domain in $\mathbb{R}^N$ $(N \geq 3)$ with smooth boundary $\partial\Omega$, $\lambda > 0$, $\theta > 0$, $0 \leq \mu < \overline{\mu} \triangleq \left(\frac{N-p}{p}\right)^p$, $0 \leq s < p$, $1 \leq q < p$, and $p^*(t) \triangleq \frac{p(N-t)}{N-p}$ is the Hardy-Sobolev critical exponent. Note that $p^*(0) = p^* = \frac{pN}{N-p}$ is the Sobolev critical exponent. We assume that $F \in C^1(\overline{\Omega} \times (\mathbb{R}^+)^2, \mathbb{R}^+)$ is positively homogeneous of degree p^*, that is, $F(x,tu,tv) = t^{p^*}F(x,u,v)$ $(t > 0)$ holds for all $(x,u,v) \in \overline{\Omega} \times (\mathbb{R}^+)^2$, $(F_u(x,u,v), F_v(x,u,v)) = \nabla F(x,u,v)$.

Problem (1) is related to the well-known Caffarelli-Kohn-Nirenberg inequality in [1]:

$$\left(\int_\Omega \frac{|u|^r}{|x|^t}dx\right)^{\frac{p}{r}} \leq C_{r,t,p}\int_\Omega |\nabla u|^p dx, \quad \text{for all} \quad u \in D_0^{1,p}(\Omega), \tag{2}$$

where $p \leq r < p^*(t)$. If $t = r = p$, the above inequality becomes the well-known Hardy inequality [1-3]:

$$\int_\Omega \frac{|u|^p}{|x|^p}dx \leq \frac{1}{\overline{\mu}}\int_\Omega |\nabla u|^p dx, \quad \text{for all} \quad u \in D_0^{1,p}(\Omega). \tag{3}$$

In the space $D_0^{1,p}(\Omega)$, we employ the following norm:

$$||u|| = ||u||_{D_0^{1,p}(\Omega)} := \left(\int_\Omega \left(|\nabla u|^p - \mu \frac{|u|^p}{|x|^p}\right)dx\right)^{\frac{1}{p}}, \quad \mu \in [0, \overline{\mu}).$$

Using the Hardy inequality (3), this norm is equivalent to the usual norm $\left(\int_\Omega |\nabla u|^p dx\right)^{\frac{1}{p}}$. The operator $L := \left(|\nabla \cdot|^{p-2}\nabla \cdot - \mu\frac{|\cdot|^{p-2}}{|x|^p}\right)$ is positive in $D^{1,p}(\Omega)$ if $0 \leq \mu < \overline{\mu}$.

Now, we define the space $W = D_0^{1,p}(\Omega) \times D_0^{1,p}(\Omega)$ with the norm

$$||(u,v)||^p = ||u||^p + ||v||^p.$$

Also, by Hardy inequality and Hardy-Sobolev inequality, for $0 \leq \mu < \overline{\mu}$, $0 \leq t < p$ and $p \leq r \leq p^*(t)$, we can define the best Hardy-Sobolev constant:

$$A_{\mu,t,r}(\Omega) = \inf_{u \in D_0^{1,p}(\Omega)\setminus\{0\}} \frac{\int_\Omega \left(|\nabla u|^p - \mu\frac{|u|^p}{|x|^p}\right)dx}{\left(\int_\Omega \frac{|u|^r}{|x|^t}dx\right)^{\frac{p}{r}}}.$$

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In the important case when $r = p^*(t)$, we simply denote $A_{\mu,t,p^*(t)}$ as $A_{\mu,t}$. Note that $A_{\mu,0}$ is the best constant in the Sobolev inequality, namely,

$$A_{\mu,0}(\Omega) = \inf_{u\in D_0^{1,p}(\Omega)\setminus\{0\}} \frac{\int_\Omega \left(|\nabla u|^p - \mu \frac{|u|^p}{|x|^p}\right) dx}{\left(\int_\Omega |u|^{p^*} dx\right)^{\frac{p}{p^*}}}.$$

Also, we denote

$$\tilde{A}_{\mu,F} = \inf_{(u,v)\in W\setminus\{(0,0)\}} \frac{\int_\Omega \left(|\nabla u|^p + |\nabla v|^p - \mu \frac{|u|^p+|v|^p}{|x|^p}\right) dx}{\left(\int_\Omega F(x,u,v)dx\right)^{\frac{p}{p^*}}}. \tag{4}$$

Throughout this paper, let R_0 be the positive constant such that $\Omega \subset B(0;R_0)$, where $B(0;R_0) = \{x \in \mathbb{R}^N : |x| < R_0\}$. By Hölder and Sobolev-Hardy inequalities, for all $u \in D_0^{1,p}(\Omega)$, we obtain

$$\begin{aligned}\int_\Omega \frac{|u|^q}{|x|^s} &\le \left(\int_{B(0;R_0)} |x|^{-s}\right)^{\frac{p^*(s)-q}{p^*(s)}} \left(\int_\Omega \frac{|u|^{p^*(s)}}{|x|^s}\right)^{\frac{q}{p^*(s)}} \\ &\le \left(\int_0^{R_0} r^{N-s+1} dr\right)^{\frac{p^*(s)-q}{p^*(s)}} A_{\mu,s}^{-\frac{q}{p}} ||u||^q \\ &\le \left(\frac{N\omega_N R_0^{N-s}}{N-s}\right)^{\frac{p^*(s)-q}{p^*(s)}} A_{\mu,s}^{-\frac{q}{p}} ||u||^q,\end{aligned} \tag{5}$$

where $\omega_N = \frac{2\pi^{\frac{N}{2}}}{N\Gamma(\frac{N}{2})}$ is the volume of the unit ball in $\mathbb{R}^N$.

Existence of nontrivial nonnegative solutions for elliptic equations with singular potentials was recently studied by several authors, but, essentially, only with a solely critical exponent. We refer, e.g., in bounded domains and for $p = 2$ in [3-6], and for general $p > 1$ in [7-11] and the references therein. For example, Kang [11] studied the following elliptic equation via the generalized Mountain Pass Theorem [12]:

$$\begin{cases} -\Delta_p u - \mu \frac{|u|^{p-2}u}{|x|^p} = \frac{|u|^{p^*(t)-2}u}{|x|^t} + \lambda \frac{|u|^{p-2}u}{|x|^s}, & x \in \Omega, \\ u = 0, & x \in \partial\Omega \end{cases} \tag{6}$$

where $\Omega \subset \mathbb{R}^N$ is a bounded domain, $1 < p < N$, $0 \le s,t < p$ and $0 \le \mu < \overline{\mu} \triangleq \left(\frac{N-p}{p}\right)^p$. Also, the authors in [13] via the Mountain Pass Theorem of Ambrosetti and Rabinowitz [14] proved that

$$-\Delta_p u - \mu \frac{u^{p-1}}{|x|^p} = |u|^{p^*-1} + \frac{u^{p^*(s)-1}}{|x|^s}, \qquad \text{in } \mathbb{R}^N$$

admits a positive solution in $\mathbb{R}^N$, whenever $\mu < \overline{\mu} \triangleq \left(\frac{N-p}{p}\right)^p$.

Also, in recent years, several authors have used the Nehari manifold to solve semilinear and quasilinear problems (see [15-22] and references therein). Brown and Zhang [23] have studied a subcritical semi-linear elliptic equation with a sign-changing weight function and a bifurcation real parameter in the case $p = 2$ and Dirichlet boundary conditions. In [22], the author studied the Equation 6 via the Nehari manifold. Exploiting the relationship between the Nehari manifold and fibering maps (i.e., maps of the form $t \mapsto J_\lambda(tu)$, where J_λ is the Euler function associated with the equation), they gave an interesting explanation of the well-known bifurcation result. In fact, the nature of the Nehari manifold changes as the parameter λ crosses the bifurcation value. In this work, we give a variational method which is similar to the fibering method (see [16,23]) to prove the existence and multiplicity of nontrivial nonnegative solutions of problem (1).

