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Mock modular forms and class number relations

Michael H Mertens

Correspondence:
mmertens@math.uni-koeln.de
Mathematisches Institut der Universität zu Köln, Weyertal 86-90, Köln 50931, Germany

Abstract

Purpose: Almost 40 years ago, H. Cohen formulated a conjecture about the modularity of a certain infinite family of functions involving the generating function of the Hurwitz class numbers of binary quadratic forms.

Methods: We use techniques from the theory of modular, mock modular, and Jacobi forms.

Result: In this paper, we prove a slight improvement of Cohen's original conjecture.

Conclusions: From our main result, we derive so far unknown recurrence relations for Hurwitz class numbers.

MSC: 11E41; 11F37; 11F30

Keywords: Class number relation; Mock modular form; Appell-Lerch sum

Background

Since the days of C.F. Gauß, it has been an important problem in number theory to determine the class numbers of binary quadratic forms. One aspect of this, which is also of interest regarding computational issues, is the so-called *class number relations*. These express certain sums of class numbers in terms of more elementary arithmetic functions which are easier to understand and computationally more feasible. The first examples of these relations are due to Kronecker [1] and Hurwitz [2,3].

Let $H(n)$ denote the Hurwitz class number of a non-negative integer n (*cf.* 'Methods' section for the definition). Then, we have the relation

$$\sum_{s\in\mathbb{Z}} H\left(4n-s^2\right)+2\lambda_1(n)=2\sigma_1(n), \tag{1}$$

where

$$\lambda_k(n):=\frac{1}{2}\sum_{d|n}\min\left(d,\frac{n}{d}\right)^k \tag{2}$$

and $\sigma_k(n):=\sum_{d|n}d^k$ is the usual k-th power divisor sum.

This was further extended by Eichler in [4]. For odd $n\in\mathbb{N}$, we have

$$\sum_{s\in\mathbb{Z}} H\left(n-s^2\right)+\lambda_1(n)=\frac{1}{3}\sigma_1(n). \tag{3}$$

Other such examples of class number relations can be obtained, e.g., from the famous Eichler-Selberg trace formula for cusp forms on $\mathrm{SL}_2(\mathbb{Z})$.

In 1975, Cohen [5] generalized the Hurwitz class number using Dirichlet's class number formula (see e.g. [6]) to a number $H(r,n)$ which is closely related to the value of a certain Dirichlet L-series at $(1-r)$ and showed that for $r \geq 2$ the generating function

$$\mathscr{H}_r(\tau) := \sum_{n=0}^{\infty} H(r,n) q^n, \quad q = e^{2\pi i\tau}, \quad \mathrm{Im}(\tau) > 0$$

is a modular form of weight $r + \frac{1}{2}$ on $\Gamma_0(4)$ ([5], Theorem 3.1). This yields many interesting relations in the shape of (1) and (3) for $H(r,n)$.

The case $r = 1$, where $H(1,n) = H(n)$, was treated around the same time by Zagier ([7], see Chapter 2 in [8]): He showed that the function $\mathscr{H}(\tau) = \mathscr{H}_1(\tau)$ is in fact *not* a modular form but can be completed by a non-holomorphic term such that the completed function transforms like a modular form of weight $\frac{3}{2}$ on $\Gamma_0(4)$.

In more recent years, this phenomenon has been understood in a broader context: The discovery of the theory behind Ramanujan's mock theta functions by Zwegers [9], Bruinier and Funke [10], Bringmann and Ono [11] and many, many others has revealed that the function $\mathscr{H}$ is an example of a weight $\frac{3}{2}$ mock modular form, i.e., the holomorphic part of a harmonic weak Maaß form (see 'Methods' section for a definition). Note that in the literature, the spelling 'Maass form' is more common, although these functions are named after the German mathematician Hans Maaß (1911 to 1992). Using this theory, some quite unexpected connections to combinatorics occur, as for example in [12], where class numbers were related to ranks of so-called overpartitions.

In [5], Cohen considered the formal power series

$$S_4^1(\tau, X) := \sum_{\substack{n=0 \\ n \text{odd}}}^{\infty} \left[\sum_{\substack{s\in\mathbb{Z} \\ s^2 \leq n}} \frac{H\left(n-s^2\right)}{1-2sX+nX^2} + \sum_{k=0}^{\infty} \lambda_{2k+1}(n) X^{2k} \right] q^n. \tag{4}$$

From Zagier's and his own results, as well as computer calculations, he conjectured that the following should be true.

Conjecture 1 ([5])**.** *The coefficient of X^ℓ in the formal power series in* (4) *is a (holomorphic) modular form of weight $\ell + 2$ on* $\Gamma_0(4)$.

The goal of this paper is to prove the following result.

Theorem 1. *Conjecture 1 is true. Moreover, for $\ell > 0$, the coefficient of X^ℓ in* (4) *is a cusp form.*

This obviously implies new relations for Hurwitz class numbers which to the author's knowledge have not been proven so far. We give some of them explicitly in Corollary 2.

The main idea of the proof of Theorem 1 is to relate both summands in the coefficient of the above power series to objects which in accordance to the nomenclature in [13] should be called *quasi mixed mock modular forms*, complete them, such that they transform like modular forms and show that the completion terms cancel each other out. The same idea is also used in a recent paper by Bringmann and Kane [14] in which they also prove several identities for sums of Hurwitz class numbers conjectured by Brown et al. in [15].

The outline of this paper is as follows: The preliminaries and notations are explained in 'Methods' section. 'Results and discussion' section contains some useful identities and other lemmas which will be used in 'Conclusions' section to prove Cohen's conjecture.

Since many of the proofs involve rather long calculations, we omit some of them here. More detailed proofs will be available in the author's PhD thesis [16].

Methods

First, we fix some notation. For this entire paper, let τ be a variable from the complex upper half plane $\mathbb{H}$ and denote $x := \mathrm{Re}(\tau)$ and $y := \mathrm{Im}(\tau)$. As usual, we set $q := e^{2\pi i\tau}$. The letters u, v denote arbitrary complex variables. The differential operators with respect to all these variables shall be renormalized by a factor of $\frac{1}{2\pi i}$; thus, we abbreviate

$$D_t := \frac{1}{2\pi i}\frac{d}{dt}.$$

An element of $\mathrm{SL}_2(\mathbb{Z})$ is always denoted by $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$. For some natural number N, we set as usual

$$\Gamma_0(N) = \left\{\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z}) \mid c \equiv 0 \pmod{N}\right\}.$$

There are two different theta series occuring in this paper. One is the theta series of the lattice $2\mathbb{Z}$,

$$\vartheta(\tau) := \sum_{n\in\mathbb{Z}} q^{n^2}, \tag{5}$$

while the other is the classical Jacobi theta series

$$\Theta(v;\tau) := \sum_{\nu\in\frac{1}{2}+\mathbb{Z}} q^{\nu^2/2} e^{2\pi i\nu(v+1/2)}. \tag{6}$$

Note that, e.g., in [9], the letter ϑ stands for the Jacobi theta series.

