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Common fixed point theorems on weakly compatible maps on dislocated metric spaces

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Abstract

Purpose: The purpose of this paper is to study a common fixed point theorem for two pairs of weakly compatible maps in dislocated metric spaces.

Methods: Using familiar techniques, we extend the results of Hitzler and Kang et al. in dislocated metric spaces.

MSC: 47H10, 54H25

Keywords: Dislocated metric, Weakly compatible maps, Common fixed point, γ-Property

Introduction

In 1922, S. Banach proved a fixed point theorem for contraction mapping in metric space. Since then a number of fixed point theorems have been proved by different authors, and many generalizations of this theorem have been established. The notion of a dislocated metric (*d-metric*) space was introduced by Pascal Hitzler in [1] as a part of the study of logic programming semantics. The study of common fixed point mappings in dislocated metric space satisfying certain contractive conditions has been at the center of vigorous research activity, see for example in [2,3].

In 1996, Jungck [4] introduced the concept of weak compatibility. Since then, many interesting fixed point theorems of compatible and weakly compatible maps under various contractive conditions have been obtained by a number of authors. We proved two common fixed point theorems for four weakly compatible maps.

Preliminaries

For convenience we start with the following definitions, lemmas, and theorems.

Definition 2.1 [5] Let X be a non empty set and let d: $X \times X \to [0,\infty)$ be a function satisfying the following conditions:

i. $d(x, y) = d(y, x)$

ii. $d(x, y) = d(y, x) = 0$ implies that $x = y$ and

iii. $d(x, y) \le d(x, z) + d(z, y)$ for all $x, y, z \in X$.

Then d is called dislocated metric (or simply d-metric) on X.

Definition 2.2 [5] A sequence $\{x_n\}$ in a d-metric space (X, d) is called a *Cauchy sequence* if given $\in > 0$ there corresponds $n_0 \in N$ such that for all $m,n \ge n_0$, we have $d(x_m, x_n) < \in$.

Definition 2.3 [5] A sequence $\{x_n\}$ in d-metric space converges with respect to d (or in d) if there exists $x \in X$ such that $d(x_n, x) \to 0$ as $n \to \infty$. In this case, x is called *d-limit of* $\{x_n\}$ and we write $x_n \to x$.

Definition 2.4 [5] A d-metric space (X,d) is called *complete* if every Cauchy sequence in it is convergent with respect to d.

Definition 2.5 [5] Let (X,d) be a d-metric space. A map $T{:}X \to X$ is called *contraction* if there exists a number λ with $0 \le \lambda < 1$ such that $d(Tx,Ty) \le \lambda d(x,y)$. We state the following lemmas without proof.

Lemma 2.6 [5] *Let (X,d) be a d-metric space. If T: $X \to X$ is a contraction function, then $\{T^n(x_0)\}$ is a Cauchy sequence for $x_0 \in X$.*

Lemma 2.7 [5] *The limits in a d-metric space are unique.*

Definition 2.8 [6] Let A and S be mappings from a metric space (X,d) into itself. Then A and S are said to be weakly compatible if they commute at their 'coincident point' x, that is, $Ax = Sx$ implies $ASx = SAx$.

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Theorem 2.9 [5] *Let* (X,d) *be a complete d-metric space and let* $T:X \to X$ *be a contraction mapping, then T has a unique fixed point.*

Results and discussion

Theorem 3.1 *Let* (X,d) *be a complete d-metric space. Let* $A,B,S,T:X \to X$ *be continuous mappings satisfying*

I. $T(X) \subset A(X)$, $S(X) \subset B(X)$
II. *The pairs* (S,A) *and* (T,B) *are weakly compatible and*
III. $d(Sx,Ty) \leq \alpha \max\{d(Ax,By), d(Ax,Sx), d(By,Ty)\}$

For all $x, y \in X$, *where* $0 \leq \alpha < \frac{1}{2}$ *then A,B,S and T have a unique common fixed point.*

Proof. Using condition (i), we define sequences $\{x_n\}$ and $\{y_n\}$ in X by the rule

$y_{2n} = Bx_{2n+1} = Sx_{2n}$ and $y_{2n+1} = Ax_{2n+2} = Tx_{2n+1}$, $n = 0,1,2\ldots$, where $x_0 = x, y_0 = y$.

