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HUR stability of a generalized Apollonius type quadratic functional equation in non-Archimedean Banach spaces

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Abstract

Using the fixed point and direct methods, we prove the generalized Hyers-Ulam stability of the following generalized Apollonius type quadratic functional equation

$$f\left(\sum_{i=1}^{m} z_i - \sum_{i=1}^{m} x_i\right) + f\left(\sum_{i=1}^{m} z_i - \sum_{i=1}^{m} y_i\right) = \frac{1}{2} f\left(\sum_{i=1}^{m} x_i - \sum_{i=1}^{m} y_i\right) + 2f\left(\sum_{i=1}^{m} z_i - \frac{\sum_{i=1}^{m} x_i + \sum_{i=1}^{m} y_i}{2}\right)$$

in non-Archimedean Banach spaces.

Keywords: Hyers-Ulam stability, Non-Archimedean normed space, p-adic field

MSC: 11J61, 32P05, 39B52, 46S10, 47S10

Introduction

The stability problem of functional equations originates from a question of Ulam [1] concerning the stability of group homomorphisms. Hyers [2] gave the first affirmative partial answer to the question of Ulam for Banach spaces. Hyers' theorem was generalized by Aoki [3] for additive mappings and by Rassias [4] for linear mappings by considering an unbounded Cauchy difference.

Theorem 1. *Let f be an approximately additive mapping from a normed vector space E into a Banach space E', i.e., f satisfies the inequality* $\|f(x+y)-f(x)-f(y)\| \leq \epsilon(\|x\|^r + \|y\|^r)$ *for all* $x, y \in E$, *where* ϵ *and r are constants with* $\epsilon > 0$ *and* $0 \leq r < 1$. *Then, the mapping* $L : E \to E'$ *defined by* $L(x) = \lim_{n\to\infty} 2^{-n} f(2^n x)$ *is the unique additive mapping which satisfies*

$$\|f(x+y) - L(x)\| \leq \frac{2\epsilon}{2-2^r}\|x\|^r$$

for all $x \in E$.

However, the following example shows that the same result of Theorem 1 is not true in non-Archimedean normed spaces.

Example 1. *Let* $p > 2$ *and let* $f : \mathbb{Q}_p \to \mathbb{Q}_p$ *be defined by* $f(x) = 2$. *Then for* $\epsilon = 1$,

$$|f(x+y) - f(x) - f(y)| = 1 \leq \epsilon$$

for all $x, y \in \mathbb{Q}_p$. *However, the sequences* $\left\{\frac{f(2^n x)}{2^n}\right\}_{n=1}^{\infty}$ *and* $\left\{2^n f\left(\frac{x}{2^n}\right)\right\}_{n=1}^{\infty}$ *are not Cauchy. In fact, by using the fact that* $|2| = 1$, *we have*

$$\left|\frac{f(2^n x)}{2^n} - \frac{f(2^{n+1}x)}{2^{n+1}}\right| = |2^{-n}\cdot 2 - 2^{-(n+1)}\cdot 2| = |2^{-n}| = 1$$

and

$$\left|2^n f\left(\frac{x}{2^n}\right) - 2^{n+1} f\left(\frac{x}{2^{n+1}}\right)\right| = |2^n.2 - 2^{(n+1)}.2| = |2^{n+1}| = 1$$

for all $x, y \in \mathbb{Q}_p$ *and* $n \in \mathbb{N}$. *Hence, these sequences are not convergent in* $\mathbb{Q}_p$.

The paper of Rassias [4] has provided a lot of influence on the development of what we call the 'Hyers-Ulam stability' or 'Hyers-Ulam-Rassias stability' of functional equations. A generalization of the Th.M. Rassias theorem

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was obtained by Găvruta [5] by replacing the unbounded Cauchy difference by a general control function in the spirit of Rassias' approach.

The functional equation $f(x+y)+f(x-y)=2f(x)+2f(y)$ is called a 'quadratic functional equation'. In particular, every solution of the quadratic functional equation is said to be a 'quadratic mapping'. A Hyers-Ulam stability problem for the quadratic functional equation was proven by Skof [6] for mappings $f : X \to Y$, where X is a normed space and Y is a Banach space. Cholewa [7] noticed that the theorem of Skof is still true if the relevant domain X is replaced by an Abelian group. Czerwik [8] proved the Hyers-Ulam stability of the quadratic functional equation. The stability problems of several functional equations have been extensively investigated by a number of authors, and there are many interesting results concerning this problem (see [3-47]).