Mock modular forms

In this subsection, we give some basic definitions about harmonic Maaß forms and mock modular forms (for details, *cf.* [10,17]). Therefore, we fix some $k \in \frac{1}{2}\mathbb{Z}$ and define for a smooth function $f : \mathbb{H} \to \mathbb{C}$ the following operators:

1. The weight k *slash operator* by

$$\left(f|_k\gamma\right)(\tau) = \begin{cases} (c\tau+d)^{-k} f\left(\frac{a\tau+b}{c\tau+d}\right) & \text{, if } k \in \mathbb{Z} \\ \left(\frac{c}{d}\right)\varepsilon_d\,(c\tau+d)^{-k} f\left(\frac{a\tau+b}{c\tau+d}\right) & \text{, if } k \in \frac{1}{2}+\mathbb{Z} \end{cases},$$

where $\left(\frac{m}{n}\right)$ denotes the extended Legendre symbol in the sense of [18], $\tau^{1/2}$ denotes the principal branch of the square root (i.e., $-\frac{\pi}{2} < \arg\left(\tau^{1/2}\right) \leq \frac{\pi}{2}$), and

$$\varepsilon_d := \begin{cases} 1 & \text{, if } d \equiv 1 \pmod{4} \\ i & \text{, if } d \equiv 3 \pmod{4}. \end{cases}$$

We shall assume $\gamma \in \Gamma_0(4)$ if $k \notin \mathbb{Z}$.

2. The weight k *hyperbolic Laplacian* by ($\tau = x + iy$)

$$\Delta_k := -y^2\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right) + iky\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right).$$

Definition 1. Let $f : \mathbb{H} \to \mathbb{C}$ be a smooth function and $k \in \frac{1}{2}\mathbb{Z}$. We call f a *harmonic weak Maaß form* of weight k on some subgroup $\Gamma \leq \mathrm{SL}_2(\mathbb{Z})$ (resp. $\Gamma_0(4)$ if $k \notin \mathbb{Z}$) of finite index if the following conditions are met:

1. $\left(f|_k\gamma\right)(\tau) = f(\tau)$ for all $\gamma \in \Gamma$ and $\tau \in \mathbb{H}$.
2. $\left(\Delta_k f\right)(\tau) = 0$ for all $\tau \in \mathbb{H}$.
3. f grows at most linearly exponentially in all cusps of Γ.

Proposition 1 ([17], Lemma 7.2 and equation (7.8))**.** *Let f be a harmonic Maaß form of weight k with $k > 0$ and $k \neq 1$. Then, there is canonical splitting*

$$f(\tau) = f^+(\tau) + f^-(\tau),$$

where for some $N, M \in \mathbb{Z}$ we have,

$$f^+(\tau) = \sum_{n=N}^{\infty} c_f^+(n)q^n$$

and

$$f^-(\tau) = c_f^-(0)\frac{(4\pi y)^{-k+1}}{k-1} + \sum_{\substack{n=M\\ n\neq 0}}^{\infty} c_f^-(-n)\Gamma\left(k-1; 4\pi ny\right)q^{-n}.$$

Here,

$$\Gamma(\alpha; x) = \int_x^\infty e^{-t}t^{\alpha-1}dt$$

is the incomplete gamma function.

Definition 2. (i) The functions f^+ and f^- in the above proposition are referred to as the *holomorphic* and *non-holomorphic part* of the harmonic weak Maaß form f.
(ii) A *mock modular form* is the holomorphic part of a harmonic weak Maaß form.

There are several generalizations of mock modular forms, e.g., *mixed mock modular forms*, which are essentially products of mock modular forms and usual holomorphic modular forms. For details, we refer the reader to Section 7.3 in [13].

Class numbers

Let d be a non-negative integer with $d \equiv 0, 3 \pmod 4$. Then, the *class number* for the discriminant $-d$ is the number of $\mathrm{SL}_2(\mathbb{Z})$ equivalence classes of primitive binary integral quadratic forms of discriminant $-d$,

$$h(-d) = \left|\left\{Q = \begin{pmatrix} 2a & b \\ b & 2c \end{pmatrix} \in \mathbb{Z}^{2\times 2} \mid \det Q = d \text{ and } \gcd(a,b,c) = 1\right\}/\mathrm{SL}_2(\mathbb{Z})\right|, \tag{7}$$

where of course $\mathrm{SL}_2(\mathbb{Z})$ acts via $(Q, \gamma) \mapsto \gamma^{tr}Q\gamma$.

The Hurwitz class number is now a weighted sum of these class numbers: Define $w_3 = 3$, $w_4 = 2$ and $w_d = 1$ for $d \neq 3, 4$. Then, the Hurwitz class number is given by

$$H(n) = \begin{cases} -\frac{1}{12} & \text{if } n = 0, \\ 0 & \text{if } n \equiv 1, 2 \pmod 4, \\ \sum\limits_{f^2 \mid n} \frac{h(-n/f^2)}{w_{n/f^2}} & \text{otherwise.} \end{cases}$$

The generating function of the Hurwitz class number shall be denoted by

$$\mathscr{H}(\tau) = \sum_{n=0}^{\infty} H(n) q^n.$$

We have the following result concerning a modular completion of the function $\mathscr{H}$ which was already mentioned in the introduction (*cf.* [8], Chapter 2 and Theorem 2).

Theorem 2. *Let*

$$\mathscr{R}(\tau) = \frac{1+i}{16\pi} \int_{-\overline{\tau}}^{i\infty} \frac{\vartheta(z)}{(z+\tau)^{3/2}} dz.$$

Then, the function

$$\mathscr{G}(\tau) = \mathscr{H}(\tau) + \mathscr{R}(\tau)$$

transforms under $\Gamma_0(4)$ *like a modular form of weight* $\frac{3}{2}$.