If $y_{2n} = y_{2n+1}$ is for some n, then $Bx_{2n+1} = Tx_{2n+1}$. Therefore x_{2n+1} is a coincident point of B and T. Also, if $y_{2n+1} = y_{2n+2}$ for some n, then $Ax_{2n+2} = Sx_{2n+2}$. Hence, x_{2n+2} is a coincident point of S and A.

Assume that $y_{2n} \neq y_{2n+1}$ for all n. Then, we have

$$\begin{aligned} d(y_{2n}, y_{2n+1}) &= d(Sx_{2n}, Tx_{2n+1}) \\ &\leq \alpha \max\{d(Ax_{2n}, Bx_{2n+1}), \\ &\quad d(Ax_{2n}, Sx_{2n}), d(Bx_{2n+1}, Tx_{2n+1})\} \\ &= \alpha \max\{d(y_{2n-1}, y_{2n}), d(y_{2n-1}, y_{2n}), \\ &\quad d(y_{2n}, y_{2n+1})\} \\ &= \alpha \max\{d(y_{2n-1}, y_{2n}), d(y_{2n}, y_{2n+1})\} \\ &= \alpha d(y_{2n-1}, y_{2n}) \end{aligned}$$

$$\therefore d(y_{2n}, y_{2n+1}) \leq \alpha d(y_{2n-1}, y_{2n});$$

$$\text{similarly, } d(y_{2n-1}, y_{2n}) \leq \alpha d(y_{2n-2}, y_{2n-1}).$$

$$\begin{aligned} \text{Hence, } \forall n \geq 1, d(y_n, y_{n+1}) &\leq \alpha d(y_{n-1}, y_n) \leq \ldots \\ &\leq \alpha^n d(y_0, y_1) \end{aligned}$$

Hence, by induction $d(y_n,y_{n+1}) \leq \alpha^n d(y_0,y_1)$.

Hence, for any integer $n \geq 1$ and $q \geq 1$,

$$\begin{aligned} d(y_n, y_{n+q}) &\leq d(y_n, y_{n+1}) + d(y_{n+1}, y_{n+2}) \\ &\quad + d(y_{n+2}, y_{n+3}) + \ldots + d(y_{n+q-1}, y_{n+q}) \\ &\leq (\alpha^n + \alpha^{n+1} + \ldots + \alpha^{n+q-1}) d(y_0, y_1) \\ &= \alpha^n (1 + \alpha + \ldots. + \alpha^{q-1}) d(y_0, y_1) \\ &= \frac{\alpha^n (1-\alpha^q)}{1-\alpha} d(y_0, y_1) \\ &< \alpha^n d(y_0, y_1), \quad \text{since } 0 \leq \alpha < 1. \end{aligned}$$

Since $\lim \alpha^n = 0$, it follows that $\{y_n\}$ is a Cauchy sequence in the complete dislocated metric space (X,d). So there exists $z \in X$ such that

$$\{y_n\} \to z.$$

Therefore, the subsequences

$$\{Sx_{2n}\} \to z, \ \{Bx_{2n+1}\} \to z, \{Tx_{2n+1}\} \to z \text{ and } \{Ax_{2n+2}\} \to z.$$

Since $T(X) \subset A(X)$, there exists $u \in X$ such that $z = Au$.

$$\begin{aligned} \text{So, } d(Su, z) &= d(Su, Tx_{2n+1}) \\ &\leq \alpha \max\{d(Au, Bx_{2n+1}), d(Au, Su), \\ &\quad d(Bx_{2n+1}, Tx_{2n+1})\}. \end{aligned}$$

