Li and Wu *Mathematical Sciences* 2012, **6**:16
http://www.iaumath.com/content/6/1/16

Mathematical Sciences
a SpringerOpen Journal

ORIGINAL RESEARCH **Open Access**

Reproducing kernel method for singular multi-point boundary value problems

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Abstract

Purpose: In this paper, we shall present an algorithm for solving more general singular second-order multi-point boundary value problems.

Methods: The algorithm is based on the quasilinearization technique and the reproducing kernel method for linear multi-point boundary value problems.

Results: Three numerical examples are given to demonstrate the efficiency of the present method.

Conclusions: Obtained results show that the present method is quite efficient.

Keywords: Reproducing kernel method; Singular multi-point boundary value problem; Numerical solutions

Background

In this paper, we consider the following second-order multi-point boundary value problems:

$$\begin{cases} a(x)u''(x)+b(x)u'(x)+c(x)u(x)=f(x,u),\ 0\le x\le 1,\\ u(0)=\sum\limits_{i=1}^{m_0}\alpha_i u(\xi_i),\ u(1)=\sum\limits_{i=1}^{m_1}\beta_i u(\eta_i), \end{cases} \tag{1}$$

where $a(x),\ b(x),\ c(x)\in C[0,1], f(x,u)$ is continuous, $0<\xi_i,\eta_i<1$, $a(0)=0$ or $a(1)=0$ and $a(x)\neq 0, x\in(0,1)$; that is, the equation may be singular at $x=0,1$.

Multi-point boundary value problems (BVPs) arise in a variety of applied mathematics and physics. For instance, the vibrations of a guy wire of uniform cross section composed of N parts of different densities can be set up as a multi-point BVP [1]; also, many problems in the theory of elastic stability can be handled by multi-point problems in the work of Timoshenko [2]. The existence and multiplicity of solutions of multi-point boundary value problems have been studied by many authors, see the works of Agarwal and Kiguradze, Du, Feng and Webb, Thompson and Tisdell [3-6], and the references therein. The shooting method is used to solve multi-point boundary value problems in the works of Kwong and Zou et al. [7,8]. However, the shooting method is a trial-and-error method and is often sensitive to initial guess. This makes the computation by the conventional shooting method expensive and ineffective. Lin [9] introduced an algorithm for solving a class of multi-point BVPs by constructing reproducing kernel (RK) satisfying multi-point boundary conditions. However, both the method for obtaining RK satisfying multi-point boundary conditions and the form of obtained RK are very complicated in this method. Tatari and Dehghan [10] introduced the Adomian decomposition method for multi-point BVPs. Yao [11] proposed a successive iteration method for multi-point BVPs. However, there seems to be little discussion about numerical solutions of singular multi-point boundary value problems. Geng [12] proposed a method for a class of second-order three-point BVPs by converting the original problem into an equivalent integro-differential equation. Li and Wu [13,14] presented RK method for singular three-point and four-point BVPs by constructing RK satisfying three-point or four-point boundary conditions. The goal of this paper is to give an effective method for solving more general singular multi-point boundary value problems.

The rest of the paper is organized as follows. In the next section, three numerical examples are presented. The conclusions of this paper are introduced in the 'Conclusions' section. The algorithm for solving singular multi-point

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boundary value problems is proposed in the section 'Methods'.

Results and discussion

In this section, three numerical examples are studied to demonstrate the accuracy of the present method. The examples are computed using Mathematica 5.0. Results obtained by the method are compared with the analytical solution of each example and are found to be in good agreement with each other.

Example 1. Consider the following linear singular multi-point boundary value problem:

$$\begin{cases} \sin x u''(x) + \frac{e^{x/2}}{2} u'(x) + \sin\sqrt{x} u(x) = f(x), \quad 0 \le x \le 1, \\ u(0) = u(1/8) + u(1/2) - 0.652287 u(7/8), u(1) \\ \qquad = u(1/8) + 2u(1/2) + 0.00775445 u(7/8), \end{cases}$$

where $f(x) = \frac{1}{2} e^{x/2} \cosh(x) + \left(\sin\left(\sqrt{x}\right) + \sin(3x)\right) \sinh(x)$. The exact solution is given by $u(x) = \sinh x$.

Using the present method, taking $x_i = \frac{i-1}{N-1}, i = 1, 2, \cdots, N$, $N = 11,\ 51$ and initial approximation $u_0(x) = 0$, the numerical results are given in Figure 1. From Figure 1, it is shown that the absolute errors is monotonically decreasing if N increases.

