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A type of uncertain differential equations with analytic solution

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Abstract

Uncertain differential equation is a type of differential equation driven by the Liu process. So far, an analytic solution of linear uncertain differential equation has been obtained. This paper aims at proposing a method to solve a type of nonlinear uncertain differential equation.

Keywords: Uncertainty theory; Uncertain differential equation; Liu process

Introduction

Uncertainty theory, as a branch of axiomatic mathematics dealing with human's belief degree, was founded by Liu [1] in 2007 and refined by Liu [2] in 2010. During the past 6 years, many researchers have contributed in this area. For example, Peng and Iwamura [3] gave a sufficient condition for the uncertainty distribution of an uncertain variable. Liu and Ha [4] gave a formula to calculate the expected value of a function of multiple uncertain variables. Chen and Dai [5] showed that a normal uncertain variable has the maximum entropy given the expected value and variance. Especially, Liu [6] proposed uncertain programming as a type of mathematical programming involving uncertain variables.

In order to model the evolution of an uncertain phenomenon, Liu [7] proposed a concept of uncertain process. Meanwhile, Liu [7] gave an uncertain renewal process as an example. After that, Liu [2] proposed an uncertain renewal reward process, and Yao and Li [8] proposed an uncertain alternating renewal process. In addition, Zhang et al. [9] proposed an uncertain delayed renewal process. In 2009, Liu [10] mathematically defined a type of uncertain process, named canonical Liu process, which has independent and stationary uncertain normal increments and of which almost all the sample paths are Lipschitz continuous. In addition, Liu [11] proved the extreme value theorems for an independent increment uncertain process.

In 2009, Liu [10] founded an uncertain calculus to deal with the integral and differential of an uncertain process with respect to Liu process, which are called Liu integral and Liu differential afterwards. Then Liu and Yao [12] studied an uncertain integral with respect to multiple Liu processes. After that, Chen and Ralescu [13] proposed an uncertain integral with respect to the general Liu process. Inspired by the Liu integral, Yao [14] proposed an uncertain calculus with respect to an uncertain renewal process.

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Uncertain differential equation was first proposed by Liu [7] as a type of differential equation driven by Liu process. Then Chen and Liu [15] gave a sufficient condition for an uncertain differential equation having a unique solution. After that, Gao [16] provided an existence and unique theorem under weaker conditions. In 2013, Yao et al. [17] gave a sufficient condition for it being stable. After that, Sheng [18] studied the stability in pth moment.

Nowadays, uncertain differential equation has been applied to many areas especially in finance. In 2009, Liu [10] assumed that the stock price follows a geometric Liu process in the short run and proposed a stock model in an uncertain environment. After that, Chen [19] derived its American option formulae. In 2011, Peng and Yao [20] proposed another model to describe the stock price displaying a mean-reverting property in the long run via uncertain differential equation. In 2013, Chen et al. [21] proposed a stock model with periodic dividends. Assuming that the interest rate follows an uncertain differential equation, Chen and Gao [22] presented an uncertain interest rate model and calculated its zero-coupon bond. Besides, Liu et al. [23] proposed an uncertain currency model via uncertain differential equation. For more recent developments in uncertain finance, please refer to Liu [24].

In 2010, Chen and Liu [15] gave an analytic solution for the linear uncertain differential equation. Then Liu [25] proposed a method to solve a special type of nonlinear uncertain differential equation. In addition, some numerical methods were designed to calculate the uncertainty distributions of the solution and the extreme values of the solution by Yao and Chen [26] and Yao [27]. In this paper, we will give an analytic method to solve another special type of uncertain differential equation. The rest of this paper is organized as follows. Some concepts and theorems about uncertainty theory, uncertain calculus, and uncertain differential equation will be introduced in the 'Uncertainty theory' section, 'Uncertain calculus' section, and 'Uncertain differential equation' section, respectively. Then a special type of uncertain differential equation is solved in the 'An analytic method' section. Finally, some remarks are made in the 'Conclusions' section.

Uncertainty theory

This section will briefly introduce some basic concepts in uncertainty theory, including uncertainty space, uncertain variable, independence, and the operational law.