The idea of the proof is to write $\mathscr{H}$ as a linear combination of Eisenstein series of weight $\frac{3}{2}$, in analogy to the proof of Theorem 3.1 in [5]. These series diverge, but using an idea of Hecke (*cf.* [19], § 2), who used it to derive the transformation law of the weight 2 Eisenstein series E_2, one finds the non-holomorphic completion term $\mathscr{R}$.

It is easy to check that $\mathscr{G}$ is indeed a harmonic weak Maaß form of weight $\frac{3}{2}$. As a mock modular form, the function $\mathscr{H}$ is rather peculiar since it is basically the only example of such an object which is holomorphic at the cusps of $\Gamma_0(4)$ (*cf.* [13], Section 7).

Appell-Lerch sums

In this subsection, we are going to recall some general facts about Appell-Lerch sums. For details, we refer the reader to [9,20].

Definition 3. Let $\tau \in \mathbb{H}$ and $u, v \in \mathbb{C} \setminus (\mathbb{Z}\tau + \mathbb{Z})$. The *Appell-Lerch sum* of level 1 is then the following expression:

$$A_1(u, v) = A_1(u, v; \tau) := a^{1/2} \sum_{n \in \mathbb{Z}} \frac{(-1)^n q^{n(n+1)/2} b^n}{1 - aq^n}$$

where $a = e^{2\pi i u}$, $b = e^{2\pi i v}$, and $q = e^{2\pi i \tau}$.

In addition, we define the following real analytic functions.

$$R(u;\tau) := \sum_{\nu\in\frac{1}{2}+\mathbb{Z}} \left\{ \operatorname{sgn}(\nu) - E\left(\left(\nu + \frac{\operatorname{Im} u}{y}\right)\sqrt{2y}\right)\right\} (-1)^{\nu-1/2} q^{-\nu^2/2} e^{-2\pi i \nu u}, \tag{8}$$

$$E(t) := 2\int_0^t e^{-\pi u^2} du = \operatorname{sgn}(t)\left(1 - \beta\left(t^2\right)\right), \tag{9}$$

$$\beta(x) := \int_x^\infty u^{-1/2} e^{-\pi u} du, \tag{10}$$

where for the second equality in (9) we refer to Lemma 1.7 in [9].

This function R has some nice properties, a few of which are collected in the following propositions.

Proposition 2 ([9], Proposition 1.9)**.** *The function R fulfills the elliptic transformation properties*

(i) $R(u+1;\tau) = -R(u;\tau)$.
(ii) $R(u;\tau) + e^{-2\pi i u - \pi i \tau} R(u+\tau;\tau) = 2e^{-\pi i u - \pi i \tau/4}$.
(iii) $R(-u) = R(u)$.

The following proposition has already been mentioned in [21]. The proof is a straightforward computation.

Proposition 3. *The function R lies in the kernel of the renormalized heat operator $2D_\tau + D_u^2$; hence,*

$$D_u^2 R = -2D_\tau R. \tag{11}$$

We now define the non-holomorphic function

$$\widehat{A}_1(u,v;\tau) = A_1(u,v;\tau) + \frac{i}{2}\Theta(v;\tau)R(u-v;\tau)$$

which will henceforth be referred to as the *completion* of the Appell-Lerch sum A_1.

Theorem 3 ([20], Theorem 2.2)**.** *The real analytic function $\widehat{A}_1$ transforms like a Jacobi form of weight 1 and index $\begin{pmatrix} -1 & 1 \\ 1 & 0 \end{pmatrix}$:*

(S) $\widehat{A}_1(-u,-v) = -\widehat{A}_1(u,v)$.
(E) $\widehat{A}_1\left(u+\lambda_1\tau+\mu_1, v+\lambda_2\tau+\mu_2\right) = (-1)^{(\lambda_1+\mu_1)} a^{\lambda_1-\lambda_2} b^{-\lambda_1} q^{\lambda_1^2/2-\lambda_1\lambda_2} \widehat{A}_1(u,v)$ *for* $\lambda_i, \mu_i \in \mathbb{Z}$.
(M) $\widehat{A}_1\left(\frac{u}{c\tau+d}, \frac{v}{c\tau+d}; \frac{a\tau+b}{c\tau+d}\right) = (c\tau+d) e^{\pi i c\left(-u^2+2uv\right)/(c\tau+d)} \widehat{A}_1(u,v;\tau)$ *for* $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z})$.

Results and discussion

As we mentioned in the introduction, we would like to relate the two summands for each coefficient in the power series in Conjecture 1 to some sort of modular object. For that purpose, we recall the definition of Rankin-Cohen brackets as given in p. 53 of [22], which differs slightly (see below) from the original one in Theorem 7.1 of [5].

Definition 4. Let f, g be smooth functions defined on the upper half plane and $k, \ell \in \mathbb{R}_{>0}$, $n \in \mathbb{N}_0$. Then, we define the n-th *Rankin-Cohen bracket* of f and g as

$$\left[f, g\right]_n = \sum_{r+s=n} (-1)^r \binom{k+n-1}{s} \binom{\ell+n-1}{r} D^r f D^s g$$

where for non-integral entries we define

$$\binom{m}{s} := \frac{\Gamma(m+1)}{\Gamma(s+1)\Gamma(m-s+1)}.$$

Here, the letter Γ denotes the usual gamma function.

It is well-known (*cf.* [5], Theorem 7.1) that if f, g transform like modular forms of weight k and ℓ, respectively, then $\left[f, g\right]_n$ transforms like a modular form of weight $k + \ell + 2n$ and that $\left[f, g\right]_0 = f \cdot g$. The interaction of the first Rankin-Cohen bracket, which itself fulfills the Jacobi identity of Lie brackets, and the regular product of modular forms give the graded algebra of modular forms the additional structure of a Poisson algebra (*cf.* [22], p. 53).

Note that our definition of the Rankin-Cohen bracket differs from the one in Theorem 7.1 of [5], by a factor of $n!\,(-2\pi i)^n$ which guarantees that if f and g have integer Fourier coefficients, so does $\left[f, g\right]_n$.