In 1897, Hensel [15] introduced a normed space which does not have the Archimedean property. It turned out that non-Archimedean spaces have many nice applications [17,18,22,48].

In this paper, we prove the Hyers-Ulam-Rassias (or generalized Hyers-Ulam) stability of the following generalized Apollonius type quadratic functional equation:

$$\begin{aligned} f\left(\sum_{i=1}^{m} z_i - \sum_{i=1}^{m} x_i\right) + f\left(\sum_{i=1}^{m} z_i - \sum_{i=1}^{m} y_i\right) &= \frac{1}{2} f\left(\sum_{i=1}^{m} x_i - \sum_{i=1}^{m} y_i\right) \\ &\quad + 2f\left(\sum_{i=1}^{m} z_i - \frac{\sum_{i=1}^{m} x_i - \sum_{i=1}^{m} y_i}{2}\right) \end{aligned} \tag{1}$$

in non-Archimedean Banach spaces. It is easy to show that the function $f(x) = x^2$ satisfies the functional Equation (1), which is called a quadratic functional equation, and every solution of the quadratic functional equation is said to be a quadratic mapping.

Definition 1. *By a* non-Archimedean field *we mean a field* $\mathbb{K}$ *equipped with a function (valuation)* $|\cdot| : \mathbb{K} \to [0,\infty)$ *such that for all* $r,s \in \mathbb{K}$, *the following conditions hold:* (a) $|r| = 0$ *if and only if* $r = 0$; (b) $|rs| = |r||s|$; *and* (c) $|r+s| \leq max\{|r|,|s|\}$.

Remark 1. *Clearly,* $|1| = |-1| = 1$ *and* $|n| \leq 1$ *for all* $n \in \mathbb{N}$.

Definition 2. *Let X be a vector space over a scalar field* $\mathbb{K}$ *with a non-Archimedean, non-trivial valuation* $|\cdot|$. *A function* $||\cdot|| : X \to \mathbb{R}$ *is a non-Archimedean norm (valuation) if it satisfies the following conditions:* (a) $||x|| = 0$ *if and only if* $x = 0$; (b) $||rx|| = |r|||x||$ $(r \in \mathbb{K}, x \in X)$; *and (c) the strong triangle inequality (ultrametric), namely*

$$||x+y|| \leq max\{||x||,||y||\}, \qquad x,y \in X.$$

Then, $(X, ||\cdot||)$ *is called a non-Archimedean space.*

Definition 3. *A sequence* $\{x_n\}$ *is Cauchy if and only if* $\{x_{n+1} - x_n\}$ *converges to zero in a non-Archimedean space. By a complete non-Archimedean space we mean one in which every Cauchy sequence is convergent.*

The most important examples of non-Archimedean spaces are p-adic numbers. A key property of p-adic numbers is that they do not satisfy the Archimedean axiom: 'for $x,y > 0$, there exists $n \in \mathbb{N}$ such that $x < ny$'.

Example 2. *Fix a prime number p. For any nonzero rational number x, there exists a unique integer* $n_x \in \mathbb{Z}$ *such that* $x = \frac{a}{b}p^{n_x}$, *where a and b are integers not divisible by p. Then,* $|x|_p := p^{-n_x}$ *defines a non-Archimedean norm on* $\mathbb{Q}$. *The completion of* $\mathbb{Q}$ *with respect to the metric* $d(x,y) = |x-y|_p$ *is denoted by* $\mathbb{Q}_p$, *which is called the p-adic number field. In fact,* $\mathbb{Q}_p$ *is the set of all formal series* $x = \sum_{k \geq n_x}^{\infty} a_k p^k$ *where* $|a_k| \leq p-1$ *are integers. The addition and multiplication between any two elements of* $\mathbb{Q}_p$ *are defined naturally. The norm* $|\sum_{k \geq n_x}^{\infty} a_k p^k|_p = p^{-n_x}$ *is a non-Archimedean norm on* $\mathbb{Q}_p$, *and it makes* $\mathbb{Q}_p$ *a locally compact field.*

Definition 4. *Let X be a set. A function* $d : X \times X \to [0,\infty]$ *is called a generalized metric on X if d satisfies the following conditions: (a)* $d(x,y) = 0$ *if and only if* $x = y$ *for all* $x,y \in X$; *(b)* $d(x,y) = d(y,x)$ *for all* $x,y \in X$; *and (c)* $d(x,z) \leq d(x,y) + d(y,z)$ *for all* $x,y,z \in X$.