Definition 1. (Liu [1]) Let $\mathcal{L}$ be a σ-algebra on a nonempty set Γ. A set function $\mathcal{M}$: $\mathcal{L} \to [0,1]$ is called an uncertain measure if it satisfies the following axioms:
Axiom 1. (Normality Axiom) $\mathcal{M}\{\Gamma\} = 1$ for the universal set Γ.
Axiom 2. (Duality Axiom) $\mathcal{M}\{\Lambda\} + \mathcal{M}\{\Lambda^c\} = 1$ for any event Λ.
Axiom 3. (Subadditivity Axiom) For every countable sequence of events $\Lambda_1, \Lambda_2, \ldots$, we have

$$\mathcal{M}\left\{\bigcup_{i=1}^{\infty} \Lambda_i\right\} \le \sum_{i=1}^{\infty} \mathcal{M}\left\{\Lambda_i\right\}.$$

Besides, the product uncertain measure on the product σ-algebre $\mathcal{L}$ is defined by Liu [10] as follows:

Axiom 4. (Product Axiom) Let $(\Gamma_k, \mathcal{L}_k, \mathcal{M}_k)$ be uncertainty spaces for $k = 1, 2, \ldots$ Then the product uncertain measure $\mathcal{M}$ on the product σ-algebra satisfies

$$\mathcal{M}\left\{\prod_{i=1}^{\infty} \Lambda_k\right\} = \bigwedge_{k=1}^{\infty} \mathcal{M}_k\{\Lambda_k\}$$

where Λ_k are arbitrarily chosen events from $\mathcal{L}_k$ for $k = 1, 2, \ldots$, respectively.

In order to represent the quantities with uncertainty, an uncertain variable was proposed as a real valued function on an uncertainty space.

Definition 2. (Liu [1]) An uncertain variable ξ is a measurable function from an uncertainty space $(\Gamma, \mathcal{L}, \mathcal{M})$ to the set of real numbers, i.e., for any Borel set B, the set $\{\gamma \mid \xi(\gamma) \in B\}$ is an event in $\mathcal{L}$.

The uncertainty distribution Φ of an uncertain variable is defined by $\Phi(x) = \mathcal{M}\{\xi \le x\}$ for any real number x. The expected value of an uncertain variable ξ is

$$E[\xi] = \int_0^{+\infty} \mathcal{M}\{\xi \ge r\} \mathrm{d}r - \int_{-\infty}^{0} \mathcal{M}\{\xi \le r\} \mathrm{d}r$$

provided that at least one of the two integrals is finite, and the variance of ξ is

$$V[\xi] = E\left[(\xi - E[\xi])^2\right].$$

An uncertain variable ξ is said to be normal if it has an uncertainty distribution

$$\Phi(x) = \left(1 + \exp\left(\frac{\pi(e - x)}{\sqrt{3}\sigma}\right)\right)^{-1}, \quad x \in \Re$$

where e and σ^2 are the expected value and variance of ξ, respectively.

Definition 3. (Liu [10]) The uncertain variables $\xi_1, \xi_2, \ldots, \xi_n$ are said to be independent if

$$\mathcal{M}\left\{\bigcap_{i=1}^{n} (\xi_i \in B_i)\right\} = \bigwedge_{i=1}^{n} \mathcal{M}\{\xi_i \in B_i\}$$

for any Borel sets $B_1, B_2, \ldots, B_n$.

Since the definition of independence for uncertain variables is quite different from that for random variables, the operational law of uncertain variables is also different.

Theorem 1. (Liu [2]) *Let $\xi_1, \xi_2, \ldots, \xi_n$ be independent uncertain variables with uncertainty distributions $\Phi_1, \Phi_2, \ldots, \Phi_n$, respectively. If $f(x_1, x_2, \ldots, x_n)$ is strictly increasing with respect to $x_1, x_2, \ldots, x_m$ and strictly decreasing with respect to $x_{m+1}, x_{m+2}, \ldots, x_n$, then $\xi = f(\xi_1, \xi_2, \ldots, \xi_n)$ is an uncertain variable with an inverse uncertainty distribution*

$$\Phi^{-1}(r) = f\left(\Phi_1^{-1}(r), \ldots, \Phi_m^{-1}(r), \Phi_{m+1}^{-1}(1 - r), \ldots, \Phi_n^{-1}(1 - r)\right).$$

Uncertain calculus

An uncertain process is essentially a sequence of uncertain variables indexed by time or space. As one of the most important types of uncertain processes, the canonical Liu process is defined as follows:

Definition 4. (Liu [10]) An uncertain process C_t is called a canonical Liu process if:

1. $C_0 = 0$ and almost all sample paths are Lipschitz continuous.
2. C_t has stationary and independent increments.
3. Every increment $C_{s+t} - C_s$ is a normal uncertain variable $\mathcal{N}(0, t)$ with expected value 0 and variance t^2 whose uncertainty distribution is

$$\Phi_t(x) = \left(1 + \exp\left(-\frac{\pi x}{\sqrt{3}t}\right)\right)^{-1}, \quad x \in \Re.$$

Based on the canonical Liu process, Liu [10] proposed an uncertain integral of an uncertain process with respect to the canonical Liu process and thus founded a theory of uncertain calculus.