Lemma 1. *The coefficient of X^{2k} in* (4) *is given by*

$$\frac{c_k}{2}\left([\mathscr{H}, \vartheta]_k (\tau) - [\mathscr{H}, \vartheta]_k \left(\tau + \frac{1}{2}\right)\right) + \Lambda_{2k+1,odd}(\tau), \tag{12}$$

where $c_k = k!\,\frac{\sqrt{\pi}}{\Gamma\left(k+\frac{1}{2}\right)}$ *and*

$$\Lambda_{\ell,odd}(\tau) := \sum_{n=0}^{\infty} \lambda_\ell(2n+1) q^{2n+1} \tag{13}$$

with λ_ℓ as in (2). *The coefficient of X^{2k+1} is identically 0.*

Proof. As in Theorem 6.1 of [5], we define for a modular form f with $f(\tau) = \sum_{n=0}^{\infty} a(n)q^n$ of weight k and an integer $D \neq 0$ the series

$$S_D^f(\tau; X) := \sum_{n=0}^{\infty} \left(\sum_{\substack{s \in \mathbb{Z} \\ s^2 \leq n}} \frac{a\left(\frac{n-s^2}{|D|}\right)}{\left(1 - 2sX + nX^2\right)^{k-\frac{1}{2}}} \right) q^n,$$

where we assume $a(n) = 0$ if $n \notin \mathbb{N}_0$. From p. 283 in [5], we immediately get the formula

$$S_D^f(\tau; X) = \sum_{n=0}^{\infty} \frac{\sqrt{\pi}\,\Gamma\left(n+k-\frac{1}{2}\right)}{\Gamma\left(n+\frac{1}{2}\right)\Gamma\left(k-\frac{1}{2}\right)} \left[g, \vartheta\right]_n(\tau),$$

where $g(\tau) := f(|D|\tau)$. This yields the assertion by plugging in $f = \mathscr{H}$, $k = \frac{3}{2}$, and $D = 1$. □

Since in the Rankin-Cohen brackets that we consider here, we have linear combinations of products of derivatives of a mock modular form and a regular modular form, one could call an object like this a *quasi mixed mock modular form.*

Lemma 2. *For odd $k \in \mathbb{N}$, the function $\Lambda_{k,odd}$ can be written as a linear combination of derivatives of Appell-Lerch sums. More precisely*

$$\Lambda_{k,odd} = \frac{1}{2}\left(D_v^k A_1^{odd}\right)\left(0, \tau + \frac{1}{2}; 2\tau\right),$$

where we define

$$\begin{aligned} A_1^{odd}(u,v;\tau) :=& a^{1/2} \sum_{\substack{n\in\mathbb{Z} \\ n odd}} \frac{(-1)^n q^{n(n+1)/2} b^n}{1-aq^n} \\ =& \frac{1}{2}\left(A_1(u,v;\tau) - A_1\left(u, v+\frac{1}{2}; \tau\right)\right), \end{aligned}$$

where again $a = e^{2\pi i u}$ and $b = e^{2\pi i v}$.

Proof. First we remark that the right-hand side of the identity to be shown is actually well-defined because as a function of u, $A_1(u,v;\tau)$ has simple poles in $\mathbb{Z}\tau + \mathbb{Z}$ (*cf.* [9], Proposition 1.4) which cancel out if the sum is only taken over odd integers. Thus, the equation actually makes sense.

Then, we write $\Lambda_{k,\mathrm{odd}}$ as a q-series

$$\begin{aligned} 2\Lambda_{k,\mathrm{odd}}(\tau) =& \sum_{\ell=0}^{\infty}\sum_{m=0}^{\infty} \min(2\ell+1, 2m+1)^k q^{(2\ell+1)(2m+1)} \\ =& 2\sum_{\ell=0}^{\infty}\left((2\ell+1)^k \sum_{r=1}^{\infty} q^{(2\ell+1)(2\ell+1+2r)}\right) + \sum_{\ell=0}^{\infty}(2\ell+1)^k q^{(2\ell+1)^2} \\ =& 2\sum_{\ell=0}^{\infty}(2\ell+1)^k q^{(2\ell+1)^2}\left(\frac{1}{1-q^{2(2\ell+1)}} - 1\right) + \sum_{\ell=0}^{\infty}(2\ell+1)^k q^{(2\ell+1)^2} \\ =& \sum_{\ell=0}^{\infty}(2\ell+1)^k\left(\frac{q^{(2\ell+1)^2} + q^{(2\ell+1)^2+2(2\ell+1)}}{1-q^{2(2\ell+1)}}\right). \end{aligned}$$

This is easily seen to be the same as $\left(D_v^k A_1^{\mathrm{odd}}\right)\left(0, \tau+\frac{1}{2}; 2\tau\right)$. □

Remark 1. Now, we can write down completions for each summand in (12), and thus, we see that the function

$$\frac{c_k}{2}\left(\left[\mathscr{G}, \vartheta\right]_k(\tau) - \left[\mathscr{G}, \vartheta\right]_k\left(\tau + \frac{1}{2}\right)\right) + \left(D_v^{2k+1}\widehat{A}_1^{\mathrm{odd}}\right)\left(0, \tau + \frac{1}{2}; 2\tau\right) \tag{14}$$

transforms like a modular form of weight $2k+2$. Because the Fourier coefficients of the holomorphic parts grow polynomially, they are holomorphic at the cusps as well.

Thus, it remains to show that the non-holomorphic parts given by

$$\frac{c_k}{2}\left([\mathscr{R},\vartheta]_k(\tau)-[\mathscr{R},\vartheta]_k\left(\tau+\frac{1}{2}\right)\right) \tag{15}$$

and

$$\begin{aligned}\frac{i}{4}\Bigg(&\sum_{\ell=0}^{2k+1}\binom{2k+1}{\ell}(-1)^{\ell}(D_u^{\ell}R)\left(-\tau-\frac{1}{2};2\tau\right)\left(D_v^{2k-\ell+1}\Theta\right)\left(\tau+\frac{1}{2};2\tau\right)\\&-\sum_{\ell=0}^{2k+1}\binom{2k+1}{\ell}(-1)^{\ell}(D_u^{\ell}R)\left(-\tau-1;2\tau\right)\left(D_v^{2k-\ell+1}\Theta\right)(\tau+1;2\tau)\Bigg)\end{aligned} \tag{16}$$

are indeed equal up to sign and that the function in (14) is modular on $\Gamma_0(4)$.

This shows that we will need some specific information about the derivatives of the Jacobi theta series and the R-function evaluated at the torsion point $\left(\tau+\frac{1}{2},2\tau\right)$.

A simple and straight forward calculation gives us the following result.