Theorem 2. *Let (X,d) be a complete generalized metric space and* $J : X \to X$ *be a strictly contractive mapping with Lipschitz constant* $L < 1$. *Then, for all* $x \in X$, *either* $d(J^n x, J^{n+1}x) = \infty$ *for all nonnegative integers n or there exists a positive integer* n_0 *such that (a)* $d(J^n x, J^{n+1}x) < \infty$ *for all* $n_0 \geq n_0$; *(b) the sequence* $\{J^n x\}$ *converges to a fixed point* y^* *of J; and (c)* y^* *is the unique fixed point of J in the set* $Y = \{y \in X : d(J^{n_0}x, y) < \infty\}$; *(d)* $d(y,y^*) \leq \frac{1}{1-L} d(y,Jy)$ *for all* $y \in Y$.

Arriola and Beyer [49] investigated the Hyers-Ulam stability of approximate additive functions $f : \mathbb{Q}_p \to \mathbb{R}$. They showed that if $f : \mathbb{Q}_p \to \mathbb{R}$ is a continuous function for which there exists a fixed ϵ: $|f(x+y) - f(x) - f(y)| \leq \epsilon$ for all $x,y \in \mathbb{Q}_p$, then there exists a unique additive function $T : \mathbb{Q}_p \to \mathbb{R}$ such that $|f(x) - T(x)| \leq \epsilon$ for all $x \in \mathbb{Q}_p$. In this paper, using the fixed point and direct method, we prove the generalized Hyers-Ulam stability of the functional equation (1) in non-Archimedean normed spaces.

Methods

Non-archimedean stability of Equation 1: fixed point method

Throughout this section, using the fixed point alternative approach, we prove the generalized Hyers-Ulam stability of functional Equation 1 in non-Archimedean normed spaces. Let X be a non-Archimedean normed space and Y be a non-Archimedean Banach space.

Remark 2. *Let* $x := \sum_{i=1}^m x_i$*,* $y := \sum_{i=1}^m y_i$*,* $z := \sum_{i=1}^m z_i$ *and* $|4| \neq 1$*.*

Theorem 3. *Let* $\zeta : X^2 \to [0,\infty)$ *be a function such that there exists* $L < 1$ *with*

$$\zeta\left(\frac{x}{2},\frac{y}{2},\frac{z}{2}\right) \leq \frac{L\zeta(x,y,z)}{|4|} \tag{2}$$

for all $x, y, z \in X$*. If* $f : X \to Y$ *is a mapping with* $f(0) = 0$ *and satisfying*

$$\left\| f(z-x) + f(z-y) - \frac{1}{2}f(x-y) - 2f\left(z - \frac{x+y}{2}\right)\right\| \leq \zeta(x,y,z) \tag{3}$$

for all $x, y, z \in X$*, then the limit* $Q(x) = \lim_{n\to\infty} 4^n f\left(\frac{x}{2^n}\right)$ *exists for all* $x \in X$ *and defines a unique quadratic mapping* $Q : X \to Y$ *such that*

$$\|f(x) - Q(x)\| \leq \frac{L\zeta(x,-x,x)}{|2| - |2|L}. \tag{4}$$

Proof. Putting $z = x$ and $y = -x$ in Equation 3, we have

$$\left\|\frac{1}{2}f(2x) - 2f(x)\right\| \leq \zeta(x,-x,x). \tag{5}$$

Replacing x by $\frac{x}{2}$ in the above inequality, we obtain

$$\left\|4f\left(\frac{x}{2}\right) - f(x)\right\| \leq |2|\zeta\left(\frac{x}{2},\frac{-x}{2},\frac{x}{2}\right) \tag{6}$$

for all $x \in X$. Consider the set $S := \{g : X \to Y;\ g(0) = 0\}$ and the generalized metric d in S defined by

$$d(f,g) = \inf\left\{\mu \in \mathbb{R}^+ : \|g(x) - h(x)\| \leq \mu\zeta(x,-x,x),\ \forall x \in X\right\}, \tag{7}$$

