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Nontrivial solutions for a boundary value problem with integral boundary conditions

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Abstract
This paper concerns the existence of nontrivial solutions for a boundary value problem with integral boundary conditions by topological degree theory. Here the nonlinear term is a sign-changing continuous function and may be unbounded from below.

1 Introduction

Consider the following Sturm-Liouville problem with integral boundary conditions

$$\begin{cases} (Lu)(t) + h(t)f(t,u(t)) = 0, \quad 0 < t < 1, \\ (\cos\gamma_0)u(0) - (\sin\gamma_0)u'(0) = \int_0^1 u(\tau)\,d\alpha(\tau), \\ (\cos\gamma_1)u(1) + (\sin\gamma_1)u'(1) = \int_0^1 u(\tau)\,d\beta(\tau), \end{cases} \tag{1.1}$$

where $(Lu)(t) = (\tilde{p}(t)u'(t))' + q(t)u(t)$, $\tilde{p}(t) \in C^1[0,1]$, $\tilde{p}(t) > 0$, $q(t) \in C[0,1]$, $q(t) < 0$, α and β are right continuous on $[0,1)$, left continuous at $t = 1$ and nondecreasing on $[0,1]$ with $\alpha(0) = \beta(0) = 0$; $\gamma_0, \gamma_1 \in [0,\pi/2]$, $\int_0^1 u(\tau)\,d\alpha(\tau)$ and $\int_0^1 u(\tau)\,d\beta(\tau)$ denote the Riemann-Stieltjes integral of u with respect to α and β, respectively. Here the nonlinear term $f : [0,1] \times (-\infty,+\infty) \to (-\infty,+\infty)$ is a continuous sign-changing function and f may be unbounded from below, $h : (0,1) \to [0,+\infty)$ with $0 < \int_0^1 h(s)\,ds < +\infty$ is continuous and is allowed to be singular at $t = 0, 1$.

Problems with integral boundary conditions arise naturally in thermal conduction problems [1], semiconductor problems [2], hydrodynamic problems [3]. Integral BCs (BCs denotes boundary conditions) cover multi-point BCs and nonlocal BCs as special cases and have attracted great attention, see [4–14] and the references therein. For more information about the general theory of integral equations and their relation with boundary value problems, we refer to the book of Corduneanu [4], Agarwal and O'Regan [5]. Yang [6], Boucherif [8], Chamberlain *et al.* [10], Feng [11], Jiang *et al.* [14] focused on the existence of positive solutions for the cases in which the nonlinear term is nonnegative. Although many papers investigated two-point and multi-point boundary value problems with sign-changing nonlinear terms, for example, [15–20], results for boundary value problems with integral boundary conditions when the nonlinear term is sign-changing are rarely seen except for a few special cases [7, 12, 13].

Inspired by the above papers, the aim of this paper is to establish the existence of nontrivial solutions to BVP (1.1) under weaker conditions. Our findings presented in this paper have the following new features. Firstly, the nonlinear term f of BVP (1.1) is allowed to be sign-changing and unbounded from below. Secondly, the boundary conditions in BVP (1.1) are the Riemann-Stieltjes integral, which includes multi-point boundary conditions in BVPs as special cases. Finally, the main technique used here is the topological degree theory, the first eigenvalue and its positive eigenfunction corresponding to a linear operator. This paper employs different conditions and different methods to solve the same BVP (1.1) as [7]; meanwhile, this paper generalizes the result in [17] to boundary value problems with integral boundary conditions. What we obtain here is different from [6–20].

2 Preliminaries and lemmas

Let $E = C[0,1]$ be a Banach space with the maximum norm $\|u\| = \max_{0\leq t\leq 1}|u(t)|$ for $u \in E$. Define $P = \{u \in E \mid u(t) \geq 0, t \in [0,1]\}$ and $B_r = \{u \in E \mid \|u\| < r\}$. Then P is a total cone in E, that is, $E = \overline{P - P}$. P^* denotes the dual cone of P, namely, $P^* = \{g \in E^* \mid g(u) \geq 0, \text{ for all } u \in P\}$. Let E^* denote the dual space of E, then by Riesz representation theorem, E^* is given by

$$E^* = \bigl\{\nu \mid \nu \text{ is right continuous on } [0,1) \text{ and is bounded variation on } [0,1] \\ \text{with } \nu(0) = 0\bigr\}.$$

We assume that the following condition holds throughout this paper.

