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Fixed point results on a class of generalized metric spaces

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Abstract

Brianciari ('A fixed point theorem of Banach-Caccioppoli type on a class of generalized metric spaces,' Publ. Math. Debrecen 57 (2000) 31-37) initiated the notion of the generalized metric space as a generalization of a metric space in such a way that the triangle inequality is replaced by the 'quadrilateral inequality,' $d(x, y) \leq d(x, a) + d(a, b) + d(b, y)$ for all pairwise distinct points x, y, a, and b of X. In this paper, we establish a fixed point result for weak contractive mappings $T : X \to X$ in complete Hausdorff generalized metric spaces. The obtained result is an extension and a generalization of many existing results in the literature.

Keywords: Fixed point, Contractive type mapping, Generalized metric spaces

AMS subject classification: 54H25; 47H10

Introduction

It is well known that the Banach contraction principle [1] is a very useful and classical tool in nonlinear analysis. Later, this principle has been generalized in many directions. For instance, a very interesting generalization of the concept of a metric space was obtained by Branciari [2] by replacing the triangle inequality of a metric space with a more general inequality. Thereafter, many authors initiated and studied many existing fixed point theorems in such spaces. For more details about the fixed point theory in generalized metric spaces, we refer the reader to [3-15].

In the sequel, the letters $\mathbb{R}$, $\mathbb{R}^+$, and $\mathbb{N}$ will denote the set of real numbers, the set of nonnegative real numbers, and the set of nonnegative integer numbers, respectively. The following definitions will be needed in the sequel.

Definition 1.1. [2] Let X be a non-empty set and $d : X \times X \to [0, +\infty)$ such that for all $x, y \in X$ and for all distinct points $u, v \in X$, each of them different from x and y, one has the following:

$$(p1)\, x = y \Leftrightarrow d(x, y) = 0,$$
$$(p2)\, d(x, y) = d(y, x),$$
$$(p3)\, d(x, y) \leq d(x, u) + d(u, v) + d(v, y).$$

Then, (X, d) is called a generalized metric space (or shortly g.m.s.).

Any metric space is a generalized metric space, but the converse is not true [2]. We confirm this by the following.

Example 1.2. Let $X = A \cup B$, where $A = \left\{\frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}\right\}$ and $B = [\frac{3}{4}, \infty)$. Define the generalized metric d on X as follows:

$$d\left(\frac{1}{2}, \frac{1}{3}\right) = d\left(\frac{1}{4}, \frac{1}{5}\right) = \frac{1}{5},$$
$$d\left(\frac{1}{2}, \frac{1}{5}\right) = d\left(\frac{1}{3}, \frac{1}{4}\right) = \frac{1}{4},$$
$$d\left(\frac{1}{2}, \frac{1}{4}\right) = d\left(\frac{1}{5}, \frac{1}{3}\right) = \frac{1}{2}, \quad and$$
$$d\left(\frac{1}{2}, \frac{1}{2}\right) = d\left(\frac{1}{3}, \frac{1}{3}\right) = d\left(\frac{1}{4}, \frac{1}{4}\right) = d\left(\frac{1}{5}, \frac{1}{5}\right) = 0,$$
$$d(x, y) = d(y, x) \text{ for all } x, y \in A,$$

$$d(x, y) = |x - y| \text{ if } \begin{cases} x \in B, y \in A, \text{ or,} \\ x \in A, y \in B, \text{ or,} \\ x, y \in B. \end{cases}$$

It is clear that d does not satisfy the triangle inequality on A. Indeed,

$$\frac{1}{2} = d\left(\frac{1}{2}, \frac{1}{4}\right) > d\left(\frac{1}{2}, \frac{1}{3}\right) + d\left(\frac{1}{3}, \frac{1}{4}\right) = \frac{9}{20}.$$

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Notice that (p3) holds, so d is a generalized metric.

Definition 1.3. [2] Let (X, d) be a g.m.s., $\{x_n\}$ be a sequence in X, and $x \in X$. We say that $\{x_n\}$ is g.m.s. convergent to x if and only if $d(x_n, x) \to 0$ as $n \to +\infty$. We denote this by $x_n \to x$.

Definition 1.4. [2] Let (X, d) be a g.m.s. and $\{x_n\}$ be a sequence in X. We say that $\{x_n\}$ is a g.m.s. Cauchy sequence if and only if for each $\varepsilon > 0$, there exists a natural number N such that $d(x_n, x_m) < N$ for all $n > m > N$.