Definition 5. (Liu [10]) Let X_t be an uncertain process and C_t be a canonical Liu process. For any partition of closed interval $[a, b]$ with $a = t_1 < t_2 < \cdots < t_{k+1} = b$, the mesh is written as

$$\Delta = \max_{1 \le i \le k} |t_{i+1} - t_i|.$$

Then the Liu integral of X_t is defined by

$$\int_a^b X_t \mathrm{d}C_t = \lim_{\Delta \to 0} \sum_{i=1}^{k} X_{t_i} \cdot (C_{t_{i+1}} - C_{t_i})$$

provided that the limit exists almost surely and is finite.

For example, the Liu integral of an integrable real function $f(t)$ is

$$\int_0^t f(s)\mathrm{d}C_s \sim \left(0, \int_0^t |f(s)|\mathrm{d}s\right)$$

at each time t.

Definition 6. (Liu [10]) Let X_t be an uncertain process and N_t be an uncertain renewal process. Then the Yao integral of X_t is defined by

$$\int_a^b X_t \mathrm{d}N_t = \sum_{a<t\le b} X_{t-}(N_t - N_{t-})$$

provided that the sum exists almost surely and is finite.

For a continuously differentiable function $h(t, c, n)$, the uncertain process $Z_t = h(t, C_t, N_t)$ has an uncertain differential

$$\mathrm{d}Z_t = \frac{\partial h}{\partial t}(t, C_t, N_t)\mathrm{d}t + \frac{\partial h}{\partial c}(t, C_t, N_t)\mathrm{d}C_t + h(t, C_t, N_t) - h(t, C_t, N_{t-}).$$

For example, the uncertain differential of an uncertain process $X_t = \mu t + \sigma C_t + \gamma N_t$ is

$$\mathrm{d}X_t = \mu \mathrm{d}Z_t + \sigma \mathrm{d}C_t + \gamma(N_t - N_{t-}),$$

and the uncertain differential of an uncertain process $Y_t = tC_tN_t$ is

$$\mathrm{d}Y_t = C_tN_t\mathrm{d}t + tN_t\mathrm{d}C_t + tC_t(N_t - N_{t-}).$$

Uncertain differential equation

Definition 7. (Liu [7]) Suppose C_t is a canonical Liu process, and f and g are two given functions. Then

$$\mathrm{d}X_t = f(t, X_t)\mathrm{d}t + g(t, X_t)\mathrm{d}C_t \tag{1}$$

is called an uncertain differential equation.

In 2010, Chen and Liu [15] solved the linear uncertain differential equation

$$\mathrm{d}X_t = (u_{1t}X_t + u_{2t})\mathrm{d}t + (v_{1t}X_t + v_{2t})\mathrm{d}C_t$$

and obtained a solution

$$X_t = Y_t\left(X_0 + \int_0^t \frac{u_{2s}}{Y_s}\mathrm{d}s + \int_0^t \frac{v_{2s}}{Y_s}\mathrm{d}C_s\right)$$

where

$$Y_t = \exp\left(\int_0^t u_{1s}\mathrm{d}s + \int_0^t v_{1s}\mathrm{d}C_s\right).$$

After that, Liu [25] provided a method to solve the following types of uncertain differential equations:

$$\mathrm{d}X_t = f(t, X_t)\mathrm{d}t + \sigma_t X_t \mathrm{d}C_t, \tag{2}$$

$$\mathrm{d}X_t = \mu_t X_t \mathrm{d}t + g(t, X_t)\mathrm{d}C_t, \tag{3}$$

and showed that uncertain differential equation (2) has a solution $X_t = Y_t^{-1}Z_t$ where

$$Y_t = \exp\left(-\int_0^t \sigma_s \mathrm{d}C_s\right)$$

and Z_t solves uncertain differential equation $\mathrm{d}Z_t = Y_t f(t, Y_t^{-1}Z_t)\mathrm{d}t$ with an initial value $Z_0 = X_0$, and uncertain differential equation (3) has a solution $X_t = Y_t^{-1}Z_t$ where

$$Y_t = \exp\left(-\int_0^t \alpha_s \mathrm{d}s\right)$$

and Z_t solves uncertain differential equation $\mathrm{d}Z_t = Y_t g(t, Y_t^{-1}Z_t)\mathrm{d}C_t$ with an initial value $Z_0 = X_0$.