Taking limits as $n \to \infty$ we get,

$$\begin{aligned} d(Su, z) &\leq \alpha \max\{d(z, Bx_{2n+1}), d(z, Su), d(z, z)\} \\ &= \alpha \max\{d(z, z), d(z, Su)\} \\ &= \alpha d(z, z) \\ &\leq \alpha [d(z, Su) + d(Su, z)] \\ &\leq 2\alpha d(z, Su) \quad \because d(z, Su) = d(Su, z) \\ &= \beta d(z, Su) \quad where\ \beta = 2\alpha < 1 \end{aligned}$$

$$\therefore d(z, Su) = 0$$

$$\therefore Su = z = Au$$

Again, since $S(X) \subset B(X)$, there exists a $v \,\varepsilon\, X$ such that $z = Bv$.

We claim that $z = Tv$, then

$$\begin{aligned} d(z, Tv) &= d(Su, Tv) \\ &\leq \alpha \max\{d(Au, Bv), d(Au, Su), d(Bv, Tv)\} \\ &= \alpha \max\{d(z, z), d(z, z), d(z, Tv)\} \\ &= \alpha d(z, z) \\ &\leq 2\alpha d(z, Tv) \end{aligned}$$

$$\therefore d(z, Tv) = 0,$$

so we get $z = Tv$.

Hence, $Su = Au = Tv = Bv = z$.

Since the pair (S,A) is weakly compatible, by definition $SAu = ASu$ implies $Sz = Az$.

Now, we show that z is a fixed point of S in the following:

$$\begin{aligned} d(Sz, z) &= d(Sz, Tv) \\ &\leq \alpha \max\{d(Az, Bv), d(Az, Sz), d(Bv, Tv)\} \qquad (1) \\ &= \alpha \max\{d(Sz, z), d(Sz, Sz), d(z, z)\} \end{aligned}$$

Since $0 \leq \alpha < 1$ and $d(Sz,z) \leq \alpha\, d\,(Sz,z)$ from Equation 1, we gset $Sz = z$. This implies that $Az = Sz = z$.

Again, the pair (T,B) are weakly compatible, by definition $TBv = BTv$ implies $Tz = Bz$.

Now, we show that z is a fixed point of T as:

$$\begin{aligned} d(z,Tz) &= d(Sz,Tz) \\ &\leq \alpha \max d(Az,Bz), d(Az,Sz), d(Bz,Tz) \} \\ &= \alpha \max d(z,Tz), d(z,z), d(Tz,Tz)\} \\ &= \alpha\, d(z,Tz) \ \because 0<\alpha<1; d(z,Tz)=0 \therefore Tz=z. \end{aligned}$$

Hence, we have $Az = Bz = Sz = Tz = z$. This shows that 'z' is a common fixed point of the self mappings A, B, S and T.

Uniqueness. Let $u \neq v$ be two common fixed points of the mappings A,B,S and T in the following:

$$\begin{aligned} d(u,v) &= d(Su,Tv) \\ &\leq \alpha \max\{d(Au,Bv), d(Au,Su), d(Bv,Tv)\} \\ &= \alpha \max\{d(u,v), d(u,u), d(v,v)\}. \qquad (2) \\ &\therefore \text{from}(2) d(u,v) \leq \alpha\, d(u,v) \\ &\therefore d(u,v) = 0. \end{aligned}$$

Since (X, d) is a dislocated metric space, so we have $u = v$.

Put $A = B = I$ an identity mapping in above Theorem 3.1 yields Corollary 3.2

Corollary 3.2 *Let (X,d) be a complete d-metric space. Let $S,T:X \to X$ be continuous mappings satisfying,*

$$d(Sx,Ty) \leq \alpha \max\{d(x,y), d(x,Sx), d(y,Ty)\} \text{ for all } x,y \in X,$$

where $0 \leq \alpha < \frac{1}{2}$, then S and T have a unique fixed point.