where $\inf \emptyset = +\infty$. It is easy to show that (S,d) is complete (see Lemma 2.1 in [20]). Now, we consider a linear mapping $J : S \to S$ such that $Jh(x) := 4h\left(\frac{x}{2}\right)$ for all $x \in X$. Let $g, h \in S$ be such that $d(g,h) = \epsilon$. Then, we have $\|g(x) - h(x)\| \leq \epsilon\zeta(x,-x,x)$ for all $x \in X$, and so,

$$\begin{aligned}\|Jg(x) - Jh(x)\| &= \left\|4g\left(\frac{x}{2}\right) - 4h\left(\frac{x}{2}\right)\right\| \leq |4|\epsilon\zeta\left(\frac{x}{2},\frac{-x}{2},\frac{x}{2}\right)\\ &\leq \frac{|4|L\epsilon\zeta(x,-x,x)}{|4|}\end{aligned}$$

for all $x \in X$. Thus, $d(g,h) = \epsilon$ implies that $d(Jg, Jh) \leq L\epsilon$. This means that $d(Jg, Jh) \leq Ld(g,h)$ for all $g, h \in S$. It follows from Equation 6 that $d(f, Jf) \leq \frac{L}{|2|}$. By Theorem 2, there exists a mapping $Q : X \to Y$ satisfying the following: (1) Q is a fixed point of J, that is,

$$Q\left(\frac{x}{2}\right) = \frac{1}{4}Q(x) \tag{8}$$

for all $x \in X$. The mapping Q is a unique fixed point of J in the set $\Omega = \{h \in S : d(g,h) < \infty\}$. This implies that Q is a unique mapping satisfying Equation 8 such that there exists $\mu \in (0,\infty)$ satisfying $\|f(x) - Q(x)\| \leq \mu\zeta(x,-x,x)$ for all $x \in X$. (2) $d(J^n f, Q) \to 0$ as $n \to \infty$. This implies the equality $\lim_{n\to\infty} 4^n f\left(\frac{x}{2^n}\right) = Q(x)$ for all $x \in X$. (3) $d(f,Q) \leq \frac{d(f,Jf)}{1-L}$ with $f \in \Omega$, which implies the inequality $d(f,Q) \leq \frac{L}{|2|-|2|L}$. This implies that the inequality (Equation 4) holds. By Equation 3, we have

$$\begin{aligned}&\left\|4^n f\left(\frac{z-x}{2^n}\right) + 4^n f\left(\frac{z-y}{2^n}\right) - \frac{4^n}{2}f\left(\frac{x-y}{2^n}\right)\right.\\ &\left.\quad -2.4^n f\left(\frac{z}{2^n} - \frac{x+y}{2^{n+1}}\right)\right\| \leq |4|^n\zeta\left(\frac{x}{2^n},\frac{y}{2^n},\frac{z}{2^n}\right)\\ &\qquad \leq \frac{|4|^n L^n\zeta(x,y,z)}{|4|^n}\end{aligned}$$

for all $x, y \in X$ and $n \geq 1$, and so, $\left\|Q(z-x) + Q(z-y) - \frac{1}{2}Q(x-y) - 2Q\left(z - \frac{x+y}{2}\right)\right\| = 0$ for all $x, y \in X$. Therefore, the mapping $Q : X \to Y$ satisfies Equation 1. On the other hand,

$$\begin{aligned}Q(2x) - 4Q(x) &= \lim_{n\to\infty} 4^n f\left(\frac{x}{2^{n-1}}\right) - 4\lim_{n\to\infty} 4^n f\left(\frac{x}{2^n}\right)\\ &= 4\left[\lim_{n\to\infty} 4^{n-1} f\left(\frac{x}{2^{n-1}}\right) - \lim_{n\to\infty} 4^n f\left(\frac{x}{2^n}\right)\right]\\ &= 0\end{aligned}$$