Definition 1.5. [2] Let (X, d) be a g.m.s. Then, (X, d) is called a complete g.m.s. if every g.m.s. Cauchy sequence is g.m.s convergent in X.

Recently, Miheţ established the following theorem, extending Kannan's Theorem [16] to generalized metric spaces.

Theorem 1.6. *Let (X, d) be a T-orbitally g.m.s. and $T : X \to X$ be a self-map. Assume that there exists $k \in [0, \frac{1}{2})$ such that*

$$d(Tx, Ty) \le k(d(x, Tx) + d(Ty, y)),$$

for all $x, y \in X$. Then, T has a unique fixed point in X.

In complete metric spaces, an important fixed point theorem has been proved by Choudhury [17].

Theorem 1.7. *Let (X, d) be a complete metric space and $T : X \to X$ be a self-map such that for all $x, y \in X$*

$$d(Tx, Ty) \le \frac{1}{2}(d(x, Ty) + d(y, Tx)) - \phi(d(x, Ty), d(y, Tx)),$$

where $\phi : [0, \infty) \times [0, \infty) \to [0, \infty)$ is continuous, and $\phi(a, b) = 0$ if and only if $a = b = 0$. Then, there exists a unique point $u \in X$ such that $u = Tu$.

Several papers attempting to generalize fixed point theorems in metric spaces to g.m.s. are plagued by the use of some false properties given in [2] (see, for example, [3,4,6-8]). This was observed first by Samet [13,18] and then by Sarma et al. [14] by assuming that the generalized metric space is Hausdorff. In this paper, we prove a fixed point result involving weak contractive mappings $T : X \to X$ in complete generalized metric spaces by assuming, in particular, that (X,d) is Hausdorff. As a corollary, we derive a Kannan-type [16,19] fixed point result in such spaces. Also, we state some examples to illustrate our results.

Main results

Our main result is the following.

Theorem 2.1. *Let (X, d) be a Hausdorff and complete generalized metric space. Suppose that $T : X \to X$ such that for all $x, y \in X$,*

$$d(Tx, Ty) \le \frac{1}{2}(d(x, Tx) + d(y, Ty)) - \phi(d(x, Tx), d(y, Ty)), \tag{2.1}$$

where $\phi : [0, \infty) \times [0, \infty) \to [0, \infty)$ is continuous, and $\phi(a, b) = 0$ if and only if $a = b = 0$. Then, there exists a unique point $u \in X$ such that $u = Tu$.

Proof. Let $x_0 \in X$ be an arbitrary point. By induction, we easily construct a sequence $\{x_n\}$ such that

$$x_{n+1} = Tx_n = T^{n+1}x_0 \quad \text{for all } n \ge 0. \tag{2.2}$$

If for some $n \in \mathbb{N}$, $x_n = x_{n+1}$, the proof is completed. For the rest, assume that $x_n \neq x_{n+1}$ for all $n \in \mathbb{N}$. □

Step 1. We claim that

$$\lim_{n\to\infty} d(x_n, x_{n+1}) = 0. \tag{2.3}$$

Substituting $x = x_n$ and $y = x_{n-1}$ in (2.1) and using the properties of ϕ, we obtain

$$\begin{aligned} d(x_{n+1}, x_n) &= d(Tx_n, Tx_{n-1}) \\ &\le \frac{1}{2}(d(x_n, Tx_n) + d(x_{n-1}, Tx_{n-1})) \\ &\quad - \phi(d(x_n, Tx_n), d(x_{n-1}, Tx_{n-1})) \\ &= \frac{1}{2}(d(x_n, x_{n+1}) + d(x_{n-1}, x_n)) \\ &\quad - \phi(d(x_n, x_{n+1}), d(x_{n-1}, x_n)) \\ &\le \frac{1}{2}(d(x_n, x_{n+1}) + d(x_{n-1}, x_n)), \end{aligned} \tag{2.4}$$

which implies that

$$d(x_{n+1}, x_n) \le d(x_n, x_{n-1}) \quad \text{for all } n \ge 1.$$

Therefore, the sequence $\{d(x_n, x_{n+1})\}$ is monotone non-increasing and bounded below. So, there exists $r \ge 0$ such that

$$\lim_{n\to\infty} d(x_n, x_{n+1}) = r.$$

Letting $n \to \infty$ in (2.4) and using the continuity of ϕ, we get $r \le \frac{1}{2}(r + r) - \phi(r, r)$, which implies that $\phi(r, r) = 0$, so $r = 0$ by a property of ϕ. Thus, (2.3) is proved.