(H_1) $u(t) \equiv 0$ *is the unique* C^2 *solution of the linear boundary value problem*

$$\begin{cases} -(Lu)(t) = 0, \quad 0 < t < 1, \\ (\cos\gamma_0)u(0) - (\sin\gamma_0)u'(0) = 0, \qquad (\cos\gamma_1)u(1) + (\sin\gamma_1)u'(1) = 0. \end{cases}$$

Let $\varphi, \psi \in C^2([0,1], \mathbf{R}^+)$ solve the following inhomogeneous boundary value problems, respectively:

$$\begin{cases} -(L\varphi)(t) = 0, \quad 0 < t < 1, \\ (\cos\gamma_0)\varphi(0) - (\sin\gamma_0)\varphi'(0) = 1, \\ (\cos\gamma_1)\varphi(1) + (\sin\gamma_1)\varphi'(1) = 0 \end{cases} \quad \text{and} \quad \begin{cases} -(L\psi)(t) = 0, \quad 0 < t < 1, \\ (\cos\gamma_0)\psi(0) - (\sin\gamma_0)\psi'(0) = 0, \\ (\cos\gamma_1)\psi(1) + (\sin\gamma_1)\psi'(1) = 1. \end{cases}$$

Let $\kappa_1 = 1 - \int_0^1 \varphi(\tau)\,d\alpha(\tau)$, $\kappa_2 = \int_0^1 \psi(\tau)\,d\alpha(\tau)$, $\kappa_3 = \int_0^1 \varphi(\tau)\,d\beta(\tau)$, $\kappa_4 = 1 - \int_0^1 \psi(\tau)\,d\beta(\tau)$.

(H_2) $\kappa_1 > 0$, $\kappa_4 > 0$, $k = \kappa_1\kappa_4 - \kappa_2\kappa_3 > 0$.

Lemma 2.1 ([7]) *If* (H_1) *and* (H_2) *hold, then BVP* (1.1) *is equivalent to*

$$u(t) = \int_0^1 G(t,s)h(s)f\bigl(s,u(s)\bigr)\,ds,$$

where $G(t,s) \in C([0,1] \times [0,1], \mathbf{R}^+)$ *is the Green function for* (1.1).

Define an operator $A : E \to E$ as follows:

$$(Au)(t) = \int_0^1 G(t,s)h(s)f\bigl(s,u(s)\bigr)\,ds, \quad u \in E. \tag{2.1}$$

It is easy to show that $A : E \to E$ is a completely continuous nonlinear operator, and if $u \in E$ is a fixed point of A, then u is a solution of BVP (1.1) by Lemma 2.1.

For any $u \in E$, define a linear operator $K : E \to E$ as follows:

$$(Ku)(t) = \int_0^1 G(t,s)h(s)u(s)\,ds, \quad u \in E. \tag{2.2}$$

It is easy to show that $K : E \to E$ is a completely continuous nonlinear operator and $K(P) \subset P$ holds. By [7], the spectral radius $r(K)$ of K is positive. The Krein-Rutman theorem [21] asserts that there are $\phi \in P \setminus \{0\}$ and $\omega \in P^* \setminus \{0\}$ corresponding to the first eigenvalue $\lambda_1 = 1/r(K)$ of K such that

$$\lambda_1 K\phi = \phi \tag{2.3}$$

and

$$\lambda_1 K^*\omega = \omega, \qquad \omega(1) = 1. \tag{2.4}$$

Here $K^* : E^* \to E^*$ is the dual operator of K given by:

$$\bigl(K^*v\bigr)(s) = \int_0^s \int_0^1 G(t,\tau)h(\tau)\,dv(t)\,d\tau, \quad v \in E^*.$$

The representation of K^*, the continuity of G and the integrability of h imply that $\omega \in C^1[0,1]$. Let $e(t) := \omega'(t)$. Then $e \in P \setminus \{0\}$, and (2.4) can be rewritten equivalently as

$$r(K)e(s) = \int_0^1 G(t,s)h(s)e(t)\,dt, \qquad \int_0^1 e(t)\,dt = 1. \tag{2.5}$$

Lemma 2.2 ([7]) *If* (H_1) *holds, then there is* $\delta > 0$ *such that* $P_0 = \{u \in P \mid \int_0^1 u(t)e(t)\,dt \geq \delta\|u\|\}$ *is a subcone of* P *and* $K(P) \subset P_0$.