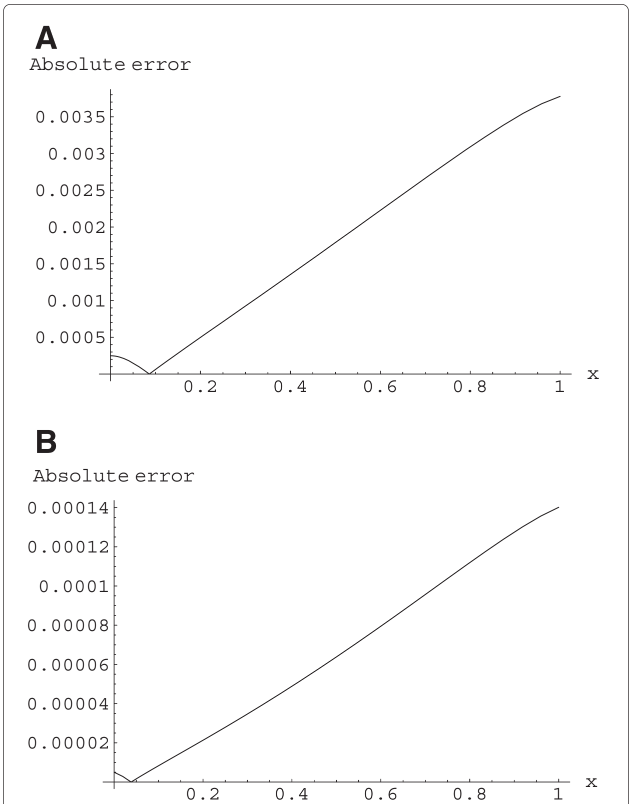

Figure 1 Figures of absolute errors.
$|u_{1,11}(x) - u(x)|$, $|u_{1,51}(x) - u(x)|$ for Example 1.

Example 2. Consider the following nonlinear singular multi-point boundary value problem:

$$\begin{cases} \sin x u''(x) + \frac{e^{x/2}}{2} u'(x) + \sin\sqrt{x} u(x) + u^4(x) = f(x), 0 \le x \le 1, \\ u(0) = u(1/8) + u(1/2) - 0.652287 u(7/8), u(1) \\ \qquad = u(1/8) + 2u(1/2) + 0.00775445 u(7/8), \end{cases}$$

where $f(x) = \sinh^4(x) + \sin\left(\sqrt{x}\right)\sinh(x) + \sin(3x)\sinh(x) + \frac{1}{2} e^{x/2} \cosh(x)$. The exact solution is given by $u(x) = \sinh x$.

Using the present method, taking $k = 3$, $x_i = \frac{i-1}{n-1}, i = 1, 2, \cdots, N$, $N = 11,\ 51$ and initial approximation $u_0(x) = 0$, the numerical results are given in Figure 2. From Figure 2, we can see that the approximate solutions are in good agreement with exact solutions.

Example 3. The method discussed in the paper is finally tested on a sandwich problem. The shear deformation $u(x)$ of sandwich beams is governed by a linear third-order differential equation with its boundary conditions at three different points [15]:

$$\begin{cases} u'''(x) - k^2 u'(x) + r = 0, \ 0 \le x \le 1, \\ u'(0) = 0, \ u(\frac{1}{2}) = 0, \ u'(1) = 0, \end{cases}$$

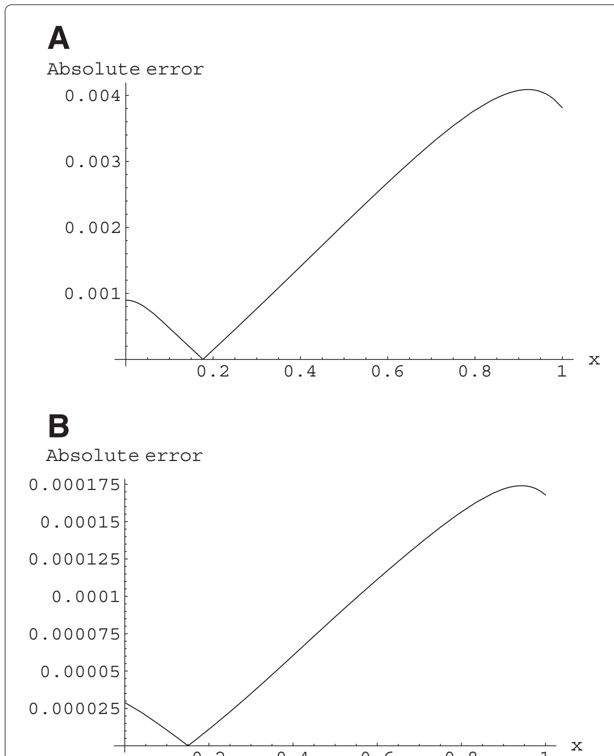

Figure 2 Figures of absolute errors.
$|u_{3,11}(x) - u(x)|$, $|u_{3,51}(x) - u(x)|$ for Example 2.