Step 2. We shall prove that

$$\lim_{n\to\infty} d(x_n, x_{n+2}) = 0. \tag{2.5}$$

By (2.1), we have

$$\begin{aligned} d(x_{n+2}, x_n) &= d(Tx_{n+1}, Tx_{n-1}) \\ &\le \frac{1}{2}(d(x_{n+1}, Tx_{n+1}) + d(x_{n-1}, Tx_{n-1})) \\ &\quad - \phi(d(x_{n+1}, Tx_{n+1}), d(x_{n-1}, Tx_{n-1})) \\ &= \frac{1}{2}(d(x_{n+1}, x_{n+2}) + d(x_{n-1}, x_n)) \\ &\quad - \phi(d(x_{n+1}, x_{n+2}), d(x_{n-1}, x_n)) \\ &\le \frac{1}{2}(d(x_{n+1}, x_{n+2}) + d(x_{n-1}, x_n)). \end{aligned} \tag{2.6}$$

By (2.3), we get that

$$\limsup_{n\to\infty} d(x_{n+2}, x_n) \le 0$$

so (2.5) is proved.

Step 3. We claim that T has a periodic point.

We argue by contradiction. Assume that T has no periodic point; then, $\{x_n\}$ is a sequence of distinct points, that is, $x_n \neq x_m$ for all $m \neq n$. We will show that in this case, $\{x_n\}$ is a g.m.s. Cauchy. Suppose to the contrary. Then, there is a $\varepsilon > 0$ such that for an integer k, there exist integers $m(k) > n(k) > k$ such that

$$d(x_{n(k)}, x_{m(k)}) > \varepsilon. \tag{2.7}$$

For every integer k, let $m(k)$ be the least positive integer exceeding $n(k)$ satisfying (2.7) and such that

$$d(x_{n(k)}, x_{m(k)-1}) \le \varepsilon. \tag{2.8}$$

Now, using (2.7) and (2.8) and the rectangular inequality (because $\{x_n\}$ is a sequence of distinct points), we find that

$$\begin{aligned} \varepsilon < d(x_{m(k)}, x_{n(k)}) &\le d(x_{m(k)}, x_{m(k)-2}) \\ &\quad + d(x_{m(k)-2}, x_{m(k)-1}) \\ &\quad + d(x_{m(k)-1}, x_{n(k)}) \\ &\le d(x_{m(k)}, x_{m(k)-2}) \\ &\quad + d(x_{m(k)-2}, x_{m(k)-1}) + \varepsilon. \end{aligned}$$

Then, by (2.3) and (2.5), it follows that

$$\lim_{k\to+\infty} d(x_{n(k)}, x_{m(k)}) = \varepsilon. \tag{2.9}$$

Applying (2.1) with $x = x_{m(k)-1}$ and $y = x_{n(k)-1}$, we have

$$\begin{aligned} d(x_{m(k)}, x_{n(k)}) &= d(Tx_{m(k)-1}, Tx_{n(k)-1}) \\ &\le \frac{1}{2}(d(x_{m(k)-1}, x_{m(k)}) + d(x_{n(k)-1}, x_{n(k)})) \\ &\quad - \phi(d(x_{m(k)-1}, x_{m(k)}), d(x_{n(k)-1}, x_{n(k)})). \end{aligned}$$

Letting $k \to \infty$ in the above inequality and using (2.3) and (2.9), we obtain

$$\varepsilon \le 0 - \phi(0, 0) = 0.$$

It is a contradiction.

Hence, $\{x_n\}$ is a g.m.s. Cauchy. Since (X, d) is a complete g.m.s., there exists $u \in X$ such that $x_n \to u$. Applying again (2.1) with $x = x_n$ and $y = u$, we obtain

$$\begin{aligned} d(x_{n+1}, Tu) = d(Tx_n, Tu) &\le \frac{1}{2}d(x_n, x_{n+1}) + d(u, Tu)) \\ &\quad - \phi(d(x_n, x_{n+1}), d(u, Tu)), \end{aligned} \tag{2.10}$$

which implies that

$$d(x_{n+1}, Tu) \le \frac{1}{2}(d(x_n, x_{n+1}) + d(u, Tu))).$$

By (2.3), it follows that

$$\limsup_{n\to\infty} d(x_{n+1}, Tu) \le \frac{1}{2}d(u, Tu). \tag{2.11}$$

Next, we shall find a contradiction of the fact that T has no periodic point in each of two following cases:

- If for all $n \ge 2$, $x_n \neq u$ and $x_n \neq Tu$. Then, by rectangular inequality

$$d(u, Tu) \le d(u, x_n) + d(x_n, x_{n+1}) + d(x_{n+1}, Tu),$$

and using (2.3), we get that

$$d(u, Tu) \le \limsup_{n\to\infty} d(x_{n+1}, Tu). \tag{2.12}$$

From (2.11) and (2.12),

$$d(u, Tu) \le \limsup_{n\to\infty} d(x_{n+1}, Tu) \le \frac{1}{2}d(u, Tu), \tag{2.13}$$

which holds unless $d(u, Tu) = 0$, so $Tu = u$, that is, u is a fixed point of T, so u is a periodic point of T. It contradicts the fact that T has no periodic point.

- If for some $q \ge 2$, $x_q = u$ or $x_q = Tu$. Since T has no periodic point, so obviously $u \neq x_0$. Indeed, if $x_q = u = x_0$, so $T^q x_0 = x_0$, i.e, x_0 is a periodic point of T. On the other hand, if $x_q = Tu$ and $x_0 = u$, so $Tx_0 = Tu = x_q = T^q x_0 = T^{q-1}(Tx_0)$, i.e, Tx_0 is a periodic point of T.

For all $n \ge 1$, we have

$$\begin{aligned} &d(T^n u, u) = d(T^n x_q, u) = d(x_{n+q}, u) \quad \text{or} \\ &d(T^n u, u) = d(T^{n-1}Tu, u) = d(T^{n-1}x_q, u) \\ &\qquad = d(x_{n+q-1}, u). \end{aligned}$$

In the two precedent identities, the integer $q \ge 2$ is fixed, so $\{x_{n+q}\}$ and $\{x_{n+q-1}\}$ are subsequences from $\{x_n\}$, and since $\{x_n\}$ g.m.s. converges to u in (X,d) which is assumed to be Hausdorff, so the two subsequence g.m.s. converge to the same unique limit u, i.e,

$$\lim_{n\to\infty} d(x_{n+q}, u) = \lim_{n\to\infty} d(x_{n+q-1}, u) = 0.$$

Thus,

$$\lim_{n\to\infty} d(T^n u, u) = 0. \tag{2.14}$$

Again, since (X,d) is Hausdorff, then by (2.14),

$$\lim_{n\to\infty} d(T^{n+2} u, u) = 0. \tag{2.15}$$

On the other hand, since T has no periodic point, then it is easy that

$$T^s u \neq T^r u \quad \text{for any } \ s, r \in \mathbb{N}, \quad s \neq r. \tag{2.16}$$

Using (2.16) and a rectangular inequality, we may write

$$\begin{aligned}|d(T^{n+1}u, Tu) - d(u, Tu)| \leq & \, d(T^{n+1}u, T^{n+2}u) \\ & + d(T^{n+2}u, u).\end{aligned}$$

Letting $n \to \infty$ in the above limit and proceeding as (2.3) (since the point x_0 is arbitrary and so the same for the point u) and using (2.15), we obtain

$$\lim_{n\to\infty} d(T^{n+1} u, Tu) = d(u, Tu). \tag{2.17}$$

Similarly,

$$\lim_{n\to\infty} d(T^{n} u, Tu) = d(u, Tu). \tag{2.18}$$

Now, by (2.1),

$$\begin{aligned} d(T^{n+1}u, Tu) \leq & \frac{1}{2}(d(T^n u, Tu) + d(u, Tu)) \\ & - \phi(d(T^n u, Tu), d(u, Tu)). \end{aligned} \tag{2.19}$$

Letting $n \to \infty$ in (2.19) and using (2.17) and (2.18), we get that

$$d(u, Tu) \leq d(u, Tu) - \phi(d(Tu, u), d(u, Tu)),$$

which holds unless $d(u, Tu) = 0$, so $Tu = u$; hence, u is a periodic point of T. It is a contradiction with the fact that T has no periodic point.
Consequently, T admits a periodic point, that is, there exists $u \in X$ such that $u = T^p u$ for some $p \geq 1$.

Step 4. Existence of a fixed point of T.