Then $S = T$ in Corollary 3.2 yields Corollary 3.3.

Corollary 3.3 *Let (X,d) be a complete d-metric space. Let $T:X \to X$ be a continuous mapping satisfying*

$d(Tx, Ty) \leq \alpha \max\{d(x, y), d(x, Tx), d(y, Ty)\}$ *for all x, $y \in X$,*

where $0 \leq \alpha < \frac{1}{2}$ then T have unique common fixed point.

Taking *A=T and B=S* in Theorem 3.1 yields Corollary 3.4.

Corollary 3.4 *Let (X,d) be a complete d-metric space. Let $S,T:X \to X$ be continuous mappings satisfying*

$$d(Sx,Ty) \leq \alpha \max\{d(Tx,Sy), d(Tx,Sx), d(Sy,Ty)\}$$

for all $x,y \in X$, where $0 \leq \alpha < \frac{1}{2}$, then S and T have unique common fixed point.

Theorem 3.5 Let (X,d) be a complete d-metric space. Let $A,B,S,T:X \to X$ be the continuous mapping satisfying,

I. $T(X) \subset A(X)$, $S(X) \subset B(X)$

II. The pairs (S,A) and (T,B) are weakly compatible and

III. $d(Sx,Ty) \leq \alpha \left\{ \frac{d(Ax,Sx)d(By,Ty)}{d(Ax,By)} \right\} + \beta\, d(Ax,By)$

for all $x,y \in X$ where $\alpha, \beta \geq 0$, $0 \leq \alpha+\beta < \frac{1}{4}$, then A,B,S and T have a unique common fixed point.

Proof. Using condition (i), we define sequences $\{x_n\}$ and $\{y_n\}$ in X by the rule,

$y_{2n} = Bx_{2n+1} = Sx_{2n}$, and $y_{2n+1} = Ax_{2n+2} = Tx_{2n+1}$, $n = 0,1,2\ldots$,

If $y_{2n} = y_{2n+1}$ for some n, then $Bx_{2n+1} = Tx_{2n+1}$. Therefore, x_{2n+1} is a coincident point of B and T. Also, if $y_{2n+1} = y_{2n+2}$ for some n, then $Ax_{2n+2} = Sx_{2n+2}$. Hence, x_{2n+2} is a coincident point of S and A.

Assume that $y_{2n} \neq y_{2n+1}$ for all n. Then, we have

$$\begin{aligned} d(y_{2n},y_{2n+1}) &= d(Sx_{2n},Tx_{2n+1}) \\ &\leq \alpha \left\{ \frac{d(Ax_{2n},Sx_{2n})d(Bx_{2n+1},Tx_{2n+1})}{d(Ax_{2n},Bx_{2n+1})} \right\} \\ &\quad + \beta\, d(Ax_{2n},Bx_{2n+1}) \\ &= \alpha \left\{ \frac{d(y_{2n-1},y_{2n})d(y_{2n},y_{2n+1})}{d(y_{2n-1},y_{2n})} \right\} \\ &\quad + \beta\, d(y_{2n-1},y_{2n}) \\ &= \alpha\{d(y_{2n},y_{2n+1})\} + \beta\, d(y_{2n-1},y_{2n}) \\ \Rightarrow d(y_{2n},y_{2n+1}) &\leq \frac{\beta}{1-\alpha}\, d(y_{2n-1},y_{2n}) \\ &= h\, d(y_{2n-1},y_{2n}), \end{aligned}$$

where $h = \frac{\beta}{1-\alpha} < 1$ and

similarly, $d(y_{2n-1},y_{2n}) \leq h\, d(y_{2n-2},y_{2n-1})$.