The exact solution is given by $u(x) = \frac{r\left(k(2x-1)-2\sinh(kx)+2\cosh(kx)\tanh\left(\frac{k}{2}\right)\right)}{2k^3}$. The method was tested for $r = 1, k = 5$, and 10, and the results are compared with the exact solution. Using our method, taking $n = 1, N = 11, 51, x_i = \frac{i-1}{N-1}, i = 1, 2, \cdots, N$, the absolute errors $\mid u_{n,N}(x) - u(x) \mid$ between the approximate solution and exact solution are given in Tables 1 and 2. Numerical results compared with that of Tirmizi et al. [15] show that the accuracy of approximate solutions obtained using the present method is higher.

Discussion

The method introduced in the work of Geng [12] can only be used to solve special multi-point BVPs with one multi-point boundary condition, whereas the present method can be extended to general nonlocal BVPs with linear nonlocal boundary conditions. Comparing with that of Lin's work [9], this paper provide a simpler method for constructing reproducing kernel satisfying linear nonlocal boundary conditions. A major advantage of the present method over the Adomian decomposition method [10] is that it can avoid unnecessary computation in determining the unknown parameters.

Conclusions

In this paper, we introduce an algorithm for solving singular second-order multi-point BVPs. The present method is based on the reproducing kernel method and the quasilinearization technique. Results of numerical examples show that the present method is an accurate and reliable analytical technique for singular multi-point BVPs.

Methods

Quasilinearization technique for singular multi-point BVP (Equation 1)

To solve Equation 1, the quasilinearization technique is used to reduce Equation 1 to a sequence of linear problems. By choosing a reasonable initial approximation $u_0(x)$ for the function $u(x)$ in $f(x,u)$ and expanding $f(x,u)$ around the function $u_0(x)$

$$f(x,u_1) = f(x,u_0) + (u_1 - u_0)\frac{\partial f}{\partial u}|_{u=u_0} + \cdots.$$

In general, one can write for $k = 1, 2, \cdots$

$$f(x,u_k) = f(x,u_{k-1}) + (u_k - u_{k-1})\frac{\partial f}{\partial u}|_{u=u_{k-1}} + \cdots.$$

Hence, we can obtain the following iteration formula for Equation 1:

$$\begin{cases} a(x)u_k''(x) + b(x)u_k'(x) + c_k(x)u_k(x) = f_k(x), \\ \quad k = 1, 2, \cdots \\ u_k(0) = \sum_{i=1}^{m_0} \alpha_i u_k(\xi_i), u_k(1) = \sum_{i=1}^{m_1} \beta_i u_k(\eta_i), \end{cases} \tag{2}$$

where $c_k(x) = c(x) - \frac{\partial f}{\partial u}|_{u=u_{k-1}}, f_k(x) = f(x,u_{k-1}) - u_{k-1}\frac{\partial f}{\partial u}|_{u=u_{k-1}}$, $u_0(x)$ is the initial approximation.

Therefore, to solve singular multi-point BVP (Equation 1), it suffices for us to solve the series of linear problem (Equation 2).

Reproducing kernel method for solving Equation 2

In order to solve Equation 2 using the RKM presented in previous works [16-21], it is necessary to construct a reproducing kernel space $W_2^3[0,1]$ in which every function satisfies the multi-point boundary conditions of Equation 2.

First, we construct the following reproducing kernel space.

Reproducing kernel Hilbert space $W^3[0,1]$ is defined as follows: $W^3[0,1] = \{u(x) \mid u''(x)$ is an absolutely continuous real value functions, $u'''(x) \in L^2[0,1]\}$. The inner product and norm in $W^3[0,1]$ are given, respectively, by the following:

$$(u(y), v(y))_{W^3} = u(0)v(0) + u'(0)v'(0) + u(1)v(1) + \int_0^1 u'''v'''dy$$

Table 1 Exact solution and absolute errors for $k = 5$ for Example 3

Node x	Exact solution	Absolute error [15]	Present method ($u_{1,11}$)	Present method ($u_{1,51}$)
0.00	-0.121071	6.65×10^{-5}	2.81×10^{-5}	1.04×10^{-6}
0.10	-0.112685	6.50×10^{-5}	2.26×10^{-5}	8.44×10^{-7}
0.20	-0.009222	5.25×10^{-5}	1.41×10^{-5}	5.27×10^{-7}
0.30	-0.006466	3.63×10^{-5}	7.32×10^{-6}	2.74×10^{-7}
0.40	-0.003320	1.87×10^{-5}	2.95×10^{-6}	1.10×10^{-7}
0.60	0.0033201	1.73×10^{-5}	2.95×10^{-6}	1.10×10^{-7}
0.70	0.0064668	3.40×10^{-5}	7.32×10^{-6}	2.74×10^{-7}
0.80	0.0092222	4.98×10^{-5}	1.41×10^{-5}	5.27×10^{-7}
0.90	0.0112685	6.20×10^{-5}	2.26×10^{-5}	8.44×10^{-7}
1.00	0.1210710	6.34×10^{-5}	2.81×10^{-5}	1.04×10^{-6}

Table 2 Exact solution and absolute errors for $k = 10$ for Example 3

Node x	Exact solution	Absolute error [15]	Present method ($u_{1,11}$)	Present method($u_{1,51}$)
0.00	-0.00400009	3.51×10^{-5}	1.14×10^{-5}	4.27×10^{-7}
0.10	-0.00363226	3.85×10^{-5}	5.73×10^{-6}	1.60×10^{-7}
0.20	-0.00286501	3.02×10^{-5}	8.65×10^{-7}	4.77×10^{-8}
0.30	-0.00195113	2.23×10^{-5}	2.22×10^{-6}	8.80×10^{-8}
0.40	-0.00098416	1.40×10^{-5}	1.39×10^{-6}	5.40×10^{-8}
0.60	0.000984164	7.00×10^{-5}	1.39×10^{-6}	5.40×10^{-8}
0.70	0.001951130	1.26×10^{-6}	2.22×10^{-6}	8.80×10^{-8}
0.80	0.002865010	1.95×10^{-5}	8.65×10^{-7}	4.77×10^{-8}
0.90	0.003632260	2.74×10^{-5}	5.73×10^{-6}	1.60×10^{-7}
1.00	0.004000090	2.40×10^{-5}	1.14×10^{-5}	4.27×10^{-7}

and

$$\| u \|_{W^3} = \sqrt{(u,u)_{W^3}},\ u,v \in W^3[0,1].$$

Using the works of Cui [16,17], it is easy to obtain its reproducing kernel:

$$k(x,y) = \begin{cases} h_1(x,y),\ y \le x, \\ h_1(y,x),\ y > x, \end{cases} \tag{3}$$

where $h_1(x,y) = \frac{1}{120}(-(x^2-1)y^5+5(x-1)xy^4-(x^5-5x^4+10x^3-366x^2+120x+120)y^2-120(x-1)xy-120x^2+120)$.

Next, we construct a reproducing kernel space $W_1^3[0,1]$ in which every function satisfies $u(0) = \sum_{i=1}^{m_0} \alpha_i u(\xi_i)$.

Definition 1. $W_1^3[0,1] = \{u(x) \mid u(x) \in W^3[0,1], u(0) = \sum_{i=1}^{m_0} \alpha_i u(\xi_i)\}$.

Clearly, $W_1^3[0,1]$ is a closed subspace of $W^3[0,1]$, and therefore, it is also a reproducing kernel space.

Put $L_1 u(x) = u(0) - \sum_{i=1}^{m_0} \alpha_i u(\xi_i)$. The following theorem give its reproducing kernel.

Theorem 1. *If $L_{1x}L_{1y}k(x,y) \neq 0$, then the reproducing kernel $k_{\alpha\beta}$ of $W_1^3[0,1]$ is given by the following:*

$$k_1(x,y) = k(x,y) - \frac{L_{1x}k(x,y)L_{1y}k(x,y)}{L_{1x}L_{1y}k(x,y)}, \tag{4}$$

where the subscript x by the operator L_1 indicates that the operator L_1 applies to the function of x.

Proof. It is easy to see that $k_1(x,0) = \sum_{i=1}^{m_0} \alpha_i k_1(x,\xi_i)$, and therefore $k_1(x,y) \in W_1^3[0,1]$.