If $p = 1$, then $u = Tu$, that is, u is a fixed point of T. Suppose now that $p > 1$. We will prove that $a = T^{p-1}u$ is a fixed point of T. Suppose that $T^{p-1}u \neq T^p u$; then, $d(T^{p-1}u, T^p u) > 0$, and so $\phi(d(T^{p-1}u, T^p u), d(T^{p-1}u, T^p u)) > 0$. Now, using the inequality (2.1), we obtain

$$\begin{aligned} d(u, Tu) &= d(T^p u, T^{p+1} u) \\ &= d(T(T^{p-1}u), T(T^p u)) \\ &\leq \frac{1}{2}(d(T^{p-1}u, T^p u) + d(T^p u), T(T^p u)) \\ &\quad - \phi(d(T^{p-1}u, T^p u), d(T^p u), T(T^p u)) \\ &< \frac{1}{2}(d(T^{p-1}u, T^p u) + d(u, Tu)) \end{aligned}$$

which implies that

$$d(u, Tu) < d(T^{p-1}u, T^p u). \tag{2.20}$$

Again, by (2.1), we have

$$\begin{aligned} d(T^{p-1}u, T^p u)) &= d(T(T^{p-2}u), T(T^{p-1}u)) \\ &\leq \frac{1}{2}(d(T^{p-2}u, T^{p-1}u) + d(T^{p-1}u, T^p u)) \\ &\quad - \phi(d(T^{p-2}u, T^{p-1}u), d(T^{p-1}u, T^p u)) \\ &\leq \frac{1}{2}(d(T^{p-2}u, T^{p-1}u) + d(T^{p-1}u, T^p u)). \end{aligned}$$

Again, this implies that

$$d(T^{p-1}u, T^p u) \leq d(T^{p-2}u, T^{p-1}u). \tag{2.21}$$

Continuing this process as in (2.20) and (2.21), we find that

$$\begin{aligned} d(u, Tu) &< d(T^{p-1}u, T^p u) \leq d(T^{p-2}u, T^{p-1}u) \\ &\leq \ldots \leq d(u, Tu), \end{aligned}$$

which is a contradiction. We deduce that $a = T^{p-1}u$ is a fixed point of T.

Step 5. Uniqueness of the fixed point of T.

Suppose that there are two points $b, c \in X$ such that $Tb = b$ and $Tc = c$. By (2.1), we obtain

$$d(b, c) = d(Tb, Tc) \leq \frac{1}{2}(d(b, b) + d(c, c)) - \phi(0, 0) = 0$$

so $b = c$. Thus, T has a unique fixed point. This completes the proof of Theorem 2.1.

Now, we state a corollary of Theorem 2.1 (a Kannan-type contraction [16,19]) given in the following.

Corollary 2.2. *Let (X, d) be a Hausdorff and complete generalized metric space. Suppose that* $T : X \to X$ *such that for all* $x, y \in X$, *there exists* $k \in [0, 1)$ *and*

$$d(Tx, Ty) \leq \frac{k}{2}(d(x, Tx) + d(y, Ty)). \tag{2.22}$$

Then, T has a unique fixed point.

Proof. It suffices to take $\phi(t,s) = \frac{1-k}{2}(t+s)$ in Theorem 2.1. □

Also, we have the following consequence from Theorem 2.1.

Corollary 2.3. *Let (X, d) be a Hausdorff and complete generalized metric space. Suppose that $T : X \to X$ such that for all $x, y \in X$*

$$d(Tx,Ty) \le \frac{1}{2}(d(x,Tx)+d(y,Ty)) - \psi\left(\frac{1}{2}(d(x,Tx)+d(y,Ty))\right), \qquad (2.23)$$

where $\psi :[0,\infty) \to [0,\infty)$ is continuous, and $\psi^{-1}(\{0\}) = \{0\}$. Then, T has a unique fixed point.

Proof. We have only to show that $\phi(t,s) = \psi\left(\frac{1}{2}(t+s)\right)$ satisfies the hypotheses of Theorem 2.1. □

Remark 2.4. (1) Corollary 2.2 corresponds to the main result of Miheţ [11], except that we assumed, in addition, that the generalized metric space is Hausdorff.
(2) Theorem 2.1 extends the results of Branciari [2], Azam and Arshad [4], and Sarma et al. [14].

We give some examples illustrating Theorem 2.1.