Before stating our result, we need the following assumptions:

(H1) $F : \overline{\Omega} \times (\mathbb{R}^+)^2 \to \mathbb{R}^+$ is a C^1 function such that $F(x, tu, tv) = t^{p^*} F(x, u, v)$ $(t > 0)$ holds for all $(x, u, v) \in \overline{\Omega} \times (\mathbb{R}^+)^2$;
(H2) $F(x, , u, 0) = F(x, , 0, v) = F_u(x, 0, v) = F_v(x, u, 0) = 0$ where $u, v \in \mathbb{R}^+$;
(H3) $F_u(x, u, v)$ and $F_v(x, u, v)$ are strictly increasing functions about $u > 0$ and $v > 0$.

Moreover, using assumption (H1), we have the so-called Euler identity

$$(u, v) \cdot \nabla F(x, u, v) = p^* F(x, u, v), \tag{7}$$

and

$$F(x, u, v) \le K \left(|u|^p + |v|^p\right)^{\frac{p^*}{p}}, \quad \text{for some constant } K > 0. \tag{8}$$

This paper is divided into three sections organized as follows: In the 'Notations and preliminaries,' we establish some elementary results. Finally, in the 'Main results and proof,' we state our main result (Theorem 1) and prove it.

Notations and preliminaries

The corresponding energy functional of problem (1) is defined by

$$J_{\lambda,\theta}(u, v) = \frac{1}{p}||(u, v)||^p - \frac{1}{p^*}\int_\Omega F(x, u, v)dx - \frac{1}{q}K_{\lambda,\theta}(u, v),$$

for each $(u, v) \in W$, where $K_{\lambda,\theta}(u, v) = \lambda \int_\Omega \frac{|u|^q}{|x|^s} dx + \theta \int_\Omega \frac{|v|^q}{|x|^s} dx$.

In order to verify $J_{\lambda,\theta} \in C^1(W, \mathbb{R})$, we need the following lemmas:

Lemma 1. *Suppose that (H3) holds. Assume that $F \in C^1(\overline{\Omega} \times \mathbb{R}^2, \mathbb{R})$ is positively homogenous of degree p^*, then $F_u, F_v \in C(\overline{\Omega} \times \mathbb{R}^{+2}, \mathbb{R}^+)$ is positively homogenous of degree $p^* - 1$.*

Moreover by Lemma 1, we get the existence of positive constant M such that

$$|F_u(x,u,v)| \leq M\left(|u|^{p^*-1} + |v|^{p^*-1}\right), \quad \forall x \in \overline{\Omega},\ u,v \in \mathbb{R}^+, \tag{9}$$

$$|F_v(x,u,v)| \leq M\left(|u|^{p^*-1} + |v|^{p^*-1}\right), \quad \forall x \in \overline{\Omega},\ u,v \in \mathbb{R}^+. \tag{10}$$

Now, we consider the functional $\psi(u,v) = \int_\Omega F(x,u,v)dx$, then by Lemma 1, (9), (10), and by similar computation as Lemma 2.2 in [24], we get the functional ψ of class $C^1(W, \mathbb{R}^+)$ and $\langle \psi'(u,v),(a,b)\rangle = \int_\Omega (F_u(x,u,v)a + F_u(x,u,v)b)dx$, where $(u,v),(a,b) \in W$. Thus, we have $J_{\lambda,\theta} \in C^1(W,\mathbb{R})$.

Now, we consider the problem on the Nehari manifold. Define the Nehari manifold (*cf.* [25]):

$$N_{\lambda,\theta} = \left\{(u,v) \in W \setminus \{(0,0)\} | \langle J'_{\lambda,\theta}(u,v),(u,v)\rangle = 0\right\},$$

where

$$\langle J'_{\lambda,\theta}(u,v),(u,v)\rangle = ||(u,v)||^p - \int_\Omega F(x,u,v)dx - K_{\lambda,\theta}(u,v).$$

Note that $N_{\lambda,\theta}$ contains every nonzero solution of (1). Define

$$\Phi_{\lambda,\theta}(u,v) = \langle J'_{\lambda,\theta}(u,v),(u,v)\rangle,$$

then for $(u,v) \in N_{\lambda,\theta}$,

$$\langle \Phi'_{\lambda,\theta}(u,v),(u,v)\rangle$$
$$= p||(u,v)||^p - p^* \int_\Omega F(x,u,v)dx - qK_{\lambda,\theta}(u,v) \tag{11}$$
$$= (p-q)||(u,v)||^p - (p^*-q)\int_\Omega F(x,u,v)dx \tag{12}$$
$$= (p-p^*)||(u,v)||^p - (q-p^*)K_{\lambda,\theta}(u,v). \tag{13}$$

Now, we split $N_{\lambda,\theta}$ into three parts:

$$N^+_{\lambda,\theta} = \left\{(u,v) \in N_{\lambda,\theta} : \langle \Phi'_{\lambda,\theta}(u,v),(u,v)\rangle > 0\right\},$$
$$N^0_{\lambda,\theta} = \left\{(u,v) \in N_{\lambda,\theta} : \langle \Phi'_{\lambda,\theta}(u,v),(u,v)\rangle = 0\right\},$$
$$N^-_{\lambda,\theta} = \left\{(u,v) \in N_{\lambda,\theta} : \langle \Phi'_{\lambda,\theta}(u,v),(u,v)\rangle < 0\right\}.$$

To state our main result, we now present some important properties of $N^+_{\lambda,\theta}$, $N^0_{\lambda,\theta}$ and $N^-_{\lambda,\theta}$.

Lemma 2. *There exists a positive number $C = C(p,q,N,S) > 0$ such that if $0 < \lambda^{\frac{p}{p-q}} + \theta^{\frac{p}{p-q}} < C$, then $N^0_{\lambda,\theta} = \emptyset$.*

Proof. Suppose otherwise, let

$$C = \left(\frac{p-q}{K(p^*-q)}\right)^{\frac{p}{p^*-p}} \left(\frac{p^*-p}{p^*-q}\right)^{\frac{p}{p-q}} \times \left(\frac{N\omega_N R_0^{N-s}}{N-s}\right)^{-\frac{p(p^*(s)-q)}{p^*(s)(p-q)}} A_{\mu,s}^{\frac{q}{p-q}} A_{\mu,0}^{\frac{p^*}{p^*-p}}.$$

Then, there exists (λ,θ) with

$$0 < \lambda^{\frac{p}{p-q}} + \theta^{\frac{p}{p-q}} < C,$$

such that $N^0_{\lambda,\theta} \neq \emptyset$. Then, for $(u,v) \in N^0_{\lambda,\theta}$, by (12) and (13), one can get

$$||(u,v)||^p = \frac{p^*-q}{p-q}\int_\Omega F(x,u,v)dx.$$

By the Sobolev imbedding theorem, the Minkowski inequality and (8),

$$\begin{aligned}\int_\Omega F(x,u,v)dx &\leq K\left(\int_\Omega \left(|u|^p + |v|^p\right)^{\frac{p^*}{p}} dx\right)^{\frac{p}{p^*}\cdot\frac{p^*}{p}} \\ &\leq K\left(\left(\int_\Omega |u|^{p^*}dx\right)^{\frac{p}{p^*}} + \left(\int_\Omega |v|^{p^*}dx\right)^{\frac{p}{p^*}}\right)^{\frac{p^*}{p}} \\ &= K\left(||u||^p_{L^{p^*}(\Omega)} + ||v||^p_{L^{p^*}(\Omega)}\right)^{\frac{p^*}{p}} \\ &\leq KA_{\mu,0}^{-\frac{p^*}{p}}\left(||u||^p + ||v||^p\right)^{\frac{p^*}{p}} \\ &= KA_{\mu,0}^{-\frac{p^*}{p}}||(u,v)||^{p^*}.\end{aligned} \tag{14}$$