So, $Q : X \to Y$ is quadratic. This completes the proof. □

Corollary 1. *Let* $\theta_1, \theta_2 \geq 0$ *and* r *be a real number with* $r \in (1,+\infty)$*. Let* $f : X \to Y$ *be a mapping with* $f(0) = 0$ *and satisfying*

$$\left\| f(z-x) + f(z-y) - \frac{1}{2}f(x-y) - 2f\left(z - \frac{x+y}{2}\right)\right\| \leq \theta_1(\|x\|^r + \|y\|^r + \|z\|^r) + \theta_2\|x\|^{\frac{r}{3}}.\|y\|^{\frac{r}{3}}\|z\|^{\frac{r}{3}} \tag{9}$$

for all $x, y, z \in X$*. Then, the limit* $Q(x) = \lim_{n\to\infty} 4^n f\left(\frac{x}{2^n}\right)$ *exists for all* $x \in X$*, and* $Q : X \to Y$ *is a unique quadratic mapping such that*

$$\|f(x) - Q(x)\| \le \frac{|4|^r(3\theta_1 + \theta_2)\|x\|^r}{|2|(1 - |4|^r)}$$

for all $x \in X$*.*

Proof. The proof follows from Theorem 3 if we take

$$\zeta(x, y, z) = \theta_1(\|x\|^r + \|y\|^r + \|z\|^r) + \theta_2\|x\|^{\frac{r}{3}}.\|y\|^{\frac{r}{3}}\|z\|^{\frac{r}{3}}$$

for all $x, y, z \in X$. In fact, if we choose $L = |4|^r$, we then get the desired result. □

Theorem 4. *Let* $\zeta : X^2 \to [0, \infty)$ *be a function such that there exists an* $L < 1$ *with* $\zeta(2x, 2y, 2z) \le |4|L\zeta(x, y, z)$ *for all* $x, y, z \in X$*. Let* $f : X \to Y$ *be mapping with* $f(0) = 0$ *and satisfying Equation 3. Then, the limit* $Q(x) \lim_{n\to\infty} \frac{f(2^n x)}{4^n}$ *exists for all* $x \in X$ *and defines a unique quadratic mapping* $Q : X \to Y$ *such that*

$$\|f(x) - Q(x)\| \le \frac{\zeta(x, -x, x)}{|2| - |2|L}.$$

Proof. It follows from Equation 5 that $\left\|f(x) - \frac{1}{4}f(2x)\right\| \le \frac{\zeta(x,-x,x)}{|2|}$ for all $x \in X$. The rest of the proof is similar to the proof of Theorem 3. □

Corollary 2. *Let* $\theta_1, \theta_2 \ge 0$ *and* r *be a real number with* $r \in (0, 1)$*. Let* $f : X \to Y$ *be a mapping with* $f(0) = 0$ *and satisfying Equation 9. Then, the limit* $Q(x) = \lim_{n\to\infty} \frac{f(2^n x)}{4^n}$ *exists for all* $x \in X$*, and* $Q : X \to Y$ *is a unique quadratic mapping such that*

$$\|f(x) - Q(x)\| \le \frac{(3\theta_1 + \theta_2)\|x\|^r}{|2|(1 - |4|^{1-r})}$$

for all $x \in X$*.*

Proof. The proof follows from Theorem 4 if we take

$$\zeta(x, y, z) = \theta_1(\|x\|^r + \|y\|^r + \|z\|^r) + \theta_2\|x\|^{\frac{r}{3}}.\|y\|^{\frac{r}{3}}\|z\|^{\frac{r}{3}}$$

for all $x, y, z \in X$. In fact, if we choose $L = |4|^{1-r}$, we then get the desired result. □

Non-archimedean stability of Equation 1: direct method

In this section, using the direct method, we prove the generalized Hyers-Ulam stability of functional Equation 1 in non-Archimedean normed spaces. Throughout this section, let G be 2-divisible.

Theorem 5. *Let* G *be an additive semigroup and* X *be a complete non-Archimedean space. Assume that* $\zeta : G^3 \to [0, +\infty)$ *is a function such that*

$$\lim_{n\to\infty} \frac{\zeta(2^n x, 2^n y, 2^n z)}{|4|^n} = 0 \tag{10}$$

for all $x, y, z \in G$*. Let, for each* $x \in G$*, the limit*

$$\pounds(x) = \lim_{n\to\infty} max\left\{\frac{\zeta(2^k x, -2^k x, 2^k x)}{|4|^k} : 0 \le k < n\right\} \tag{11}$$

exists for all $x \in G$*. Suppose that* $f : G \to X$ *is a mapping with* $f(0) = 0$ *and satisfying the inequality*