An analytic method

In this section, we will propose an analytic method to solve uncertain differential equations

$$\mathrm{d}X_t = f(t, X_t)\mathrm{d}t + \sigma_t \mathrm{d}C_t$$

and

$$\mathrm{d}X_t = \mu_t \mathrm{d}t + g(t, X_t)\mathrm{d}C_t.$$

Model I

Theorem 2. *Let f be a function of two variables, and σ_t be an integrable function on $\Re^+$. Then the uncertain differential equation*

$$\mathrm{d}X_t = f(t, X_t)\mathrm{d}t + \sigma_t \mathrm{d}C_t$$

has a solution

$$X_t = Y_t + Z_t$$

where

$$Y_t = \int_0^t \sigma_s \mathrm{d}C_s$$

and Z_t *is the solution of uncertain differential equation*

$$\mathrm{d}Z_t = f(t, Y_t + Z_t)\mathrm{d}t, \quad Z_0 = X_0.$$

Proof. The uncertain process Y_t has an uncertain differential

$$\mathrm{d}Y_t = \mu_t \mathrm{d}t.$$

Then we have

$$\mathrm{d}(X_t - Y_t) = \mathrm{d}X_t - \mathrm{d}Y_t = f(t, X_t)\mathrm{d}t + \sigma_t \mathrm{d}C_t - \sigma_t \mathrm{d}C_t.$$

That is,

$$\mathrm{d}(X_t - Y_t) = f(t, X_t)\mathrm{d}t.$$

Defining $Z_t = X_t - Y_t$, we obtain $X_t = Y_t + Z_t$ and $\mathrm{d}Z_t = f(t, Y_t + Z_t)\mathrm{d}t$. Furthermore, since $Y_0 = 0$, the initial value Z_0 is just X_0. The theorem is verified. □

Remark 1. If σ_t degenerates to a constant σ, then $Y_t = \sigma C_t$. The uncertain differential equation

$$\mathrm{d}X_t = f(t, X_t)\mathrm{d}t + \sigma \mathrm{d}C_t$$

has a solution

$$X_t = Z_t + \sigma C_t$$

where Z_t solves the uncertain differential equation

$$\mathrm{d}Z_t = f(t, Z_t + \sigma C_t)\mathrm{d}t, \quad Z_0 = X_0.$$

Example 1. Let m_t, μ_t, and σ_t be some real functions on $\Re$. Consider the uncertain differential equation

$$\mathrm{d}X_t = (m_t - \mu_t X_t)\mathrm{d}t + \sigma_t \mathrm{d}C_t.$$

At first, we have

$$Y_t = \int_0^t \sigma_s \mathrm{d}C_s$$

and Z_t solves the uncertain differential equation

$$\mathrm{d}Z_t = \left(m_t - \mu_t \left(\int_0^t \sigma_s \mathrm{d}s + Z_t\right)\right)\mathrm{d}t,$$

so

$$\mathrm{d}Z_t + \mu_t Z_t \mathrm{d}t = \left(m_t - \mu_t \int_0^t \sigma_s \mathrm{d}C_s\right)\mathrm{d}t.$$

It follows from the fundamental theorem of uncertain calculus that

$$\mathrm{d}\left(\exp\left(\int_0^t \mu_s \mathrm{d}s\right) Z_t\right) = \exp\left(\int_0^t \mu_s \mathrm{d}s\right)\left(m_t - \mu_t \int_0^t \sigma_s \mathrm{d}C_s\right)\mathrm{d}t.$$

That is,

$$\exp\left(\int_0^t \mu_s \mathrm{d}s\right) Z_t - Z_0 = \int_0^t \exp\left(\int_0^s \mu_\nu \mathrm{d}\nu\right)\left(m_s - \mu_s \int_0^s \sigma_\nu \mathrm{d}C_\nu\right)\mathrm{d}s.$$