It follows that

$$||(u,v)|| \geq \left(\frac{p-q}{K(p^*-q)}A_{\mu,0}^{\frac{p^*}{p}}\right)^{\frac{1}{p^*-p}},$$

and

$$\begin{aligned}\frac{p^*-p}{p^*-q}||(u,v)||^p = K_{\lambda,\theta}(u,v) &= \lambda\int_\Omega \frac{|u|^q}{|x|^s}dx + \theta\int_\Omega \frac{|v|^q}{|x|^s}dx \\ &\leq \left(\frac{N\omega_N R_0^{N-s}}{N-s}\right)^{\frac{p^*(s)-q}{p^*(s)}} \\ &\quad \times A_{\mu,s}^{-\frac{q}{p}}\left(\lambda^{\frac{p}{p-q}} + \theta^{\frac{p}{p-q}}\right)^{\frac{p-q}{p}} ||(u,v)||^q.\end{aligned}$$

Thus,

$$||(u,v)|| \leq \left(\frac{p^*-q}{p^*-p}\right)^{\frac{1}{p-q}} \left(\frac{N\omega_N R_0^{N-s}}{N-s}\right)^{\frac{p^*(s)-q}{p^*(s)(p-p)}} \times A_{\mu,s}^{-\frac{q}{p(p-p)}} \left(\lambda^{\frac{p}{p-q}} + \theta^{\frac{p}{p-q}}\right)^{\frac{1}{p}}.$$

This implies

$$\lambda^{\frac{p}{p-q}} + \theta^{\frac{p}{p-q}} \geq C.$$

This is a contradiction! Here,

$$C = \left(\frac{p-q}{K(p^*-q)}\right)^{\frac{p}{p^*-p}} \left(\frac{p^*-p}{p^*-q}\right)^{\frac{p}{p-q}} \times \left(\frac{N\omega_N R_0^{N-s}}{N-s}\right)^{-\frac{p(p^*(s)-q)}{p^*(s)(p-q)}} A_{\mu,s}^{\frac{q}{p-q}} A_{\mu,0}^{\frac{p^*}{p^*-p}}.$$

□

Lemma 3. *The energy functional $J_{\lambda,\theta}$ is coercive and bounded below on $N_{\lambda,\theta}$.*

Proof. If $(u,v) \in N_{\lambda,\theta}$, then by (5),

$$\begin{aligned} J_{\lambda,\theta}(u,v) &= \frac{1}{p}||(u,v)||^p - \frac{1}{p^*}\int_\Omega F(x,u,v)dx - \frac{1}{q}K_{\lambda,\theta}(u,v) \\ &\geq \frac{p^*-p}{pp^*}||(u,v)||^p - \left(\frac{p^*-q}{p^*q}\right)\left(\frac{N\omega_N R_0^{N-s}}{N-s}\right)^{\frac{p^*(s)-q}{p^*(s)}} \\ &\quad \times \left(\lambda^{\frac{p}{p-q}} + \theta^{\frac{p}{p-q}}\right)^{\frac{p-q}{p}} A_{\mu,s}^{-\frac{q}{p}} ||(u,v)||^q. \end{aligned}$$

Since $0 \leq s < N$, $1 < q < p < p^*$, we see that $J_{\lambda,\theta}$ is coercive and bounded below on $N_{\lambda,\theta}$. □

Furthermore, similar to the argument in Brown and Zhang (see[23], Theorem 2.3 or see Binding et al. [26]), we can conclude the following result:

Lemma 4. *Assume that (u_0,v_0) is a local minimizer for $J_{\lambda,\theta}$ on $N_{\lambda,\theta}$ and that $(u_0,v_0) \notin N_{\lambda,\theta}^0$, then $J'_{\lambda,\theta}(u_0,v_0) = 0$ in W^{-1}.*

Now, by Lemma 2, we let

$$\Theta_{C_0} = \left\{(\lambda,\theta) \in \mathbb{R}^2 \setminus \{(0,0)\} : \ 0 < \lambda^{\frac{p}{p-q}} + \theta^{\frac{p}{p-q}} < C\right\},$$

where $C_0 = \left(\frac{q}{p}\right)^{\frac{p}{p-q}} C < C$. If $(\lambda,\theta) \in \Theta_{C_0}$, we have $N_{\lambda,\theta} = N_{\lambda,\theta}^+ \cup N_{\lambda,\theta}^-$. Define

$$\xi_{\lambda,\theta} = \inf_{(u,v)\in N_{\lambda,\theta}} J_{\lambda,\theta}(u,v)$$
$$\xi_{\lambda,\theta}^+ = \inf_{(u,v)\in N_{\lambda,\theta}^+} J_{\lambda,\theta}(u,v)$$
$$\xi_{\lambda,\theta}^- = \inf_{(u,v)\in N_{\lambda,\theta}^-} J_{\lambda,\theta}(u,v)$$

Lemma 5. *There exists a positive number C_0 such that if $(\lambda,\theta) \in \Theta_{C_0}$, then*

(i) $\xi_{\lambda,\theta} \leq \xi_{\lambda,\theta}^+ < 0$;
(ii) there exists $d_0 = d_0(p,q,N,K,S,\lambda,\theta) > 0$ such that $\xi_{\lambda,\theta}^- > d_0$.

Proof. (i) For $(u,v) \in N_{\lambda,\theta}^+$, by (13), we have

$$K_{\lambda,\theta}(u,v) \geq \frac{p^*-p}{p^*-q}||(u,v)||^p,$$

and so

$$\begin{aligned} J_{\lambda,\theta}(u,v) &= \left(\frac{1}{p}-\frac{1}{p^*}\right)||(u,v)||^p - \left(\frac{1}{q}-\frac{1}{p^*}\right)K_{\lambda,\theta}(u,v) \\ &\leq \left(\frac{1}{p}-\frac{1}{p^*}\right)||(u,v)||^p - \left(\frac{1}{q}-\frac{1}{p^*}\right)\frac{p^*-p}{p^*-q}||(u,v)||^p \\ &= \frac{p^*-p}{p^*}\left(\frac{1}{p}-\frac{1}{q}\right)||(u,v)||^p < 0. \end{aligned}$$

Thus, from the definition of $\xi_{\lambda,\theta}$ and $\xi_{\lambda,\theta}^+$, we can deduce that $\xi_{\lambda,\theta} < \xi_{\lambda,\theta}^+ < 0$.
(ii) For $(u,v) \in N_{\lambda,\theta}^-$, by Lemma 2,

$$||(u,v)|| \geq \left(\frac{p-q}{K(p^*-q)} A_{\mu,0}^{\frac{p^*}{p}}\right)^{\frac{1}{p^*-p}}.$$