$$\left\| f(z-x) + f(z-y) - \frac{1}{2}f(x-y) - 2f\left(z - \frac{x+y}{2}\right) \right\| \le \zeta(x, y, z) \tag{12}$$

for all $x, y, z \in G$*. Then, the limit* $\alpha(x) := \lim_{n\to\infty} \frac{f(2^n x)}{4^n}$ *exists for all* $x \in G$*, and* $\alpha(x) : G \to X$ *is a quadratic mapping satisfying*

$$||f(x) - \alpha(x)|| \le |2|^{-1}\pounds(x) \tag{13}$$

for all $x \in G$*. Moreover, if*

$$\lim_{j\to\infty}\lim_{n\to\infty} max\left\{\frac{\zeta(2^k x, -2^k x, 2^k x)}{|4|^k} : j \le k < j+n\right\} = 0 \tag{14}$$

then, $\alpha(x)$ *is the unique mapping satisfying Equation 13.*

Proof. Putting $z = x$ and $y = -x$ in Equation 12, we have

$$\left\|f(x) - \frac{f(2x)}{4}\right\| \le \frac{\zeta(x, -x, x)}{|2|}. \tag{15}$$

for all $x \in G$. Replacing x by $2^n x$ in Equation 15, we get

$$\left\|\frac{f(2^{n+1}x)}{4^{n+1}} - \frac{f(2^n x)}{4^n}\right\| \le \frac{\zeta(2^n x, -2^n x, 2^n x)}{|2|.|4|^n}. \tag{16}$$

It follows from Equations 10 and 16 that the sequence $\left\{\frac{f(2^n x)}{4^n}\right\}_{n=1}^{\infty}$ is a Cauchy sequence. Since X is complete, $\left\{\frac{f(2^n x)}{4^n}\right\}_{n=1}^{\infty}$ is convergent. Set $\alpha(x) := \lim_{n\to\infty} \frac{f(2^n x)}{4^n}$. Using induction, we see that

$$\left\|\frac{f(2^n x)}{4^n} - f(x)\right\| \le \frac{max\left\{\frac{\zeta(2^k x, -2^k x, 2^k x)}{|4|^k} : 0 \le k < n\right\}}{|2|}. \tag{17}$$

Indeed, Equation 17 holds for $n = 1$ by Equation 15. Now, if Equation 17 holds for n, then by Equation 16, we obtain

$$\begin{aligned}\left\|\frac{f(2^{n+1}x)}{4^{n+1}} - f(x)\right\| &= \left\|\frac{f(2^{n+1}x)}{4^{n+1}} \pm \frac{f(2^n x)}{4^n} - f(x)\right\| \\ &\leq max\left\{\left\|\frac{f(2^{n+1}x)}{4^{n+1}} - \frac{f(2^n x)}{4^n}\right\|,\right. \\ &\left.\left\|\frac{f(2^n x)}{4^n} - f(x)\right\|\right\} \leq \frac{1}{|2|} max\left\{\frac{\zeta(2^n x, -2^n x, 2^n x)}{|4|^n},\right. \\ &\left. max\left\{\frac{\zeta(2^k x, -2^k x, 2^k x)}{|4|^k} : 0 \leq k < n\right\}\right\} \\ &= \frac{1}{|2|} max\left\{\frac{\zeta(2^k x, -2^k x, 2^k x)}{|4|^k} : 0 \leq k < n+1\right\}.\end{aligned} \tag{18}$$

So for all $n \in \mathbb{N}$ and all $x \in G$, Equation 17 holds. By taking n to approach infinity in Equation 17, one obtains Equation 13. If $\beta(x)$ is another mapping that satisfies Equation 13, then for all $x \in G$, we get

$$\begin{aligned}||\alpha(x) - \beta(x)|| &= \lim_{k\to\infty}\left\|\frac{\alpha(2^k x)}{4^k} - \frac{\beta(2^k x)}{4^k}\right\| \\ &\leq \lim_{k\to\infty} max\left\{\left\|\frac{\alpha(2^k x)}{4^k} - \frac{f(2^k x)}{4^k}\right\|,\right. \\ &\left.\left\|\frac{f(2^k x)}{4^k} - \frac{\beta(2^k x)}{4^k}\right\|\right\} \\ &\leq \lim_{j\to\infty}\lim_{n\to\infty} max\left\{\frac{\zeta(2^k x, -2^k x, 2^k x)}{|4|^k} :\right. \\ &\left. j \leq k < j+n\right\} = 0.\end{aligned}$$