For $\forall\ u(y) \in W_1^3[0,1]$, obviously, $L_{1y}u(y) = 0$; it follows that

$$(u(y), k_1(x,y))_{W^3} = (u(y), k(x,y))_{W^3} = u(x),$$

that is, $k_1(x,y)$ is of 'reproducing property'. Thus, $k_1(x,y)$ is the reproducing kernel of $W_1^3[0,1]$, and the proof is complete. □

Similarly, we construct a reproducing kernel space which is a closed subspace of $W_1^3[0,1]$.

Definition 2. $W_2^3[0,1] = \{u(x) \mid u(x) \in W_1^3[0,1], u(1) = \sum_{i=1}^{m_1} \beta_i u(\eta_i)\}$, that is, $W_2^3[0,1] = \{u(x) \mid u(x) \in W^3[0,1], u(0) = \sum_{i=1}^{m_0} \alpha_i u(\xi_i), u(1) = \sum_{i=1}^{m_1} \beta_i u(\eta_i)\}$.

Put $L_2 u(x) = u(1) - \sum_{i=1}^{m_1} \beta_i u(\eta_i)$. By the proof of Theorem 1, it is easy to see that

Theorem 2. *The reproducing kernel $k_{\alpha\beta}$ of $W_2^3[0,1]$ is given by the following:*

$$k_2(x,y) = k_1(x,y) - \frac{L_{2x}k_1(x,y)L_{2y}k_1(x,y)}{L_{2x}L_{2y}k_1(x,y),} \tag{5}$$

In Equation 2, put $Lv(x) = a(x)v''(x) + b(x)v'(x) + c_k(x)v(x)$, it is clear that $L: W_2^3[0,1] \to W^1[0,1]$ is a bounded linear operator. Put $\varphi_i(x) = \overline{k}(x_i,x)$ and $\psi_i(x) = L^*\varphi_i(x)$ where $\overline{k}(x_i,x)$ is the RK of $W^1[0,1]$, L^* is the adjoint operator of L. The orthonormal system $\{\overline{\psi}_i(x)\}_{i=1}^{\infty}$ of $W_2^3[0,1]$ can be derived from the Gram-Schmidt orthogonalization process of $\{\psi_i(x)\}_{i=1}^{\infty}$: c

$$\overline{\psi}_i(x) = \sum_{k=1}^{i} \beta_{ik}\psi_k(x), (\beta_{ii} > 0, i = 1,2,...). \tag{6}$$

By the RKM presented in previous works [16-19], we have the following theorem.

Theorem 3. *For Equation 2, if* $\{x_i\}_{i=1}^{\infty}$ *is dense on* $[0,1]$*, then* $\{\psi_i(x)\}_{i=1}^{\infty}$ *is the complete system of* $W_2^3[0,1]$ *and* $\psi_i(x) = L_s k_2(x,s)|_{s=x_i}$.

Theorem 4. *If* $\{x_i\}_{i=1}^{\infty}$ *is dense on* $[0,1]$ *and the solution of Equation 2 is unique, then the solution of Equation 2 is as follows:*

$$u_k(x) = \sum_{j=1}^{\infty} A_j \overline{\psi}_j(x), \tag{7}$$

where $A_j = \sum_{l=1}^{j} \beta_{jl} f_k(x_l)$.

Now, the approximate solution $u_k(x)$ can be obtained by taking finitely many terms in the series representation of $u_k(x)$ and

$$u_{k,N}(x) = \sum_{j=1}^{N} A_j \overline{\psi}_j(x). \tag{8}$$

Lemma 1. *If* $u(x) \in W_2^3[0,1]$*, then there exists a constant c such that* $|u(x)| \leq c \parallel u(x) \parallel_{W_2^3}$, $|u'(x)| \leq c \parallel u(x) \parallel_{W_2^3}$.

From Lemma 1, by convergence of $u_{k,N}(x)$ in the sense of norm, it is easy to obtain the following theorem.

Theorem 5. *The approximate solution* $u_{k,N}(x)$ *and its derivatives* $u'_{k,N}(x)$ *are all uniformly convergent.*

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
XL carried out the nonlocal BVPs studies, participated in the sequence alignment, and drafted the manuscript. BW carried out the numerical tests. Both authors read and approved the final manuscript.

Acknowledgements
The authors would like to express their thanks to the unknown referees for their careful reading and helpful comments. This work is supported by the National Science Foundation of China (11126222).

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Received: 6 May 2012 Accepted: 3 July 2012
Published: 20 July 2012

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doi:10.1186/2251-7456-6-16
Cite this article as: Li and Wu: **Reproducing kernel method for singular multi-point boundary value problems.** *Mathematical Sciences* 2012 **6**:16.