Moreover, by Lemma 3,

$$\begin{aligned} J_{\lambda,\theta}(u,v) &\geq \frac{p^*-p}{pp^*}||(u,v)||^p \\ &\quad - \left(\frac{p^*-q}{p^*q}\right)\left(\frac{N\omega_N R_0^{N-s}}{N-s}\right)^{\frac{p^*(s)-q}{p^*(s)}} \\ &\quad \times \left(\lambda^{\frac{p}{p-q}} + \theta^{\frac{p}{p-q}}\right)^{\frac{p-q}{p}} A_{\mu,s}^{-\frac{q}{p}} ||(u,v)||^q \end{aligned}$$

$$= ||(u,v)||^q \left[\frac{p^*(t)-p}{pp^*}||(u,v)||^{p-q} - \left(\frac{p^*-q}{p^*q}\right)\left(\frac{N\omega_N R_0^{N-s}}{N-s}\right)^{\frac{p^*(s)-q}{p^*(s)}} \times A_{\mu,s}^{-\frac{q}{p}}\left(\lambda^{\frac{p}{p-q}}+\theta^{\frac{p}{p-q}}\right)^{\frac{p-q}{p}}\right]$$

$$\geq \left(\frac{p-q}{K(p^*-q)}A_{\mu,0}^{\frac{p^*}{p}}\right)^{\frac{q}{p^*-p}}\left[\frac{p^*-p}{pp^*}||(u,v)||^{p-q} - \left(\frac{p^*-q}{p^*q}\right)\left(\frac{N\omega_N R_0^{N-s}}{N-s}\right)^{\frac{p^*(s)-q}{p^*(s)}} \times A_{\mu,s}^{-\frac{q}{p}}\left(\lambda^{\frac{p}{p-q}}+\theta^{\frac{p}{p-q}}\right)^{\frac{p-q}{p}}\right].$$

Thus, if $0 < \lambda^{\frac{p}{p-q}} + \theta^{\frac{p}{p-q}} < C_0$, then for each $(u,v) \in N^-_{\lambda,\theta}$ one can get

$$J_{\lambda,\theta}(u,v) \geq d_0 = d_0(p,q,N,K,S,\lambda,\theta) > 0.$$

For each $(u,v) \in W \setminus \{(0,0)\}$ such that $\int_\Omega F(x,u,v)dx > 0$, let

$$t_{\max} = \left(\frac{(p-q)||(u,v)||^p}{(p^*-q)\int_\Omega F(x,u,v)dx}\right)^{\frac{1}{p^*-p}}.$$

□

Lemma 6. *Assume that* $0 < \lambda^{\frac{p}{p-q}} + \theta^{\frac{p}{p-q}} < C_0$. *Then, for every* $(u,v) \in W$ *with* $\int_\Omega F(x,u,v)dx > 0$, *there exists* $t_{\max} > 0$ *such that there are unique* t^+ *and* t^- *with* $0 < t^+ < t_{\max} < t^-$ *such that* $(t^\pm u, t^\pm v) \in N^\pm_{\lambda,\theta}$ *and*

$$J_{\lambda,\theta}(t^+u,t^+v) = \inf_{0\leq t\leq t_{\max}} J_{\lambda,\theta}(tu,tv),$$
$$J_{\lambda,\theta}(t^-u,t^-v) = \sup_{t\geq t_{\max}} J_{\lambda,\theta}(tu,tv).$$

Proof. The proof is similar to Lemma 2.6 in [17] and is omitted. □

Remark 1. *If*

$$0 < \lambda^{\frac{p}{p-q}} + \theta^{\frac{p}{p-q}} < C_0,$$

then, by Lemmas 5 and 6 for every $(u,v) \in W$ *with* $\int_\Omega F(x,u,v)dx > 0$, *we can easily deduce that there exists* $t_{\max} > 0$ *such that there are unique* t^- *with* $t_{\max} < t^-$ *such that* $(t^-u,t^-v) \in N^-_{\lambda,\theta}$ *and*

$$J_{\lambda,\theta}(t^-u,t^-v) = \sup_{t\geq 0} J_{\lambda,\theta}(tu,tv) \geq \xi^-_{\lambda,\theta} > 0.$$

Main results and proof

We are now ready to state our main result.

Theorem 1. *Assume that* $0 \leq s < p$, $N \geq 3$, $0 \leq \mu < \overline{\mu}$ *and* $1 \leq q < p$. *Then, we have the following results:*

(i) If $\lambda,\theta > 0$ *satisfy* $\lambda^{\frac{p}{p-q}} + \theta^{\frac{p}{p-q}} < C$, *then (1) has at least one positive solution in* W.
(ii) If $\lambda,\theta > 0$ *satisfy* $0 < \lambda^{\frac{p}{p-q}} + \theta^{\frac{p}{p-q}} < C_0$, *then (1) has at least two positive solutions in* W.

Now, we give an example to illustrate the result of Theorem 1.

Example 1. *Consider the problem*

$$\begin{cases} -\Delta_p u - \mu\frac{|u|^{p-2}u}{|x|^p} = \frac{\alpha}{\alpha+\beta}|u|^{\alpha-2}|v|^\beta u + \lambda\frac{|u|^{q-2}u}{|x|^s}, & x \in \Omega,\\ -\Delta_p v - \mu\frac{|v|^{p-2}v}{|x|^p} = \frac{\beta}{\alpha+\beta}|u|^{\alpha}|v|^{\beta-2} v + \theta\frac{|v|^{q-2}v}{|x|^s}, & x \in \Omega,\\ u = v = 0, & x \in \partial\Omega, \end{cases} \quad (15)$$

where $1 < \alpha,\beta < p-1$, *and* $\alpha+\beta = p^*$. *Then, all conditions of Theorem 1 hold. Hence, the system (15) has at least one positive solution if* $\lambda^{\frac{p}{p-q}} + \theta^{\frac{p}{p-q}} < C$ *and has at least two positive solutions if* $0 < \lambda^{\frac{p}{p-q}} + \theta^{\frac{p}{p-q}} < C_0$.

First, we get the following result:

Lemma 7. *(i) If* $0 < \lambda^{\frac{p}{p-q}} + \theta^{\frac{p}{p-q}} < C$, *then there exists a* $(PS)_{\xi_{\lambda,\theta}}$*-sequence* $\{(u_n,v_n)\} \subset N_{\lambda,\theta}$ *in* W *for* $J_{\lambda,\theta}$;
(ii) If $0 < \lambda^{\frac{p}{p-q}} + \theta^{\frac{p}{p-q}} < C_0$, *then there exists a* $(PS)_{\xi^-_{\lambda,\theta}}$*-sequence* $\{(u_n,v_n)\} \subset N^-_{\lambda,\theta}$ *in* W *for* $J_{\lambda,\theta}$,
where C *is the positive constant given in Lemma 2, and* $C_0 = \left(\frac{q}{p}\right)^{\frac{p}{p-q}} C$.

Proof. The proof is similar to Proposition 9 in [19] and is omitted. □

Theorem 2. *Assume that* $0 \leq s < p$, $N \geq 3$, $0 \leq \mu < \overline{\mu}$, *and* $1 \leq q < p$. *If* $0 < \lambda^{\frac{p}{p-q}} + \theta^{\frac{p}{p-q}} < C$, *then there exists* $(u_0^+,v_0^+) \in N^+_{\lambda,\theta}$ *such that*

(i) $J_{\lambda,\theta}(u_0^+,v_0^+) = \xi_{\lambda,\theta} = \xi^+_{\lambda,\theta}$.
(ii) (u_0^+,v_0^+) *is a positive solution of (1),*
(iii) $J_{\lambda,\theta}(u_0^+,v_0^+) \to 0$ *as* $\lambda \to 0^+, \theta \to 0^+$.