Lemma 3. *For $r\in\mathbb{N}_0$, one has*

$$\left(D_v^r\Theta\right)\left(\tau+\frac{1}{2};2\tau\right)=-q^{-1/4}\sum_{s=0}^{\lfloor\frac{r}{2}\rfloor}\binom{r}{2s}\left(-\frac{1}{2}\right)^{r-2s}\left(D_\tau^s\vartheta\right)(\tau) \tag{17}$$

and

$$\left(D_v^r\Theta\right)(\tau+1;2\tau)=iq^{-1/4}\sum_{s=0}^{\lfloor\frac{r}{2}\rfloor}\binom{r}{2s}\left(-\frac{1}{2}\right)^{r-2s}\left(D_\tau^s\vartheta\right)\left(\tau+\frac{1}{2}\right), \tag{18}$$

with Θ as in (6) *and ϑ as in* (5)*.*

Lemma 4. *The following identities are true:*

$$R\left(-\tau-\frac{1}{2};2\tau\right)=iq^{1/4} \tag{19}$$

$$R(-\tau-1;2\tau)=-q^{1/4} \tag{20}$$

$$(D_uR)\left(-\tau-\frac{1}{2};2\tau\right)=\frac{-1+i}{4\pi}q^{1/4}\int_{-\overline{\tau}}^{i\infty}\frac{\vartheta(z)}{(z+\tau)^{3/2}}dz-\frac{i}{2}q^{1/4} \tag{21}$$

$$(D_uR)(-\tau-1;2\tau)=-\frac{1+i}{4\pi}q^{1/4}\int_{-\overline{\tau}}^{i\infty}\frac{\vartheta\left(z+\frac{1}{2}\right)}{(z+\tau)^{3/2}}dz+\frac{1}{2}q^{1/4}. \tag{22}$$

Proof. The identities (19) and (20) follow directly by applying the transformation properties (*iii*), (*i*), and (*ii*) of R in Proposition 2.

We only show (21), since (22) then also follows from the obvious fact that $R(u;\tau+1) = e^{-\pi i/4}R(u;\tau)$. From the definition of R in (8) and (9), we see that

$$(D_uR)\left(-\tau-\frac{1}{2};2\tau\right) = iq^{1/4}\sum_{n\in\mathbb{Z}}\left(\frac{1}{\sqrt{4y}\pi}e^{-4\pi n^2y} - \operatorname{sgn}(n)\left(n+\frac{1}{2}\right)\beta\left(4n^2y\right)\right)q^{-n^2},$$

with β as in (10). Note that for convenience, we define $\operatorname{sgn}(0) := 1$.

By partial integration, one gets for all $x \in \mathbb{R}_{\geq 0}$ that

$$\beta(x) = \frac{1}{\pi}x^{-1/2}e^{-\pi x} - \frac{1}{2\sqrt{\pi}}\Gamma\left(-\frac{1}{2};\pi x\right),$$

where again, $\Gamma(\alpha;x)$ denotes the incomplete gamma function. Using the well-known fact that for $\tau \in \mathbb{H}$ and $n \in \mathbb{N}$ it holds that

$$\int_{-\overline{\tau}}^{i\infty}\frac{e^{2\pi inz}}{(-i(z+\tau))^{3/2}}dz = i\left(2\pi n\right)^{1/2}q^{-n}\Gamma\left(-\frac{1}{2};4\pi ny\right),$$

we get the assertion by a straightforward calculation. □

Now, we take a closer look at (16).

Lemma 5. *For all $k \in \mathbb{N}_0$, it holds true that*

$$\begin{aligned}&\sum_{\ell=0}^{2k+1}(-1)^\ell\binom{2k+1}{\ell}\left(D_u^\ell R\right)\left(-\tau-\frac{1}{2};2\tau\right)\left(D_v^{2k-\ell+1}\Theta\right)\left(\tau+\tfrac{1}{2};2\tau\right)\\ &\quad= q^{-1/4}\sum_{m=0}^{k}\sum_{\ell=0}^{k-m}\left[\frac{1}{2}\left(D_u^{2\ell}R\right)\left(-\tau-\frac{1}{2};2\tau\right)+\frac{2(k-\ell-m)+1}{2\ell+1}\left(D_u^{2\ell+1}R\right)\left(-\tau-\frac{1}{2};2\tau\right)\right]\\ &\qquad\qquad\times b_{k,\ell,m}\left(\frac{1}{2}\right)^{2(k-\ell-m)}\left(D_\tau^m\vartheta\right)(\tau)\end{aligned}$$

and

$$\begin{aligned}&\sum_{\ell=0}^{2k+1}(-1)^\ell\binom{2k+1}{\ell}\left(D_u^\ell R\right)(-\tau-1;2\tau)\left(D_v^{2k-\ell+1}\Theta\right)(\tau+1;2\tau)\\ &\quad= -iq^{-1/4}\sum_{m=0}^{k}\sum_{\ell=0}^{k-m}\left[\frac{1}{2}\left(D_u^{2\ell}R\right)(-\tau-1;2\tau)+\frac{2(k-\ell-m)+1}{2\ell+1}\left(D_u^{2\ell+1}R\right)(-\tau-1;2\tau)\right]\\ &\qquad\qquad\times b_{k,\ell,m}\left(\frac{1}{2}\right)^{2(k-\ell-m)}\left(D_\tau^m\vartheta\right)\left(\tau+\frac{1}{2}\right),\end{aligned}$$

where

$$b_{k,\ell,m} := \frac{(2k+1)!}{(2\ell)!\,(2m)!\,(2(k-\ell-m)+1)!} = \binom{2k+1}{2\ell,\ 2m,\ 2(k-\ell-m)+1}.$$

Proof. Again, we only show the former equation, the latter follows from the transformation laws. For simplicity, we omit the arguments of the functions considered.