This shows that

$$d(y_n,y_{n+1}) \leq h\, d(y_{n-1},y_n) \leq \ldots \leq h^n d(y_0,y_1).$$

For every integer $q > 0$, we have

$$\begin{aligned} d(y_n,y_{n+q}) &\leq d(y_n,y_{n+1}) + d(y_{n+1},y_{n+2}) \\ &\quad + d(y_{n+2},y_{n+3}) + \ldots + d(y_{n+q-1},y_{n+q}) \\ &= (1+h+h^2+\ldots+h^{q-1})d(y_0,y_1) \\ &= \frac{h^n}{1-h} d(y_0,y_1), \end{aligned}$$

since $0 < h < 1$, $h^n \to 0$ *as* $n \to \infty$.

So, when we get $d(y_n, y_{n+q}) \to 0$, this implies that $\{y_n\}$ is a Cauchy sequence in the complete dislocated metric space. So, there exists $z \in X$ such that $\{y_n\} \to z$.

Therefore, the subsequences

$$\{Sx_{2n}\} \to z, \ \{Bx_{2n+1}\} \to z, \{Tx_{2n+1}\} \to z, \text{and } \{Ax_{2n+2}\} \to z \ \text{ exist.}$$

Since $T(X) \subset A(X)$, there exists $u \in X$ such that $z = Au$.

So,

$$\begin{aligned} d(Su,z) &= d(Su,Tx_{2n+1}) \\ &\leq \alpha\left\{\frac{d(Au,Su),d(Bx_{2n+1},Tx_{2n+1})}{d(Au,Bx_{2n+1})}\right\}+\beta d(Au,Bx_{2n+1}) \\ &= \alpha\left\{\frac{d(z,Su),d(Bx_{2n+1},Tx_{2n+1})}{d(z,Bx_{2n+1})}\right\}+\beta d(z,Bx_{2n+1}). \end{aligned}$$

Taking limits as n→∞ we get,>

$$\begin{aligned} d(Su,z) &\leq \alpha\left\{\frac{d(z,Su),d(z,z)}{d(z,z)}\right\}+\beta d(z,z) \\ &\leq \alpha d(z,Su)+2\beta d(z,Su) \\ &= (\alpha+2\beta)d(z,Su) \end{aligned}$$

$\therefore\ d(z,Su)=0$ since $\alpha+2\beta<1$.

So we have $Su=z=Au$.

Again, since $S(X)\subset B(X)$ there exists $v\in X$, such that $z=Bv$.

We claim that $z=Tv$ as

$$\begin{aligned} d(z,Tv) &= d(Su,Tv) \\ &\leq \alpha\left\{\frac{d(Au,Su),d(Bv,Tv)}{d(Au,Bv)}\right\}+\beta d(Au,Bv) \\ &= \alpha\left\{\frac{d(z,z),d(z,z)}{d(z,z)}\right\}+\beta d(z,z) \\ &\leq 2\alpha d(z,Tv)+2\beta d(z,Tv) \\ &= [2\alpha+2\beta]d(z,Tv) \end{aligned}$$

$\therefore d(z,Tv)=0.$ Since $2\alpha+2\beta<1$,

so we get $z=Tv$ and hence, $Su=Au=Tv=Bv=z$.

Since the pair (S,A) is weakly compatible so by definition $SAu=ASu$ implies $Sz=Az$.

Now, we show that z is the fixed point of S.

$$\begin{aligned} d(Sz,z) &= d(Sz,Tz) \\ &\leq \alpha\left\{\frac{d(Az,Sz),d(Bv,Tv)}{d(Az,Bv)}\right\}+\beta d(Az,Bv) \\ &= \alpha\left\{\frac{d(Sz,Sz),d(z,z)}{d(Sz,z)}\right\}+\beta d(Sz,z) \\ &\leq (4\alpha+\beta)d(Sz,z) \end{aligned}$$

$$\begin{aligned} &d(Sz,z)=0, \quad \text{since } 4\alpha+\beta<1 \\ &\Rightarrow Sz=z \\ &\therefore Az=Sz=z. \end{aligned}$$

Again, the pair (T,B) are weakly compatible, so by definition, $TBv=BTv$ and this also implies $Tz=Bz$.