Proof. By Lemma 7, there exists a minimizing sequence $\{(u_n,v_n)\}$ for $J_{\lambda,\theta}$ on $N_{\lambda,\theta}$ such that

$$J_{\lambda,\theta}(u_n,v_n) = \xi_{\lambda,\theta} + o(1) \quad \text{and} \quad J'_{\lambda,\theta}(u_n,v_n) = o(1) \quad \text{in} \quad W^{-1}. \quad (16)$$

Since $J_{\lambda,\theta}$ is coercive on $N_{\lambda,\theta}$ (see Lemma 3), we get $\{(u_n, v_n)\}$ is bounded in W. Thus, there is a subsequence $\{(u_n, v_n)\}$ and $(u_0^+, v_0^+)) \in W$ such that

$$\begin{cases} u_n \rightharpoonup u_0^+, v_n \rightharpoonup v_0^+, & \text{weakly in } D_0^{1,p}(\Omega), \\ u_n \rightharpoonup u_0^+, v_n \rightharpoonup v_0^+, & \text{weakly in } L^{p^*}(\Omega), \\ u_n \to u_0^+, v_n \to v_0^+, & \text{strongly in } L^q(\Omega, |x|^{-s}), \\ & \quad \text{for } 1 \le q < p^*(s), \\ u_n \to u_0^+, v_n \to v_0^+, & \text{a.e. in } \Omega. \end{cases} \tag{17}$$

This implies that

$$K_{\lambda,\theta}(u_n, v_n) \to K_{\lambda,\theta}(u_0^+, v_0^+), \quad \text{as} \quad n \to \infty.$$

By (16) and (17), it is easy to prove that (u_0^+, v_0^+) is a weak solution of problem (1). Since

$$J_{\lambda,\theta}(u_n, v_n) = \frac{p^* - p}{pp^*}||(u_n, v_n)||^p - \frac{p^* - q}{qp^*}K_{\lambda,\theta}(u_n, v_n) \ge -\frac{p^* - q}{qp^*}K_{\lambda,\theta}(u_n, v_n),$$

and by Lemma 5(i),

$$J_{\lambda,\theta}(u_n, v_n) \to \xi_{\lambda,\theta} < 0 \ \text{ as } \ n \to \infty.$$

Letting $n \to \infty$, we see that $K_{\lambda,\theta}(u_0^+, v_0^+) > 0$. Now, we prove that $u_n \to u_0^+$, $v_n \to v_0^+$ strongly in $D_0^{1,p}(\Omega)$ and $J_{\lambda,\theta}(u_0^+, v_0^+) = \xi_{\lambda,\theta}$.

By applying Fatou's lemma and $(u_0^+, v_0^+) \in N_{\lambda,\theta}$, we get

$$\begin{aligned} \xi_{\lambda,\theta} \le J_{\lambda,\theta}(u_0^+, v_0^+) &= \frac{p^* - p}{p^*p}||(u_0^+, v_0^+)||^p - \frac{p^* - q}{qp^*}K_{\lambda,\theta}(u_0^+, v_0^+) \\ &\le \liminf_{n\to\infty}\left(\frac{p^* - p}{p^*p}||(u_n, v_n)||^p - \frac{p^* - q}{qp^*}K_{\lambda,\theta}(u_n, v_n)\right) \\ &\le \liminf_{n\to\infty} J_{\lambda,\theta}(u_n, v_n) = \xi_{\lambda,\theta}. \end{aligned}$$

This implies that

$$J_{\lambda,\theta}(u_0^+, v_0^+) = \xi_{\lambda,\theta}, \quad \lim_{n\to\infty} ||(u_n, v_n)||^p = ||(u_0^+, v_0^+)||^p.$$

Then, $u_n \to u_0^+$ and $v_n \to v_0^+$ strongly in $D_0^{1,p}(\Omega)$.

Moreover, we have $(u_0^+, v_0^+) \in N_{\lambda,\theta}^+$. In fact, if $(u_0^+, v_0^+) \in N_{\lambda,\theta}^-$, by Lemma 6, there are unique t_0^+ and t_0^- such that $(t_0^+ u_0^+, t_0^+ v_0^+) \in N_{\lambda,\theta}^+$, $(t_0^- u_0^+, t_0^- v_0^+) \in N_{\lambda,\theta}^-$ and $t_0^+ < t_0^- = 1$. Since

$$\frac{d}{dt}J_{\lambda,\theta}(t_0^+ u_0^+, t_0^+ v_0^+) = 0 \ \text{ and } \ \frac{d^2}{dt^2}J_{\lambda,\theta}(t_0^+ u_0^+, t_0^+ v_0^+) > 0,$$

there exist $t_0^+ < \bar{t} \le t_0^-$ such that $J_{\lambda,\theta}(t_0^+ u_0^+, t_0^+ v_0^+) < J_{\lambda,\theta}(\bar{t}_0 u_0^+, \bar{t}_0 v_0^+)$. By Lemma 6, we have

$$\begin{aligned} J_{\lambda,\theta}(t_0^+ u_0^+, t_0^+ v_0^+) < J_{\lambda,\theta}(\bar{t}_0 u_0^+, \bar{t}_0 v_0^+) &\le J_{\lambda,\theta}(t_0^- u_0^+, t_0^- u_0^+) \\ &= J_{\lambda,\theta}(u_0^+, v_0^+) \end{aligned}$$

which contradicts $J_{\lambda,\theta}(u_0^+, v_0^+) = \xi_{\lambda,\theta}^+$.

Since $J_{\lambda,\theta}(u_0^+, v_0^+) = J_{\lambda,\theta}(|u_0^+|, |v_0^+|)$ and $(|u_0^+|, |v_0^+|) \in N_{\lambda,\theta}^+$, by Lemma 4, we may assume that (u_0^+, v_0^+) is a nonnegative solution of problem (1).

Moreover, by Lemmas 3 and 5, we have

$$\begin{aligned} 0 > \xi_{\lambda,\theta} &= J_{\lambda,\theta}(u_0^+, v_0^+) \\ &\ge -\left(\frac{p^* - q}{p^*q}\right)\left(\frac{N\omega_N R_0^{N-s}}{N - s}\right)^{\frac{p^*(s)-q}{p^*(s)}} \\ &\quad \times \left(\lambda^{\frac{p}{p-q}} + \theta^{\frac{p}{p-q}}\right)^{\frac{p-q}{p}} A_{\mu,s}^{-\frac{q}{p}}||(u_0^+, v_0^+)||^q. \end{aligned}$$

This implies that $J_{\lambda,\theta}(u_0^+, v_0^+) \to 0$ as $\lambda \to 0^+$, $\theta \to 0^+$. □

Also, we need the following version of Brèzis-Lieb lemma [27].

Lemma 8. *Consider $F \in C^1((\mathbb{R}^+)^2, \mathbb{R}^+)$ with $F(0,0) = 0$ and $|F_u(x,u,v)|, |F_v(x,u,v)| \le C_1\left(|u|^{p-1} + |v|^{p-1}\right)$ for some $1 \le p < \infty$, $C_1 > 0$. Let (u_n, v_n) be bounded sequence in $L^p(\overline{\Omega}, (\mathbb{R}^+)^2)$, and such that $(u_n, v_n) \rightharpoonup (u, v)$ weakly in W_k. Then, one has*

$$\int_\Omega F(u_n, v_n)dx \to \int_\Omega F(u_n - u, v_n - v)dx + \int_\Omega F(u, v)dx \quad \text{as} \ \ n \to \infty.$$

Lemma 9. *Assume that $0 \le s < p$, $1 \le q < p$, and $0 \le \mu < \overline{\mu}$. If $\{(u_n, v_n)\} \subset W$ is a $(PS)_c$-sequence for $J_{\lambda,\theta}$ for all $0 < c < c^* := \frac{1}{N}(\widetilde{A}_{\mu,F})^{\frac{N}{p}}$, then there exists a subsequence of $\{(u_n, v_n)\}$ converging weakly to a nonzero solution of (1).*