As a result,

$$\begin{aligned} Z_t &= \exp\left(-\int_0^t \mu_s \mathrm{d}s\right) Z_0 + \int_0^t \exp\left(-\int_s^t \mu_\nu \mathrm{d}\nu\right)\left(m_s - \mu_s \int_0^s \sigma_\nu \mathrm{d}C_\nu\right)\mathrm{d}s \\ &= \exp\left(-\int_0^t \mu_s \mathrm{d}s\right) X_0 + \int_0^t \exp\left(-\int_s^t \mu_\nu \mathrm{d}\nu\right)\left(m_s - \mu_s \int_0^s \sigma_\nu \mathrm{d}C_\nu\right)\mathrm{d}s. \end{aligned}$$

By Theorem 2, we have

$$X_t = Y_t + Z_t = \exp\left(-\int_0^t \mu_s \mathrm{d}s\right) X_0 + \int_0^t \sigma_s \mathrm{d}C_s + \int_0^t \exp\left(-\int_s^t \mu_\nu \mathrm{d}\nu\right)\left(m_s - \mu_s \int_0^s \sigma_\nu \mathrm{d}C_\nu\right)\mathrm{d}s.$$

Example 2. Let μ and σ be real numbers with $\mu \neq 0$. Consider the uncertain differential equation

$$\mathrm{d}X_t = \mu \exp(X_t)\mathrm{d}t + \sigma \mathrm{d}C_t.$$

At first, we have $Y_t = \sigma C_t$ and Z_t satisfies the uncertain differential equation

$$\mathrm{d}Z_t = \mu \exp(\sigma C_t + Z_t)\mathrm{d}t,$$

so

$$\exp(-Z_t)\mathrm{d}Z_t = \mu \exp(\sigma C_t)\mathrm{d}t.$$

It follows from the fundamental theorem of uncertain calculus that

$$\mathrm{d}\exp(-Z_t) = -\mu \exp(\sigma C_t)\mathrm{d}t.$$

That is,

$$\exp(-Z_t) - \exp(-Z_0) = -\mu \int_0^t \exp(\sigma C_s)\mathrm{d}s.$$

As a result,

$$Z_t = Z_0 - \ln\left(1 - \mu \int_0^t \exp\left(Z_0 + \sigma C_s\right)\mathrm{d}s\right) = X_0 - \ln\left(1 - \mu \int_0^t \exp\left(X_0 + \sigma C_s\right)\mathrm{d}s\right).$$

By Theorem 2, we have

$$X_t = Y_t + Z_t = X_0 + \sigma C_t - \ln\left(1 - \mu \int_0^t \exp\left(X_0 + \sigma C_s\right)\mathrm{d}s\right).$$

Model II

Theorem 3. *Let μ_t be an integrable function on $\Re^+$, and g be a function of two variables. Then the uncertain differential equation*

$$\mathrm{d}X_t = \mu_t \mathrm{d}t + g(t, X_t)\mathrm{d}C_t$$

has a solution

$$X_t = Y_t + Z_t$$

where

$$Y_t = \int_0^t \mu_s \mathrm{d}s$$

and Z_t is the solution of uncertain differential equation

$$\mathrm{d}Z_t = g(t, Y_t + Z_t)\mathrm{d}C_t, \quad Z_0 = X_0.$$

Proof. The uncertain process Y_t has an uncertain differential

$$\mathrm{d}Y_t = \mu_t \mathrm{d}t.$$

Then we have

$$\mathrm{d}(X_t - Y_t) = \mathrm{d}X_t - \mathrm{d}Y_t = \mu_t \mathrm{d}t + g(t, X_t)\mathrm{d}C_t - \mu_t \mathrm{d}C_t.$$

That is,

$$\mathrm{d}(X_t - Y_t) = g(t, X_t)\mathrm{d}C_t.$$

Defining $Z_t = X_t - Y_t$, we obtain $X_t = Y_t + Z_t$ and $\mathrm{d}Z_t = g(t, Y_t + Z_t)\mathrm{d}C_t$. Furthermore, since $Y_0 = 0$, the initial value Z_0 is just X_0. The theorem is verified. □