Example 2.5. Following [4,12], let $X = \{1, 2, 3, 4\}$ and define $d : X \times X \to \mathbb{R}$ as follows:

$$\begin{aligned} &d(1,2) = d(2,1) = 3\\ &d(2,3) = d(3,2) = d(1,3) = d(3,1) = 1\\ &d(1,4) = d(4,1) = d(2,4) = d(4,2) = d(3,4)\\ &\qquad = d(4,3) = 4. \end{aligned}$$

Then, (*X*, *d*) is a complete generalized metric space, but (*X*, *d*) is not a metric space because it lacks the triangular property:

$$3 = d(1,2) > d(1,3) + d(3,2) = 1 + 1 = 2.$$

Now, define a mapping $T : X \to X$ as follows:

$$Tx = 3 \quad \text{if } x \neq 4 \quad \text{and} \quad T4 = 1.$$

Take $\phi(a,b) = \frac{1}{4}a + \frac{1}{8}b$ for all $a, b \ge 0$. It is easy that all hypotheses of Theorem 2.1 are satisfied, and $u = 3$ is the unique fixed of *T*.

On the other hand, Banach's theorem [1] is not applicable (for the metric $d_0|x,y) = |x-y|$ for all $x, y \in X$). Indeed, taking $x = 2$ and $y = 4$, we have

$$d_0(T2, T4) = 2 > 2k = kd_0(2,4) \quad \text{for all } k \in [0,1).$$

Also, Theorem 1.7 is not applicable by taking, for example, $x = 3$ and $y = 4$,

$$d_0(T3,T4) = 2 > \frac{3}{2} - \phi(2,1) = \frac{1}{2}(d_0(3,T4) + d_0(4,T3)) - \phi(d_0(3,T4), d_0(4,T3)),$$

for all ϕ (given as our Theorem 2.1), and so in particular, it is the same for Chatterjea's theorem [20].

Example 2.6. Let $X = A \cup B$, where $A = \left\{\frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}\right\}$ and $B = [\frac{3}{4}, 1]$. Define the generalized metric d on X as follows:

$$d\left(\frac{1}{2},\frac{1}{3}\right) = \frac{1}{2},$$

$$d\left(\frac{1}{2},\frac{1}{4}\right) = d\left(\frac{1}{3},\frac{1}{4}\right) = \frac{1}{5},$$

$$d\left(\frac{1}{4},\frac{1}{5}\right) = d\left(\frac{1}{2},\frac{1}{5}\right) = d\left(\frac{1}{5},\frac{1}{3}\right) = \frac{1}{6}, \quad \text{and}$$

$$d\left(\frac{1}{2},\frac{1}{2}\right) = d\left(\frac{1}{3},\frac{1}{3}\right) = d\left(\frac{1}{4},\frac{1}{4}\right) = d\left(\frac{1}{5},\frac{1}{5}\right) = 0,$$

$$d(x,y) = d(y,x) \text{ for all } x, y \in A,$$

$$d(x,y) = |x-y| \text{ if } \begin{cases} x \in B, y \in A, \text{ or,} \\ x \in A, y \in B, \text{ or,} \\ x, y \in B. \end{cases}$$

It is clear that d does not satisfy triangle inequality on A. Indeed,

$$\frac{1}{2} = d\left(\frac{1}{2},\frac{1}{3}\right) > d\left(\frac{1}{2},\frac{1}{4}\right) + d\left(\frac{1}{4},\frac{1}{3}\right) = \frac{2}{5}, \text{ or}$$

$$\frac{1}{2} = d\left(\frac{1}{2},\frac{1}{3}\right) > d\left(\frac{1}{2},\frac{1}{5}\right) + d\left(\frac{1}{5},\frac{1}{3}\right) = \frac{1}{3}.$$

Notice that (p3) holds, so d is a generalized metric.
Let $T : X \to X$ be defined as

$$Tx = \begin{cases} \frac{1}{2} & \text{if } x \in \left[\frac{3}{4}, 1\right] \\ \frac{1}{5} & \text{if } x \in \left\{\frac{1}{3}, \frac{1}{4}, \frac{1}{5}\right\} \\ \frac{1}{3} & \text{if } x = \frac{1}{2}. \end{cases}$$

Choose $\phi(a,b) = \frac{1}{15}(a+b)$. Then, it easy to check that T satisfies the conditions of Theorem 2.1 and has a unique fixed point on X, i.e., $u = \frac{1}{5}$.

Note that, Banach's theorem [1] is not applicable (it suffices to take $x = \frac{1}{2}$ and $y = \frac{1}{3}$). Also, we couldn't apply Theorem 1.7 by taking, for example, $x = \frac{1}{2}$ and $y = \frac{1}{3}$.

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
All authors read and approved the final manuscript.

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