We obtain

$$
\begin{aligned}
&\sum_{\ell=0}^{2k+1}(-1)^{\ell}\binom{2k+1}{\ell}\left(D_u^{\ell}R\right)\left(D_v^{2k-\ell+1}\Theta\right)\\
&=\sum_{\ell=0}^{k}\binom{2k+1}{2\ell}\left(D_u^{2\ell}R\right)\left(D_v^{2(k-\ell)+1}\Theta\right)-\sum_{\ell=0}^{k}\binom{2k+1}{2\ell+1}\left(D_u^{2\ell+1}R\right)\left(D_v^{2(k-\ell)}\Theta\right)\\
&\overset{(17)}{=}-q^{-1/4}\left[\sum_{\ell=0}^{k}\sum_{m=0}^{k-\ell}\underbrace{\binom{2k+1}{2\ell}\binom{2(k-\ell)+1}{2m}}_{=b_{k,\ell,m}}\left(-\frac{1}{2}\right)^{2(k-\ell-m)+1}\left(D_u^{2\ell}R\right)\left(D_\tau^{m}\vartheta\right)\right.\\
&\qquad\left.-\sum_{\ell=0}^{k}\sum_{m=0}^{k-\ell}\underbrace{\binom{2k+1}{2\ell+1}\binom{2(k-\ell)}{2m}}_{=\frac{2(k-\ell-m)+1}{2\ell+1}b_{k,\ell,m}}\left(-\frac{1}{2}\right)^{2(k-\ell-m)}\left(D_u^{2\ell+1}R\right)\left(D_\tau^{m}\vartheta\right)\right]\\
&=q^{-1/4}\sum_{\ell=0}^{k}\sum_{m=0}^{k-\ell}\left[\frac{1}{2}\left(D_u^{2\ell}R\right)+\frac{2(k-\ell-m)+1}{2\ell+1}\left(D_u^{2\ell+1}R\right)\right]b_{k,\ell,m}\left(\frac{1}{2}\right)^{2(k-\ell-m)}\left(D_\tau^{m}\vartheta\right).
\end{aligned}
$$

Interchanging the sums gives the desired result. □

Corollary 1. *Conjecture 1 is true if the identity*

$$
\begin{aligned}
\left(D_\tau^m\mathscr{R}\right)(\tau)=-\frac{i}{4}q^{-1/4}(-1)^m\sum_{\ell=0}^{m}&\left[\frac{1}{2}\left(D_u^{2\ell}R\right)\left(-\tau-\frac{1}{2};2\tau\right)\right.\\
&\left.+\frac{2(m-\ell)+1}{2\ell+1}\left(D_u^{2\ell+1}R\right)\left(-\tau-\frac{1}{2};2\tau\right)\right]\cdot\binom{2m+1}{2\ell}\left(\frac{1}{4}\right)^{m-\ell}
\end{aligned}
\tag{23}
$$

holds true for all $m\in\mathbb{N}_0$ and the function in (14) *is modular on* $\Gamma_0(4)$.

Proof. Lemma 5 gives us that Conjecture 1 holds true if the identity

$$
\begin{aligned}
&c_k(-1)^{k-m}\binom{k+\frac{1}{2}}{m}\binom{k-\frac{1}{2}}{k-m}D_\tau^{k-m}\mathscr{R}(\tau)\\
&=-\frac{i}{4}q^{-1/4}\sum_{\ell=0}^{k-m}\left[\frac{1}{2}\left(D_u^{2\ell}R\right)\left(-\tau-\frac{1}{2};2\tau\right)+\frac{2(k-\ell-m)+1}{2\ell+1}\left(D_u^{2\ell+1}R\right)\left(-\tau-\frac{1}{2};2\tau\right)\right]\\
&\qquad\times b_{k,\ell,m}\left(\frac{1}{2}\right)^{2(k-\ell-m)}
\end{aligned}
\tag{24}
$$

does as well.

We can simpify this a little further: We have

$$c_k\binom{k+\frac{1}{2}}{m}\binom{k-\frac{1}{2}}{k-m}=\binom{k}{m}\frac{\sqrt{\pi}\,\Gamma\left(k+\frac{3}{2}\right)}{\Gamma\left(k-m+\frac{3}{2}\right)\Gamma\left(m+\frac{1}{2}\right)},$$

and using Legendre's duplication formula for the gamma function, we obtain after a little calculation that

$$\frac{\left(\frac{1}{2}\right)^{2(k-\ell-m)}b_{k,\ell,m}}{c_k\binom{k+\frac{1}{2}}{m}\binom{k-\frac{1}{2}}{k-m}}=\binom{2(k-m)+1}{2\ell}\left(\frac{1}{4}\right)^{k-\ell-m}$$

and hence the corollary. □

Remark 2. Since

$$\begin{pmatrix}1&-\frac{1}{2}\\0&1\end{pmatrix}\Gamma_0(4)\begin{pmatrix}1&\frac{1}{2}\\0&1\end{pmatrix}=\Gamma_0(4)$$

we see that for any (not necessarily holomorphic) modular form f of even weight k on $\Gamma_0(4)$, the function $g=f|_k\begin{pmatrix}1&\frac{1}{2}\\0&1\end{pmatrix}$ is a modular form of the same weight on $\Gamma_0(4)$ as well. In particular, this applies to $[\mathscr{G},\vartheta]_k$ for all $k\in\mathbb{N}_0$.

Lemma 6. *The second summand in* (14) *transforms like a modular form on* $\Gamma_0(4)$.

Proof. Looking at the $(2\ell+1)$-st derivative of $\widehat{A}_1^{\mathrm{odd}}(0,\nu;\tau)$ with respect to ν, one immediately sees that this has the modular transformation properties of a Jacobi form of weight $2\ell+2$ and index 0 on $\mathrm{SL}_2(\mathbb{Z})$. By Theorem 1.3 in [23], it follows that $\mathcal{A}(\tau):=\left(D_\nu^{2\ell+1}\widehat{A}_1^{\mathrm{odd}}\right)\left(0,\frac{\tau}{2}+\frac{1}{2};\tau\right)$ transforms (up to some power of q) like a modular form of weight $\ell+1$ on the group

$$\Gamma:=\left\{\gamma\in\mathrm{SL}_2(\mathbb{Z})\mid\frac{a-1}{2}+\frac{c}{2}\in\mathbb{Z}\text{ and }\frac{b}{2}+\frac{d-1}{2}\in\mathbb{Z}\right\}.$$

We are interested in $\left(D_\nu^{2\ell+1}\widehat{A}_1^{\mathrm{odd}}\right)\left(0,\tau+\frac{1}{2};\tau\right)=\frac{1}{2^{\ell+1}}\mathcal{A}|_{2\ell+2}\begin{pmatrix}2&0\\0&1\end{pmatrix}$, and since one easily checks that

$$\begin{pmatrix}2&0\\0&1\end{pmatrix}\Gamma_0(4)\begin{pmatrix}\frac{1}{2}&0\\0&1\end{pmatrix}\leq\Gamma,$$

the assertion follows. □

Conclusions

We now prove Theorem 1 using Corollary 1. The proof is an induction on m. Since the base case $m=0$ gives an alternative proof of the class number relation (3) by Eichler, we give this as a proof of an additional theorem.