Lemma 2.3 ([22]) *Let* E *be a real Banach space and* $\Omega \subset E$ *be a bounded open set with* $0 \in \Omega$. *Suppose that* $A : \bar{\Omega} \to E$ *is a completely continuous operator.* (1) *If there is* $y_0 \in E$ *with* $y_0 \neq 0$ *such that* $u \neq Au + \mu y_0$ *for all* $u \in \partial\Omega$ *and* $\mu \geq 0$, *then* $\deg(I - A, \Omega, 0) = 0$. (2) *If* $Au \neq \mu u$ *for all* $u \in \partial\Omega$ *and* $\mu \geq 1$, *then* $\deg(I - A, \Omega, 0) = 1$. *Here* deg *stands for the Leray-Schauder topological degree in* E.

Lemma 2.4 *Assume that* (H_1), (H_2) *and the following assumptions are satisfied*:

- (C_1) *There exist* $\phi \in P \setminus \{0\}$, $\omega \in P^* \setminus \{0\}$ *and* $\delta > 0$ *such that* (2.3), (2.4) *hold and* K *maps* P *into* P_0.
- (C_2) *There exists a continuous operator* $H : E \to P$ *such that*

$$\lim_{\|u\| \to +\infty} \frac{\|Hu\|}{\|u\|} = 0.$$

- (C_3) *There exist a bounded continuous operator* $F : E \to E$ *and* $u_0 \in E$ *such that* $Fu + u_0 + Hu \in P$ *for all* $u \in E$.
- (C_4) *There exist* $v_0 \in E$ *and* $\zeta > 0$ *such that* $KFu \geq \lambda_1(1 + \zeta)Ku - KHu - v_0$ *for all* $u \in E$.

Let $A = KF$, then there exists $R > 0$ such that

$$\deg(I - A, B_R, 0) = 0,$$

where $B_R = \{u \in E \mid \|u\| < R\}$.

Proof Choose a constant $L_0 = (\delta\lambda_1)^{-1}(1 + \zeta^{-1}) + \|K\| > 0$. From (C_2), for $0 < \varepsilon_0 < L_0^{-1}$, there exists $R_1 > 0$ such that $\|u\| > R_1$ implies

$$\|Hu\| < \varepsilon_0\|u\|. \tag{2.6}$$

Now we shall show

$$u \neq KFu + \mu\phi \quad \text{for any } u \in \partial B_R \text{ and } \mu \geq 0, \tag{2.7}$$

provided that R is sufficiently large.

In fact, if (2.7) is not true, then there exist $u_1 \in \partial B_R$ and $\mu_1 \geq 0$ satisfying

$$u_1 = KFu_1 + \mu_1\phi. \tag{2.8}$$

Since $\phi \in P \setminus \{0\}$, $e(t) \in P \setminus \{0\}$, $\int_0^1 \phi(t)e(t)\,dt > 0$. Multiply (2.8) by $e(t)$ on both sides and integrate on $[0,1]$. Then, by (C_4), (2.5), we get

$$\begin{aligned}
&\int_0^1 u_1(t)e(t)\,dt \\
&\quad= \int_0^1 (KFu_1)(t)e(t)\,dt + \mu_1\int_0^1 \phi(t)e(t)\,dt \\
&\quad\geq \lambda_1(1+\zeta)\int_0^1\int_0^1 G(t,s)h(s)u_1(s)\,ds e(t)\,dt - \int_0^1 (KHu_1)(t)e(t)\,dt - \int_0^1 v_0(t)e(t)\,dt \\
&\quad= \lambda_1(1+\zeta)\int_0^1\int_0^1 G(t,s)h(s)u_1(s)e(t)\,ds\,dt \\
&\qquad - \int_0^1\int_0^1 G(t,s)h(s)(Hu_1)(s)e(t)\,ds\,dt - \int_0^1 v_0(t)e(t)\,dt \\
&\quad= \lambda_1(1+\zeta)\int_0^1\left[\int_0^1 G(t,s)h(s)e(t)\,dt\right]u_1(s)\,ds \\
&\qquad - \int_0^1\left[\int_0^1 G(t,s)h(s)e(t)\,dt\right](Hu_1)(s)\,ds - \int_0^1 v_0(t)e(t)\,dt \\
&\quad= \lambda_1(1+\zeta)r(K)\int_0^1 e(s)u_1(s)\,ds - r(K)\int_0^1 (Hu_1)(s)e(s)\,ds - \int_0^1 v_0(t)e(t)\,dt \\
&\quad= (1+\zeta)\int_0^1 u_1(t)e(t)\,dt - r(K)\int_0^1 (Hu_1)(t)e(t)\,dt - \int_0^1 v_0(t)e(t)\,dt.
\end{aligned} \tag{2.9}$$