Therefore, for all $x \in G$, we obtain $\alpha(x) = \beta(x)$. □

Corollary 3. *Let* $\xi :[0,\infty) \to [0,\infty)$ *be a function satisfying*

$$\xi(|2|t) \leq \xi(|2|)\xi(t) \quad (t \geq 0), \quad \xi(|2|) < |4|.$$

Let $\kappa > 0$ *and* $f : G \to X$ *be a mapping with* $f(0) = 0$ *and satisfying the inequality*

$$\left\| f(z-x) + f(z-y) - \frac{1}{2}f(x-y) - 2f\left(z - \frac{x+y}{2}\right)\right\| \leq \kappa\left(\xi(|x|) + \xi(|y|) + \xi(|z|)\right)$$

for all $x, y, z \in G$. *Then the limit* $\alpha(x) := \lim_{n\to\infty}\frac{f(2^n x)}{4^n}$ *exists for all* $x \in G$, *and* $\alpha(x) : G \to X$ *is a unique quadratic mapping satisfying*

$$\|f(x) - \alpha(x)\| \leq \frac{3\kappa\xi(|x|)}{|2|}$$

for all $x \in G$.

Proof. Define $\zeta : G^3 \to [0,\infty)$ by $\zeta(x,y,z) := \kappa\,(\xi(|x|) + \xi(|y|) + \xi(|z|))$. Since $\frac{\xi(|2|)}{|4|} < 1$, we have $\lim_{n\to\infty}\frac{\zeta(2^n x, 2^n y, 2^n z)}{|4|^n} \leq \lim_{n\to\infty}\left(\frac{\xi(|2|)}{|4|}\right)^n \zeta(x,y,z) = 0$ for all $x, y, z \in G$. Also, for all $x \in G$

$$\begin{aligned}\pounds(x) &= \lim_{n\to\infty} max\left\{\frac{\zeta(2^k x, -2^k x, 2^k x)}{|4|^k} : 0 \leq k < n\right\} \\ &= 3\kappa\xi(|x|)\end{aligned}$$

exists for all $x \in G$. Moreover, $\lim_{j\to\infty}\lim_{n\to\infty} max\left\{\frac{\zeta(2^k x, -2^k x, 2^k x)}{|4|^k} : j \leq k < j+n\right\} = \lim_{j\to\infty}\frac{\zeta(2^j x, -2^j x, 2^j x)}{|4|^j} = 0$ for all $x \in G$. Applying Theorem 5, we get the desired results. □

Theorem 6. *Let* $\zeta : G^3 \to [0,+\infty)$ *be a function such that*

$$\lim_{n\to\infty}|4|^n\zeta\left(\frac{x}{2^n}, \frac{y}{2^n}, \frac{z}{2^n}\right) = 0 \tag{19}$$

for all $x, y, z \in G$. *Let the limit*

$$\pounds(x) = \lim_{n\to\infty} max\left\{|4|^k\zeta\left(\frac{x}{2^{k+1}}, \frac{-x}{2^{k+1}}, \frac{x}{2^{k+1}}\right) : 0 \leq k < n\right\} \tag{20}$$

exist for each $x \in G$. *Suppose that* $f : G \to X$ *is a mapping with* $f(0) = 0$ *and satisfying the inequality*

$$\left\| f(z-x) + f(z-y) - \frac{1}{2}f(x-y) - 2f\left(z - \frac{x+y}{2}\right)\right\| \leq \zeta(x,y,z) \tag{21}$$

for all $x, y, z \in G$. *Then the limit* $\alpha(x) := \lim_{n\to\infty} 4^n f\left(\frac{x}{2^n}\right)$ *exists for all* $x \in G$, *and* $\alpha : G \to X$ *is a quadratic mapping satisfying*

$$||f(x) - \alpha(x)|| \leq |2|\pounds(x) \tag{22}$$

for all $x \in G$. *Moreover, if*

$$\lim_{k\to\infty}\lim_{n\to\infty} max\left\{|4|^k\zeta\left(\frac{x}{2^{k+1}}, \frac{-x}{2^{k+1}}, \frac{x}{2^{k+1}}\right) : j \leq k < n+j\right\} = 0$$