Theorem 4 ([4])**.** *For odd numbers $n\in\mathbb{N}$, we have the class number relation*

$$\sum_{s\in\mathbb{Z}}H\left(n-s^2\right)+\lambda_1(n)=\frac{1}{3}\sigma_1(n).$$

Proof. Let

$$F_2(\tau) = \sum_{n=0}^{\infty} \sigma_1(2n+1) q^{2n+1}.$$

We recall that this function is a modular form of weight 2 on $\Gamma_0(4)$ (*cf.* e.g. [5], Proposition 1.1).

Plugging in $m = 0$ into (23) gives us the equation

$$\mathscr{R}(\tau) = -\frac{i}{4} q^{-1/4} \left[\frac{1}{2} R\left(-\tau - \frac{1}{2}; 2\tau\right) + (D_u R)\left(-\tau - \frac{1}{2}; 2\tau\right) \right].$$

This equality holds true by Lemma 4. Hence, we know by Corollary 1, Lemma 2, Remark 2, and Lemma 6 that

$$\frac{1}{2}\left(\mathscr{H}(\tau)\vartheta(\tau) - \mathscr{H}\left(\tau + \frac{1}{2}\right)\vartheta\left(\tau + \frac{1}{2}\right)\right) + \Lambda_{1,\mathrm{odd}}(\tau)$$

is indeed a holomorphic modular form of weight 2 on $\Gamma_0(4)$ as well.

Since the space of modular forms of weight 2 on $\Gamma_0(4)$ is two-dimensional, the assertion follows by comparing the first two Fourier coefficients of the function above and $\frac{1}{3} F_2(\tau)$. $\square$

The proof of this given in [4] involves topological arguments about the action of Hecke operators on the Riemann surface associated to $\Gamma_0(2)$ on the one hand and arithmetic of quaternion orders on the other hand.

Proof. The base case of our induction is treated above, thus suppose that (23) holds true for one $m \in \mathbb{N}_0$.

For simplicity, we omit again the argument $\left(-\tau - \frac{1}{2}; 2\tau\right)$ in the occuring R derivatives.

By the induction hypothesis we see that

$$\begin{aligned} D_\tau^{m+1}\mathscr{R}(\tau) &= D_\tau\left(D_\tau^m \mathscr{R}(\tau)\right) \\ &= -\frac{i}{4} q^{-1/4} (-1)^m \left\{ -\frac{1}{4} \sum_{\ell=0}^{m} \left[\frac{1}{2}\left(D_u^{2\ell} R\right) + \frac{2(m-\ell)+1}{2\ell+1}\left(D_u^{2\ell+1} R\right) \right] \cdot \binom{2m+1}{2\ell} \left(\frac{1}{4}\right)^{m-\ell} \right. \\ &\quad \left. + \sum_{\ell=0}^{m} \left[\frac{1}{2}\left(D_\tau D_u^{2\ell} R\right) + \frac{2(m-\ell)+1}{2\ell+1}\left(D_\tau D_u^{2\ell+1} R\right) \right] \cdot \binom{2m+1}{2\ell} \left(\frac{1}{4}\right)^{m-\ell} \right\}. \end{aligned}$$

By Schwarz' theorem (sometimes also attributed to Clairaut), the partial derivatives interchange and thus the total differential D_τ is given by

$$D_\tau\left(D_u^\ell R\left(-\tau - \frac{1}{2}; 2\tau\right)\right) = -\left(D_u^{\ell+1} R\right)\left(-\tau - \frac{1}{2}; 2\tau\right) + 2\left(D_\tau D_u^\ell R\right)\left(-\tau - \frac{1}{2}; 2\tau\right).$$

Now, Proposition 3 implies that the above equals

$$
\begin{aligned}
&-\frac{i}{4}q^{-1/4}(-1)^{m+1}\left\{\sum_{\ell=0}^{m}\left[\frac{1}{8}\left(D_u^{2\ell}R\right)+\left(\frac{1}{4}\frac{2(m-\ell)+1}{2\ell+1}+\frac{1}{2}\right)\left(D_u^{2\ell+1}R\right)\right.\right.\\
&\quad\left.\left.+\left(\frac{1}{2}+\frac{2(m-\ell)+1}{2\ell+1}\right)\left(D_u^{2\ell+2}R\right)+\frac{2(m-\ell)+1}{2\ell+1}\left(D_u^{2\ell+3}R\right)\right]\cdot\binom{2m+1}{2\ell}\left(\frac{1}{4}\right)^{m-\ell}\right\}\\
&=-\frac{i}{4}q^{-1/4}(-1)^{m+1}\left\{\left[\frac{1}{2}R+(2m+3)(D_uR)\right]\left(\frac{1}{4}\right)^{m+1}+\sum_{\ell=1}^{m}\binom{2m+1}{2\ell}\left(\frac{1}{4}\right)^{m-\ell+1}\right.\\
&\quad\times\left[\left(\frac{1}{2}+\frac{2(2m-\ell)+5}{4\ell-2}\cdot\frac{(2\ell-1)(2\ell)}{(2(m-\ell)+2)(2(m-\ell)+3)}\right)\left(D_u^{2\ell}R\right)\right.\\
&\quad\left.+\left(\frac{2(m+\ell)+3}{2\ell+1}+\frac{2(m-\ell)+3}{2\ell-1}\cdot\frac{(2\ell-1)(2\ell)}{(2(m-\ell)+2)(2(m-\ell)+3)}\right)\left(D_u^{2\ell+1}R\right)\right]\\
&\quad\left.+\left[\frac{2m+3}{4m+2}\left(D_u^{2m+2}R\right)+\frac{1}{2m+1}\left(D_u^{2m+3}R\right)\right]\binom{2m+1}{2m}\right\}.
\end{aligned}
$$