Thus,

$$\int_0^1 u_1(t)e(t)\,dt \leq \zeta^{-1}\left(r(K)\int_0^1 (Hu_1)(t)e(t)\,dt + \int_0^1 v_0(t)e(t)\,dt\right). \tag{2.10}$$

By (2.9), $\int_0^1 (KHu_1)(t)e(t)\,dt = r(K)\int_0^1 (Hu_1)(t)e(t)\,dt$ holds. Then (2.3), (2.6) and (2.10) imply

$$\begin{aligned}
&\int_0^1 \bigl(u_1(t) + (KHu_1)(t) + (Ku_0)(t)\bigr)e(t)\,dt \\
&\quad\le \zeta^{-1}\left(r(K)\int_0^1 (Hu_1)(t)e(t)\,dt + \int_0^1 v_0(t)e(t)\,dt\right) \\
&\qquad + r(K)\int_0^1 (Hu_1)(t)e(t)\,dt + \int_0^1 (Ku_0)(t)e(t)\,dt \\
&\quad\le \zeta^{-1}(1+\zeta)r(K)\int_0^1 (Hu_1)(t)e(t)\,dt + \zeta^{-1}\int_0^1 v_0(t)e(t)\,dt + \int_0^1 (Ku_0)(t)e(t)\,dt \\
&\quad\le \zeta^{-1}(1+\zeta)r(K)\varepsilon_0\|u_1\| + L_1,
\end{aligned} \tag{2.11}$$

where $L_1 = \zeta^{-1}\int_0^1 v_0(t)e(t)\,dt + \int_0^1 (Ku_0)(t)e(t)\,dt$ is a constant.

(C_3) shows $Fu_1 + u_0 + Hu_1 \in P$ and (C_1) implies $\mu_1\phi = \mu_1\lambda_1 K\varphi_1 \in P_0$. Then (C_1), (2.8) and Lemma 2.2 tell us that

$$u_1 + KHu_1 + Ku_0 = KFu_1 + \mu_1\phi + KHu_1 + Ku_0 = K(Fu_1 + Hu_1 + u_0) + \mu_1\phi \in P_0.$$

The definition of P_0 yields

$$\begin{aligned}
\int_0^1 (u_1 + KHu_1 + Ku_0)(t)e(t)\,dt &\ge \delta\|u_1 + KHu_1 + Ku_0\| \\
&\ge \delta\|u_1\| - \delta\|KHu_1\| - \delta\|Ku_0\|.
\end{aligned} \tag{2.12}$$

It follows from (2.6), (2.11) and (2.12) that

$$\begin{aligned}
\|u_1\| &= \delta^{-1}\int_0^1 (u_1 + KHu_1 + Ku_0)(t)e(t)\,dt + \|KHu_1\| + \|Ku_0\| \\
&\le \varepsilon_0(\delta\lambda_1)^{-1}\bigl(1+\zeta^{-1}\bigr)\|u_1\| + L_1\delta^{-1} + \varepsilon_0\|K\|\cdot\|u_1\| + \|Ku_0\| \\
&= \varepsilon_0 L_0\|u_1\| + L_2,
\end{aligned} \tag{2.13}$$

where $L_2 = \|Ku_0\| + L_1\delta^{-1}$ is a constant.

Since $0 < \varepsilon_0 L_0 < 1$, then (2.13) deduces that (2.7) holds provided that R is sufficiently large such that $R > \max\{L_2/(1-\varepsilon_0 L_0), R_1\}$. By (2.13) and Lemma 2.3, we have

$$\deg(I - A, B_R, 0) = 0. \qquad \square$$

3 Main results

Theorem 3.1 *Assume that* (H_1), (H_2) *hold and the following conditions are satisfied*:

(A_1) *There exist two nonnegative functions* $b(t), c(t) \in C[0,1]$ *with* $c(t) \not\equiv 0$ *and one continuous even function* $B : \mathbf{R} \to \mathbf{R}^+$ *such that* $f(t,x) \ge -b(t) - c(t)B(x)$ *for all* $x \in \mathbf{R}$. *Moreover, B is nondecreasing on* $\mathbf{R}^+$ *and satisfies* $\lim_{x\to+\infty} \frac{B(x)}{x} = 0$.