then $\alpha(x)$ *is the unique mapping satisfying Equation 22.*

Proof. Proof. By Equation 6, we know that

$$\left\|4f\left(\frac{x}{2}\right) - f(x)\right\| \leq |2|\zeta\left(\frac{x}{2}, \frac{-x}{2}, \frac{x}{2}\right) \tag{23}$$

for all $x \in G$. Replacing x by $\frac{x}{2^n}$ in Equation 23, we get

$$\left\|4^{n+1}f\left(\frac{x}{2^{n+1}}\right) - 4^n f\left(\frac{x}{2^n}\right)\right\| \leq |2|.|4|^n\zeta\left(\frac{x}{2^{n+1}}, \frac{-x}{2^{n+1}}, \frac{x}{2^{n+1}}\right). \tag{24}$$

for all $x \in G$. It follows from Equations 19 and 24 that the sequence $\left\{4^n f\left(\frac{x}{2^n}\right)\right\}_{n=1}^{\infty}$ is a Cauchy sequence. Since X is complete, $\left\{4^n f\left(\frac{x}{2^n}\right)\right\}_{n=1}^{\infty}$ is convergent. It follows from Equation 24 that

$$\begin{aligned}\left\|4^n f\left(\frac{x}{2^n}\right) - 4^p f\left(\frac{x}{2^p}\right)\right\| &= \left\|\sum_{k=p}^{n} 4^{k+1} f\Big(\frac{x}{2^{k+1}}\Big) - 4^k f\Big(\frac{x}{2^k}\Big)\right\| \\ &\leq max\left\{\left\|4^{k+1} f\Big(\frac{x}{2^{k+1}}\Big) - 4^k f\Big(\frac{x}{2^k}\Big)\right\| : p \leq k < n\right\} \\ &\leq |2| max\left\{|4|^k \zeta\Big(\frac{x}{2^{k+1}}, \frac{-x}{2^{k+1}}, \frac{x}{2^{k+1}}\Big) : p \leq k < n\right\}\end{aligned}$$

for all $x \in G$ and all nonnegative integers n, p with $n > p \geq 0$. Letting $p = 0$ and passing the limit $n \to \infty$ in the last inequality, we obtain Equation 22. The rest of the proof is similar to the proof of Theorem 5. □

Corollary 4. *Let* $\xi : [0, \infty) \to [0, \infty)$ *be a function satisfying*

$$\xi(|2|^{-1} t) \leq \xi(|2|^{-1}) \xi(t) \quad (t \geq 0), \quad \xi(|2|^{-1}) < |4|^{-1}.$$

Let $\kappa > 0$ *and* $f : G \to X$ *be a mapping with* $f(0) = 0$ *and satisfying the inequality*

$$\begin{aligned}&\left\| f(z-x) + f(z-y) - \frac{1}{2} f(x-y) - 2f\left(z - \frac{x+y}{2}\right)\right\| \\ &\quad \leq \kappa\, \big(\xi(|x|).\xi(|y|).\xi(|z|)\big)\end{aligned}$$

for all $x, y, z \in G$. *Then the limit* $\alpha(x) := \lim_{n\to\infty} 4^n f\left(\frac{x}{2^n}\right)$ *exists for all* $x \in G$, *and* $\alpha : G \to X$ *is a unique quadratic mapping satisfying*

$$||f(x) - \alpha(x)|| \leq \frac{|2| \kappa \xi^3(|x|)}{|4|^3}$$

for all $x \in G$.

Proof. Define $\zeta : G^3 \to [0, \infty)$ by $\zeta(x, y, z) := \kappa\, (\xi(|x|).\xi(|y|).\xi(|z|))$. The rest of the proof is similar to the proof of Corollary 3. □

Results and discussion

We linked here four different disciplines, namely, non-Archimedean Banach spaces, functional equations, direct method and fixed point theory. We established the Hyers-Ulam-Rassias stability of the functional Equation 1 in Archimedean Banach spaces by using direct and fixed point methods.

Conclusions

Throughout this paper, using the fixed point and direct method we proved the Hyers-Ulam-Rassias stability of a generalized Apollonius type quadratic functional equation in non-Archimedean Banach spaces.

Competing interests

The authors declare that they have no competing interests.

Authors' contributions

Both authors conceived of the study, participated in its design and coordination, drafted the manuscript, participated in the sequence alignment, and read and approved the final manuscript.

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