Remark 2. If μ_t degenerates to a constant μ, then $Y_t = \mu t$, and the uncertain differential equation

$$\mathrm{d}X_t = \mu \mathrm{d}t + g(t, X_t)\mathrm{d}C_t$$

has a solution

$$X_t = Z_t + \mu t$$

where Z_t solves the uncertain differential equation

$$\mathrm{d}Z_t = g(t, Z_t + \mu t)\mathrm{d}t, \quad Z_0 = X_0.$$

Example 3. Let m_t, μ_t, and σ_t be some real functions on $\Re$. Consider the uncertain differential equation

$$\mathrm{d}X_t = \mu_t \mathrm{d}t + (m_t - \sigma_t X_t)\mathrm{d}C_t.$$

At first, we have

$$Y_t = \int_0^t \mu_s \mathrm{d}s$$

and Z_t solves the uncertain differential equation

$$\mathrm{d}Z_t = \left(m_t - \sigma_t \left(\int_0^t \mu_s \mathrm{d}s + Z_t\right)\right) \mathrm{d}C_t,$$

so

$$\mathrm{d}Z_t + \sigma_t Z_t \mathrm{d}C_t = \left(m_t - \sigma_t \int_0^t \mu_s \mathrm{d}s\right) \mathrm{d}C_t.$$

It follows from the fundamental theorem of uncertain calculus that

$$\mathrm{d}\left(\exp\left(\int_0^t \sigma_s \mathrm{d}C_s\right) Z_t\right) = \exp\left(\int_0^t \sigma_s \mathrm{d}C_s\right) \left(m_t - \sigma_t \int_0^t \mu_s \mathrm{d}s\right) \mathrm{d}C_t.$$

That is,

$$\exp\left(\int_0^t \sigma_s \mathrm{d}C_s\right) Z_t - Z_0 = \int_0^t \exp\left(\int_0^s \sigma_v \mathrm{d}C_v\right) \left(m_s - \sigma_s \int_0^s \mu_v \mathrm{d}v\right) \mathrm{d}C_s.$$

As a result,

$$Z_t = \exp\left(-\int_0^t \sigma_s \mathrm{d}C_s\right) Z_0 + \int_0^t \exp\left(-\int_s^t \sigma_v \mathrm{d}C_v\right)\left(m_s - \sigma_s \int_0^s \mu_v \mathrm{d}v\right)\mathrm{d}C_s$$
$$= \exp\left(-\int_0^t \sigma_s \mathrm{d}C_s\right) X_0 + \int_0^t \exp\left(-\int_s^t \sigma_v \mathrm{d}C_v\right)\left(m_s - \sigma_s \int_0^s \mu_v \mathrm{d}v\right)\mathrm{d}C_s.$$

By Theorem 3, we have

$$X_t = Y_t + Z_t = \exp\left(-\int_0^t \sigma_s \mathrm{d}C_s\right) X_0 + \int_0^t \mu_s \mathrm{d}s + \int_0^t \exp\left(-\int_s^t \sigma_v \mathrm{d}C_v\right)\left(m_s - \sigma_s \int_0^s \mu_v \mathrm{d}v\right)\mathrm{d}C_s.$$

Example 4. Let μ and σ be real numbers with $\sigma \neq 0$. Consider the uncertain differential equation

$$\mathrm{d}X_t = \mu \mathrm{d}t + \sigma \exp(X_t)\mathrm{d}C_t.$$

At first, we have $Y_t = \mu t$ and Z_t satisfies the uncertain differential equation

$$\mathrm{d}Z_t = \sigma \exp(\mu t + Z_t)\mathrm{d}C_t,$$

so

$$\exp(-Z_t)\mathrm{d}Z_t = \sigma \exp(\mu t)\mathrm{d}C_t.$$

It follows from the fundamental theorem of uncertain calculus that

$$\mathrm{d}\exp(-Z_t) = -\sigma \exp(\mu t)\mathrm{d}C_t.$$

That is,

$$\exp(-Z_t) - \exp(-X_0) = -\sigma \int_0^t \exp(\mu s)\mathrm{d}C_s.$$

As a result,

$$Z_t = Z_0 - \ln\left(1 - \sigma \int_0^t \exp(Z_0 + \mu s)\,\mathrm{d}C_s\right) = X_0 - \ln\left(1 - \sigma \int_0^t \exp(X_0 + \mu s)\,\mathrm{d}C_s\right).$$

By Theorem 3, we have

$$X_t = Y_t + Z_t = X_0 + \mu t - \ln\left(1 - \sigma \int_0^t \exp(X_0 + \mu s)\,\mathrm{d}C_s\right).$$

Conclusions

This paper proposed a method to solve a special type of uncertain differential equation and employed some examples to illustrate the method.

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