Proof. Suppose $(u_n, v_n)\} \subset W$ satisfies $J_{\lambda,\theta}(u_n, v_n) \to c$ and $J'_{\lambda,\theta}(u_n, v_n) \to 0$ with $c < c^*$. It is easy to show that $\{(u_n, v_n)\}$ is bounded in W and there exists (u, v) such that $(u_n, v_n) \rightharpoonup (u, v)$ up to a subsequence. Moreover, we may assume

$$\begin{cases} u_n \rightharpoonup u, \quad v_n \rightharpoonup v, & \text{weakly in } D_0^{1,p}(\Omega), \\ u_n \rightharpoonup u, \quad v_n \rightharpoonup v, & \text{weakly in } L^{p^*}(\Omega), \\ u_n \to u, \quad v_n \to v, & \text{strongly in } L^q(\Omega, |x|^{-s}), \\ & \quad \text{for all } \ 1 \le q < p, \\ u_n \to u, \quad v_n \to v, & \text{a.e. on } \Omega. \end{cases}$$

Hence, we have $J'_{\lambda,\theta}(u) = 0$ by the weak continuity of $J_{\lambda,\theta}$ and

$$K_{\lambda,\theta}(u_n, v_n) \to K_{\lambda,\theta}(u, v). \tag{18}$$

Let $\widetilde{u}_n = u_n - u$ and $\widetilde{v}_n = v_n - v$. Then, by Brèzis-Lieb lemma [27], we obtain

$$||(\widetilde{u}_n, \widetilde{v}_n)||^p \to ||(u_n, v_n)||^p - ||(u, v)||^p, \quad \text{as} \quad n \to \infty, \tag{19}$$

and by Lemma 8,

$$\int_\Omega F(x, \widetilde{u}_n, \widetilde{v}_n)dx \to \int_\Omega F(x, u_n, v_n)dx - \int_\Omega F(x, u, v)dx \quad \text{as} \quad n \to \infty. \tag{20}$$

Since $J_{\lambda,\theta}(u_n, v_n) = c + o(1)$, $J'_{\lambda,\theta}(u_n, v_n) = o(1)$ and (18) to (20), we can deduce that

$$\frac{1}{p}||(\widetilde{u}_n, \widetilde{v}_n)||^p - \frac{1}{p^*}\int_\Omega F(x, \widetilde{u}_n, \widetilde{v}_n)dx = c - J_{\lambda,\theta}(u, v) + o(1),$$

and

$$||(\widetilde{u}_n, \widetilde{v}_n)||^p - \int_\Omega F(x, \widetilde{u}_n, \widetilde{v}_n)dx = o(1).$$

Now, we define

$$l := \lim_{n\to\infty} \int_\Omega F(x, \widetilde{u}_n, \widetilde{v}_n)dx, \quad l := \lim_{n\to\infty} ||(\widetilde{u}_n, \widetilde{v}_n)||^p. \tag{21}$$

From the definition of $\widetilde{A}_{\mu,F}$ and (21), one can get

$$\begin{aligned}\widetilde{A}_{\mu,F} l^{\frac{p}{p^*}} &= \widetilde{A}_{\mu,F} \lim_{n\to\infty} \left(\int_\Omega F(x, \widetilde{u}_n, \widetilde{v}_n)dx\right)^{\frac{p}{p^*}} \\ &\le \lim_{n\to\infty} ||(\widetilde{u}_n, \widetilde{v}_n)||^p = l,\end{aligned}$$

which implies that either

$$l = 0 \quad \text{or} \quad l \ge (\widetilde{A}_{\mu,F})^{\frac{p^*}{p^*-p}} = (\widetilde{A}_{\mu,F})^{\frac{N-t}{p-t}}. \tag{22}$$

Note that $\langle J'_{\lambda,\theta}(u, v), (u, v)\rangle = 0$ and

$$J_{\lambda,\theta}(u, v) = J_{\lambda,\theta}(u, v) - \frac{1}{p}\langle J'_{\lambda,\theta}(u, v), (u, v)\rangle \ge 0. \tag{23}$$

From (21) and (23), we get

$$\begin{aligned} c &= J(u_n, v_n) + o(1) = J_{\lambda,\theta}(\widetilde{u}_n, \widetilde{v}_n) + J_{\lambda,\theta}(u, v) + o(1) \\ &\ge \frac{1}{p}||(\widetilde{u}_n, \widetilde{v}_n)||^p - \frac{1}{p^*}\int_\Omega F(x, \widetilde{u}_n, \widetilde{v}_n)dx \\ &= \frac{p^* - p}{pp^*} l + o(1) = \frac{1}{N} l + o(1). \end{aligned} \tag{24}$$

By (22) to (24) and the assumption $c < c^*$, we deduce that $l = 0$. Up to a subsequence, $(u_n, v_n) \to (u, v)$ strongly in W. □

Lemma 10. *[28] Assume that $1 < p < N$, $0 \le t < p$, and $0 \le \mu < \overline{\mu}$. Then, the limiting problem*

$$\begin{cases} -\Delta_p u - \mu\dfrac{|u|^{p-1}}{|x|^p} = \dfrac{|u|^{p^*(t)-1}}{|x|^t}, & \text{in } \mathbb{R}^N \setminus \{0\}, \\ u \in W^{1,p}(\mathbb{R}^N), \quad u > 0, & \text{in } \mathbb{R}^N \setminus \{0\}, \end{cases}$$

has positive radial ground states

$$V_\epsilon(x) \triangleq \epsilon^{\frac{p-N}{p}} U_{p,\mu}\left(\frac{x}{\epsilon}\right) = \epsilon^{\frac{p-N}{p}} U_{p,\mu}\left(\frac{|x|}{\epsilon}\right), \quad \forall \epsilon > 0, \tag{25}$$

that satisfy

$$\begin{aligned}\int_\Omega \left(|\nabla V_\epsilon(x)|^p - \mu\frac{|V_\epsilon(x)|^p}{|x|^p}\right)dx &= \int_\Omega \frac{|V_\epsilon(x)|^{p^*(t)}}{|x|^t}dx \\ &= (A_{\mu,t})^{\frac{N-t}{p-t}},\end{aligned}$$

where $U_{p,\mu}(x) = U_{p,\mu}(|x|)$ is the unique radial solution of the limiting problem with

$$U_{p,\mu}(1) = \left(\frac{(N-t)(\overline{\mu} - \mu)}{N - p}\right)^{\frac{1}{p^*(t)-p}}.$$

Furthermore, $U_{p,\mu}$ have the following properties:

$$\begin{aligned}&\lim_{r\to 0} r^{a(\mu)} U_{p,\mu}(r) = C_1 > 0, \\ &\lim_{r\to +\infty} r^{b(\mu)} U_{p,\mu}(r) = C_2 > 0, \\ &\lim_{r\to 0} r^{a(\mu)+1}|U'_{p,\mu}(r)| = C_1 a(\mu) \ge 0, \\ &\lim_{r\to +\infty} r^{b(\mu)+1}|U'_{p,\mu}(r)| = C_2 b(\mu) > 0,\end{aligned}$$

where $C_i (i = 1, 2)$ are positive constants and $a(\mu)$ and $b(\mu)$ are zeros of the function

$$f(\zeta) = (p-1)\zeta^p - (N-p)\zeta^{p-1} + \mu, \quad \zeta \ge 0, \quad 0 \le \mu < \overline{\mu},$$

that satisfy

$$0 \le a(\mu) < \frac{N-p}{p} < b(\mu) \le \frac{N-p}{p-1}.$$

Now, we will give some estimates on the extremal function $V_\epsilon(x)$ defined in (25). For $m \in \mathbb{N}$ large, choose $\varphi(x) \in C_0^\infty(\mathbb{R}^N)$, $0 \le \varphi(x) \le 1$, $\varphi(x) = 1$ for $|x| \le \frac{1}{2m}$, $\varphi(x) = 0$ for $|x| \ge \frac{1}{m}$, $||\nabla\varphi(x)||_{L^p(\Omega)} \le 4m$, set $u_\epsilon(x) = \varphi(x)V_\epsilon(x)$.