Now, we show that z is the fixed point of T.

$$\begin{aligned} d(z,Tz) &= d(Sz,Tz) \\ &\leq \alpha\left\{\frac{d(Az,Sz),d(Bz,Tz)}{d(Az,Bz)}\right\}+\beta d(Az,Bz) \\ &= \alpha\left\{\frac{d(z,z),d(Tz,Tz)}{d(z,Tz)}\right\}+\beta d(z,Tz) \\ &\leq 4\alpha\left\{\frac{d(z,Tz),d(z,Tz)}{d(z,Tz)}\right\}+\beta d(z,Tz) \\ &\leq (4\alpha+\beta)d(z,Tz) \end{aligned}$$

$$\begin{aligned} &d(z,Tz)=0, \quad \text{since } 4\alpha+\beta<1 \\ &\Rightarrow z=Tz \\ &\Rightarrow Az=Bz=Sz=Tz=z \end{aligned}$$

This shows that z is a common fixed point of the self mappings A,B,S and T.

Uniqueness. Let $u\neq v$ be two common fixed points of the mappings A,B,S, and T, then we have

$$\begin{aligned} d(u,v) &= d(Su,Tv) \\ &\leq \alpha\left\{\frac{d(Au,Su),d(Bv,Tv)}{d(Au,Bv)}\right\}+\beta d(Au,Bv) \\ &= \alpha\left\{\frac{d(u,u),d(v,v)}{d(u,v)}\right\}+\beta d(u,v) \\ &\leq 4\alpha d(u,v)+\beta d(u,v) \\ &= (4\alpha+\beta)d(u,v) \end{aligned}$$

$\therefore d(u,v)=0, \quad$ since $4\alpha+\beta<1$.

Since (X,d) is a dislocated metric space, so we have $u=v$.

Putting $A=B=I$ an identity mapping in above Theorem 3.5 yields Corollary 3.6.

Corollary 3.6 *Let (X,d) be a complete d-metric space. Let $S,T{:}X\to X$ be continuous mappings satisfying,*

$$d(Sx,Ty)\leq\alpha\left\{\frac{d(x,Sx)\,d(y,Ty)}{d(x,y)}\right\}+\beta d(x,y)$$

for all $x,y\in X$ where $\alpha,\beta\geq 0, 0\leq\alpha+\beta<\frac{1}{4}$ then S and T have a unique fixed point. If S = T in Corollary 3.6 yields Corollary 3.7.

Corollary 3.7 *Let (X,d) be a complete d-metric space. Let $T{:}X\to X$ be a continuous mapping satisfying, $d(Tx,Ty)\leq\alpha\left\{\frac{d(x,Tx)\,d(y,Ty)}{d(x,y)}\right\}+\beta d(x,y)$ for all $x,y\in X$,* where $\alpha,\beta\geq 0, 0\leq\alpha+\beta<\frac{1}{4}$, then T have unique common fixed point.

Taking $A=T$ and $B=S$ in Theorem 3.5 yields Corollary 3.8. This is the Theorem 2.11 in [7].

Corollary 3.8 *Let (X,d) be a complete d-metric space. Let S,T: $X\to X$ be continuous mappings satisfying $d(Sx,Ty)\leq\alpha\left\{\frac{d(Tx,Sx)\ d(Sy,Ty)}{d(Tx,Sy)}\right\}+\beta d(Tx,Sy)$, for all $x,y\in X$,where $\alpha,\beta\geq 0, 0\leq\alpha+\beta<\frac{1}{4}$, then S and T have unique common fixed point.*

Conclusion

The results in this study enable others in applying our results when one has to deal with dislocated metric spaces.

Competing interests

The authors declare that they have no competing interests.

Authors' contributions

IR and PSK formulated the problem and VVK drafted and aligned the manuscript sequentially. The three authors read and approved the final manuscript.

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