It is easily seen that the last summand equals

$$
\left[\frac{1}{2}\left(D_u^{2m+2}R\right)+\frac{1}{2m+3}\left(D_u^{2m+3}R\right)\right]\cdot(2m+3)
$$

and a direct but rather tedious calculation gives that

$$
\binom{2m+1}{2\ell}\cdot\left(\frac{1}{2}+\frac{2(2m-\ell)+5}{4\ell-2}\cdot\frac{(2\ell-1)(2\ell)}{(2(m-\ell)+2)(2(m-\ell)+3)}\right)=\frac{1}{2}\binom{2m+3}{2\ell}
$$

and

$$
\begin{aligned}
&\binom{2m+1}{2\ell}\cdot\left(\frac{2(m+\ell)+3}{2\ell+1}+\frac{2(m-\ell)+3}{2\ell-1}\cdot\frac{(2\ell-1)(2\ell)}{(2(m-\ell)+2)(2(m-\ell)+3)}\right)\\
&\quad=\frac{2(m-\ell)+3}{2\ell+1}\binom{2m+3}{2\ell}.
\end{aligned}
$$

In summary, we therefore get

$$
\begin{aligned}
D_\tau^{m+1}\mathscr{R}(\tau)=-\frac{i}{4}q^{-1/4}(-1)^{m+1}\sum_{\ell=0}^{m+1}&\left[\frac{1}{2}\left(D_u^{2\ell}R\right)\left(-\tau-\frac{1}{2};2\tau\right)\right.\\
&\left.+\frac{2(m-\ell)+3}{2\ell+1}\left(D_u^{2\ell+1}R\right)\left(-\tau-\frac{1}{2};2\tau\right)\right]\cdot\binom{2m+3}{2\ell}\left(\frac{1}{4}\right)^{m-\ell+1}
\end{aligned}
$$

which proves Conjecture 1.

The fact that we actually get a cusp form can be seen in the following way:

By Corollary 7.2 in [5], we see that the function $\tau \mapsto \frac{c_k}{2}\left([\mathscr{H},\vartheta]_k(\tau)-[\mathscr{H},\vartheta]_k\left(\tau+\frac{1}{2}\right)\right)$ is a non-holomorphic cusp form if $k\geq 1$. We use the same argument as there to see that $\left(D_u^{2\ell+1}\widehat{A}_1^{\mathrm{odd}}\right)\left(0,\tau+\frac{1}{2};2\tau\right)$ is a cusp form as well. Because we know by Lemma 6 that we have for $\gamma\in\mathrm{SL}_2(\mathbb{R})$ that

$$
\left(D_v^{2\ell+1}\widehat{A}_1^{\mathrm{odd}}\right)|_{2\ell+2}\gamma=D_v^{2\ell+1}\left(\widehat{A}_1^{\mathrm{odd}}|_1\gamma\right),
$$

and by definition $\left(D_v^{2\ell+1}\widehat{A}_1^{\mathrm{odd}}\right)\left(0,\tau+\frac{1}{2};2\tau\right)$ vanishes at the cusp $i\infty$ for all $\ell\in\mathbb{N}_0$. So by the above equation, it vanishes at every cusp of $\Gamma_0(4)$. $\square$

Corollary 2. *By comparing the first few Fourier coefficients of the modular forms in Theorem 1, one finds for all odd $n \in \mathbb{N}$ the following class number relations:*

$$\sum_{s\in\mathbb{Z}} \left(4s^2 - n\right) H\left(n - s^2\right) + \lambda_3(n) = 0,$$

$$\sum_{s\in\mathbb{Z}} g_4(s,n) H\left(n - s^2\right) + \lambda_5(n) = -\frac{1}{12} \sum_{n=x^2+y^2+z^2+t^2} \mathcal{Y}_4(x,y,z,t),$$

$$\sum_{s\in\mathbb{Z}} g_6(s,n) H\left(n - s^2\right) + \lambda_7(n) = -\frac{1}{3} \sum_{n=x^2+y^2+z^2+t^2} \mathcal{Y}_6(x,y,z,t),$$

$$\sum_{s\in\mathbb{Z}} g_8(s,n) H\left(n - s^2\right) + \lambda_9(n) = -\frac{1}{70} \sum_{n=x^2+y^2+z^2+t^2} \mathcal{Y}_8(x,y,z,t)$$

where $g_\ell(n,s)$ is the ℓ-th Taylor coefficient of $\left(1 - sX + nX^2\right)^{-1}$ and $\mathcal{Y}_d(x,y,z,t)$ is a certain harmonic polynomial of degree d in four variables. Explicitly, we have

$$\begin{aligned} g_4(s,n) &= \left(16s^4 - 12ns^2 + n^2\right), \\ g_6(s,n) &= \left(64s^6 - 80s^4n + 24s^2n^2 - n^3\right), \\ g_8(s,n) &= \left(256s^8 - 448s^6n + 240s^4n^2 - 40s^2n^3 + n^4\right), \end{aligned}$$

and

$$\begin{aligned} \mathcal{Y}_4(x,y,z,t) &= \left(x^4 - 6x^2y^2 + y^4\right), \\ \mathcal{Y}_6(x,y,z,t) &= \left(x^6 - 5x^4y^2 - 10x^4z^2 + 30x^2y^2z^2 + 5x^2z^4 - 5y^2z^4\right), \\ \mathcal{Y}_8(x,y,z,t) &= \left(13x^8 + 63x^6y^2 - 490x^6z^2 + 63x^6t^2 - 630x^4y^2z^2 - 315x^4y^2t^2 + 1435x^4z^4\right. \\ &\quad - 630x^4z^2t^2 + 315x^2y^2z^4 + 1890x^2y^2z^2t^2 - 616x^2z^6 + 315x^2z^4t^2 \\ &\quad \left. -315t^2y^2z^4 + 22z^8\right). \end{aligned}$$

The first two of the above relations were already mentioned in [5].

Remark 3. The formula (4) looks indeed very similar to the Eichler-Selberg trace formula as given in [5], so one might ask whether our result gives a similar trace formula for Hecke operators on the space $\mathcal{S}_k(\Gamma_0(4))$ of cusp forms of weight k on $\Gamma_0(4)$. Computer experiments suggest that in fact for $k \geq 1$, the coefficient of X^{2k} in (4) equals $-3\sum_{n\,\mathrm{odd}} \mathrm{trace}\left(T_n^{(2k+2)}\right) q^n$, where $T_n^{(\ell)}$ denotes the nth Hecke operator on $\mathcal{S}_\ell(\Gamma_0(4))$. This will be shown in an upcoming publication [24], since it requires different methods than the ones applied here.

Competing interests
The authors declare that they have no competing interests.

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