For $\epsilon \to 0$, the behavior of u_ϵ has to be the same as that of V_ϵ, but we need precise estimates of the error terms. For $1 < p < N$, $0 \le s,t < p$ and $1 < q < p^*(s)$, we have the following estimates [28]:

$$\int_\Omega \left(|\nabla u_\epsilon|^p - \mu \frac{|u_\epsilon|^p}{|x|^p} \right) dx = (A_{\mu,t})^{\frac{N-t}{p-t}} + O\left(\epsilon^{b(\mu)p+p-N}\right), \tag{26}$$

$$\int_\Omega \frac{|u_\epsilon|^{p^*(t)}}{|x|^t} dx = (A_{\mu,t})^{\frac{N-t}{p-t}} + O\left(\epsilon^{b(\mu)p^*(t)-N+t}\right), \tag{27}$$

$$\int_\Omega \frac{|u_\epsilon|^q}{|x|^s} dx \ge \begin{cases} C\epsilon^{N-s+(1-\frac{N}{p})q}, & q > \dfrac{N-s}{b(\mu)}, \\ C\epsilon^{N-s+(1-\frac{N}{p})q} |\ln \epsilon|, & q = \dfrac{N-s}{b(\mu)}, \\ C\epsilon^{q(b(\mu)+1-\frac{N}{p})q}, & q < \dfrac{N-s}{b(\mu)}. \end{cases} \tag{28}$$

Lemma 11. *Assume that $0 \le s < p$, $1 \le q < p$, and $0 \le \mu < \overline{\mu}$. There exists a nonnegative function $(u,v) \in W \setminus \{(0,0)\}$ and $\delta_1 > 0$ such that for $\lambda, \theta > 0$ satisfy $0 < \lambda^{\frac{p}{p-q}} + \theta^{\frac{p}{p-q}} < \delta_1$, we have*

$$\sup_{\tau \ge 0} J(\tau u, \tau v) < c^* := \frac{1}{N} (\widetilde{A}_{\mu,F})^{\frac{N}{p}}. \tag{29}$$

In particular, $\xi_{\lambda,\theta} < \frac{1}{N}(\widetilde{A}_{\mu,F})^{\frac{N}{p}}$ for all $0 < \lambda^{\frac{p}{p-q}} + \theta^{\frac{p}{p-q}} < \delta_1$.

Proof. Set $u = e_1 u_\epsilon$, $v = e_2 u_\epsilon$, and $(u,v) \in W$, where $(e_1, e_2) \in (\mathbb{R}^+)^2$, $e_1^p + e_2^p = 1$ and $\inf_{x \in \overline{\Omega}} F(x, e_1, e_2) \ge K$. Then, we consider the functions

$$\begin{aligned} g(\tau) &= J_{\lambda,\theta}(\tau e_1 u_\epsilon, \tau e_2 u_\epsilon) = \frac{\tau^p}{p} ||(e_1 u_\epsilon, e_2 u_\epsilon)||^p \\ &\quad - \frac{\tau^q}{q} K_{\lambda,\theta}(\tau e_1 u_\epsilon, \tau e_2 u_\epsilon) \\ &\quad - \frac{\tau^{p^*}}{p^*} \int_\Omega F(x, e_1 u_\epsilon, e_2 u_\epsilon) dx, \\ g_1(\tau) &= \frac{\tau^p}{p} ||(e_1 u_\epsilon, e_2 u_\epsilon)||^p - \frac{\tau^{p^*}}{p^*} \int_\Omega F(x, e_1 u_\epsilon, e_2 u_\epsilon) dx. \end{aligned}$$

By (26), (27) for $t = 0$, (4) and the fact that

$$\sup_{\tau \ge 0} \left(\frac{\tau^p}{p} A - \frac{\tau^{p^*}}{p^*} B \right) = \frac{1}{N} \left(\frac{A}{B^{\frac{p}{p^*}}} \right)^{\frac{N}{p}}, \quad A, B > 0, \tag{30}$$

we conclude that

$$\begin{aligned} \sup_{\tau \ge 0} g_1(\tau) &\le \frac{1}{N} \left(\frac{(e_1^p + e_2^p) \int_\Omega (|\nabla u_\epsilon|^p - \mu \frac{|u_\epsilon|^p}{|x|^p}) dx}{(\int_\Omega F(x, e_1 u_\epsilon, e_2 u_\epsilon) dx)^{\frac{p}{p^*}}} \right)^{\frac{N}{p}} \\ &\le \frac{1}{N} \left(\frac{\int_\Omega (|\nabla u_\epsilon|^p - \mu \frac{|u_\epsilon|^p}{|x|^p}) dx}{K^{\frac{p}{p^*}} (\int_\Omega |u_\epsilon|^{p^*} dx)^{\frac{p}{p^*}}} \right)^{\frac{N}{p}} \\ &\le \frac{1}{N} \left(\frac{1}{K^{\frac{p}{p^*}}} \right)^{\frac{N}{p}} \left(\frac{(A_{\mu,0})^{\frac{N}{p}} + O(\epsilon^{b(\mu)p+p-N})}{((A_{\mu,0})^{\frac{N}{p}} + O(\epsilon^{b(\mu)p^*-N}))^{\frac{p}{p^*}}} \right)^{\frac{N}{p}} \\ &\le \frac{1}{N} \left(\frac{1}{K^{\frac{p}{p^*}}} \right)^{\frac{N}{p}} \left(A_{\mu,0} + O(\epsilon^{b(\mu)p+p-N}) \right)^{\frac{N}{p}} \\ &= \frac{1}{N} \left(\frac{1}{K^{\frac{p}{p^*}}} \right) \left((A_{\mu,0})^{\frac{N}{p}} + O(\epsilon^{b(\mu)p+p-N}) \right) \\ &\le \frac{1}{N} (\widetilde{A}_{\mu,F})^{\frac{N}{p}} + O(\epsilon^{b(\mu)p+p-N}). \end{aligned} \tag{31}$$

On the other hand, using the definitions of g and u_ϵ, we get

$$g(\tau) = J_{\lambda,\theta}(\tau e_1 u_\epsilon, \tau e_2 u_\epsilon) \le \frac{\tau^p}{p} ||(e_1 u_\epsilon, e_2 u_\epsilon)||^p,$$
$$\text{for all } \quad \tau \ge 0 \quad \text{and} \quad \lambda > 0, \theta > 0.$$

Combining this with (26) and let $\epsilon \in (0,1)$, then there exists $\tau_0 \in (0,1)$ independent of ϵ such that

$$\sup_{0 \le \tau \le \tau_0} g(\tau) < \frac{1}{N} (\widetilde{A}_{\mu,F})^{\frac{N}{p}}, \quad \text{for all } \ 0 < \lambda^{\frac{p}{p-q}} + \theta^{\frac{p}{p-q}} < \delta_1. \tag{32}$$

Hence, as $0 < \lambda^{\frac{p}{p-q}} + \theta^{\frac{p}{p-q}} < \delta_1$, $1 \le q < p$, by (31), we have that

$$\begin{aligned} \sup_{\tau \ge \tau_0} g(\tau) &= \sup_{\tau \ge \tau_0} \left(g_1(\tau) - \frac{\tau^q}{q} K_{\lambda,\theta}(e_1 u_\epsilon, e_2 u_\epsilon) \right) \\ &\le \frac{1}{N} (\widetilde{A}_{\mu,F})^{\frac{N}{p}} + o\left(\epsilon^{b(\mu)p+p-N}\right) \\ &\quad - \frac{\tau_0^q}{q} \left(e_1^q \lambda + e_2^q \theta \right) \int_\Omega \frac{|u_\epsilon|^q}{|x|^s} dx. \end{aligned} \tag{33}$$

(i) If $1 \le q < \frac{N-s}{b(\mu)}$, then by (28), we have that

$$\int_\Omega \frac{|u_\epsilon|^q}{|x|^s} dx \ge C\epsilon^{q(b(\mu)p+1-\frac{N}{p})},$$

and since $b(\mu) > \frac{N-p}{p}$, then

$$(b(\mu)p + p - N) > q(b(\mu)p + 1 - \frac{N}{p}).$$

Combining this with (32) and (33), for any $\lambda, \theta > 0$ which $0 < \lambda^{\frac{p}{p-q}} + \theta^{\frac{p}{p-q}} < \delta_1$, we can choose ϵ small enough such that

$$\sup_{\tau \geq 0} J(\tau e_1 u_\epsilon, \tau e_2 u_\epsilon) < \frac{1}{N} (\widetilde{A}_{\mu,F})^{\frac{N}{p}}.$$

(ii) If $\frac{N-s}{b(\mu)} \leq q < p$, then by (28) and $b(\mu) > \frac{N-p}{p}$ we have that

$$\int_\Omega \frac{|u_\epsilon|^q}{|x|^s} dx \geq \begin{cases} C\epsilon^{N-s+(1-\frac{N}{p})q}, & q > \dfrac{N-s}{b(\mu)}, \\ C\epsilon^{N-s+(1-\frac{N}{p})q} |\ln \epsilon|, & q = \dfrac{N-s}{b(\mu)}, \end{cases}$$

and

$$(b(\mu)p + p - N) > N - s + (1 - \frac{N}{p})q.$$

Combining this with (32) and (33), for any $\lambda, \theta > 0$ which $0 < \lambda^{\frac{p}{p-q}} + \theta^{\frac{p}{p-q}} < \delta_1$, we can choose ϵ small enough such that

$$\sup_{\tau \geq 0} J(\tau e_1 u_\epsilon, \tau e_2 u_\epsilon) < \frac{1}{N} (\widetilde{A}_{\mu,F})^{\frac{N}{p}}.$$

From (i) and (ii), (29) holds.

From Lemma 6, (29) and the definitions of $\xi^-_{\lambda,\theta}$, for any $\lambda, \theta > 0$ which $0 < \lambda^{\frac{p}{p-q}} + \theta^{\frac{p}{p-q}} < \delta_1$, we obtain that there exists $\tau^-_{\lambda,\theta}$ such that $(\tau^-_{\lambda,\theta} e_1 u_\epsilon, \tau^-_{\lambda,\theta} e_2 u_\epsilon) \in N^-_{\lambda,\theta}$ and

$$\begin{aligned} \xi^-_{\lambda,\theta} \leq J_{\lambda,\theta}(\tau^-_{\lambda,\theta} e_1 u_\epsilon, \tau^-_{\lambda,\theta} e_2 u_\epsilon) &\leq \sup_{\tau \geq 0} J(\tau e_1 u_\epsilon, \tau e_2 u_\epsilon) \\ &< \frac{1}{N} (\widetilde{A}_{\mu,F})^{\frac{N}{p}}. \end{aligned}$$

The proof is complete. □

Theorem 3. *Assume that $0 \leq s < p$, $1 \leq q < p$, and $0 \leq \mu < \overline{\mu}$. There exists $\Lambda > 0$ such that for any $\lambda, \theta > 0$ satisfy $0 < \lambda^{\frac{p}{p-q}} + \theta^{\frac{p}{p-q}} < \Lambda$, the functional $J_{\lambda,\theta}$ has a minimizer (U, V) in $N^-_{\lambda,\theta}$ and satisfies the following:*

(i) $J_{\lambda,\theta}(U, V) = \xi^-_{\lambda,\theta}$,
(ii) (U, V) is a positive solution of (1),

where $\Lambda = \min\{C_0, \delta_1\}$

Proof. If $0 < \lambda^{\frac{p}{p-q}} + \theta^{\frac{p}{p-q}} < C_0 = \left(\frac{q}{p}\right)^{\frac{p}{p-q}} C$, then by Lemmas 5(ii), 7, and 11, there exists a $(PS)_{\xi_{\lambda,\theta}}$-sequence $\{(u_n, v_n)\} \subset N^-_{\lambda,\theta}$ in W for $J_{\lambda,\theta}$ with $\xi^-_{\lambda,\theta} \in \left(0, \frac{1}{N}(\widetilde{A}_{\mu,F})^{\frac{N}{p}}\right)$. By Lemma 3, $\{(u_n, v_n)\}$ is bounded in W. From Lemma 9, there exists a subsequence denoted by $\{(u_n, v_n)\}$ and nontrivial solution $(U, V) \in W$ of (1) such that $u_n \rightharpoonup U$, $v_n \rightharpoonup V$ weakly in $D^{1,p}_0(\Omega)$.

First, we prove that $(U, V) \in N^-_{\lambda,\theta}$. Arguing by contradiction, we assume $(U, V) \in N^+_{\lambda,\theta}$. Since $N^-_{\lambda,\theta}$ is closed in $W^{1,p}_0(\Omega)$, we have $||(U, V)|| < \liminf_{n\to\infty} ||(u_n, v_n)||$. Thus, by Lemma 6, there exists a unique τ^- such that $(\tau^- U, \tau^- V) \in N^-_{\lambda,\theta}$. If $(u, v) \in N^-_{\lambda,\theta}$, then it is easy to see that

$$J_{\lambda,\theta}(u, v) = \frac{1}{N} ||(u, v)||^p - \frac{p^* - q}{qp^*} K_{\lambda,\theta}(u, v). \tag{34}$$

From Remark 1, $(u_n, v_n) \in N^-_{\lambda,\theta}$, $||(U, V)|| < \liminf_{n\to\infty} ||(u_n, v_n)||$ and (34), we can get

$$\begin{aligned} \xi^-_{\lambda,\theta} \leq J_{\lambda,\theta}(\tau^- U, \tau^- V) &\leq \lim_{n\to\infty} J_{\lambda,\theta}(\tau^- u_n, \tau^- v_n) \\ &< \lim_{n\to\infty} J_{\lambda,\theta}(u_n, v_n) = \xi^-_{\lambda,\theta}. \end{aligned}$$

This is a contradiction. Thus, $(U, V) \in N^-_{\lambda,\theta}$, Next, by the same argument as that in Theorem 2, we get that $(u_n, v_n) \to (U, V)$ strongly in W and $J_{\lambda,\theta}(U, V) = \xi^-_{\lambda,\theta} > 0$ for all $0 < \lambda^{\frac{p}{p-q}} + \theta^{\frac{p}{p-q}} < C_0 = \left(\frac{q}{p}\right)^{\frac{p}{p-q}} C$. Since $J_{\lambda,\theta}(U, V) = J_{\lambda,\theta}(|U|, |V|)$ and $(|U|, |V|) \in N^-_{\lambda,\theta}$, by Lemma 4 we may assume that (U, V) is a nontrivial nonnegative solution of (1). Finally, by the maximum principle [29], we obtain that (U, V) is a positive solution of (1). The proof is complete. □

Proof of Theorem 1. The part (i) of Theorem 1 immediately follows from Theorem 2. When $0 < \lambda^{\frac{p}{p-q}} + \theta^{\frac{p}{p-q}} < C_0 = \left(\frac{q}{p}\right)^{\frac{p}{p-q}} C < C$, by Theorems 2 and 3, we obtain (1) has at least two positive solutions (u_0, v_0) and (U, V) such that $(u_0, v_0) \in N^+_{\lambda,\theta}$ and $(U, V) \in N^-_{\lambda,\theta}$. Since $N^+_{\lambda,\theta} \cap N^-_{\lambda,\theta} = \emptyset$, this implies that $N^+_{\lambda,\theta}$ and $N^-_{\lambda,\theta}$ are distinct. This completes the proof of Theorem 1.

Competing interests

The author has no competing interests.

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