(A_2) $f : [0,1] \times \mathbf{R} \to \mathbf{R}$ *is continuous*.

(A_3) $\liminf_{x\to+\infty} \frac{f(t,x)}{x} > \lambda_1$ *uniformly on* $t \in [0,1]$.
(A_4) $\limsup_{x\to 0} |\frac{f(t,x)}{x}| < \lambda_1$ *uniformly on* $t \in [0,1]$.

Here λ_1 *is the first eigenvalue of the operator* K *defined by* (2.2).
Then BVP (1.1) *has at least one nontrivial solution.*

Proof We first show that all the conditions in Lemma 2.4 are satisfied. By Lemma 2.2, condition (C_1) of Lemma 2.4 is satisfied. Obviously, $B: E \to P$ is a continuous operator. By (A_1), for any $\varepsilon > 0$, there is $L > 0$ such that when $x > L$, $B(x) < \varepsilon x$ holds. Thus, for $u \in E$ with $\|u\| > L$, $B(\|u\|) < \varepsilon\|u\|$ holds. The fact that B is nondecreasing on $\mathbf{R}^+$ yields $(Bu)(t) \leq B(\|u\|)$ for any $u \in P$, $t \in [0,1]$. Since $B: \mathbf{R} \to \mathbf{R}^+$ is an even function, for any $u \in E$ and $t \in [0,1]$, $(Bu)(t) \leq B(\|u\|)$ holds, which implies $\|Bu\| \leq B(\|u\|)$ for $u \in E$. Therefore,

$$\|Bu\| \leq B\bigl(\|u\|\bigr) < \varepsilon\|u\|, \quad \forall u \in E \text{ with } \|u\| > L,$$

that is, $\lim_{\|u\|\to+\infty} \frac{\|Bu\|}{\|u\|} = 0$. Take $Hu = c_0 Bu$, for any $u \in E$, where $c_0 = \max_{t\in[0,1]} c(t) > 0$. Obviously, $\lim_{\|u\|\to+\infty} \frac{\|Hu\|}{\|u\|} = 0$ holds. Therefore H satisfies condition (C_2) in Lemma 2.4.

Take $u_0(t) \equiv b = \max_{t\in[0,1]} b(t) > 0$ and $(Fu)(t) = f(t,u(t))$ for $t \in [0,1]$, $u \in E$, then it follows from (A_1) that

$$Fu + u_0 + Hu \in P \quad \text{for all } u \in E,$$

which shows that condition (C_3) in Lemma 2.4 holds.

By (A_3), there exist $\varepsilon_1 > 0$ and a sufficiently large number $l_1 > 0$ such that

$$f(t,x) \geq \lambda_1(1+\varepsilon_1)x, \quad \forall x \geq l_1. \tag{3.1}$$

Combining (3.1) with (A_1), there exists $b_1 \geq 0$ such that

$$f(t,x) \geq \lambda_1(1+\varepsilon_1)x - b_1 - c_0B(x) \quad \text{for all } x \in \mathbf{R},$$

and so

$$Fu \geq \lambda_1(1+\varepsilon_1)u - b_1 - Hu \quad \text{for all } u \in E. \tag{3.2}$$

Since K is a positive linear operator, from (3.2) we have

$$(KFu)(t) \geq \lambda_1(1+\varepsilon_1)(Ku)(t) - Kb_1 - (KHu)(t), \quad \forall t \in [0,1], u \in E.$$

So condition (C_4) in Lemma 2.4 is satisfied.

According to Lemma 2.4, we derive that there exists a sufficiently large number $R > 0$ such that

$$\deg(I - A, B_R, 0) = 0. \tag{3.3}$$

From (A_4) it follows that there exist $0 < \varepsilon_2 < 1$ and $0 < r < R$ such that

$$\bigl|f(t,x)\bigr| \leq (1-\varepsilon_2)\lambda_1|x|, \quad \forall t \in [0,1], x \in \mathbf{R} \text{ with } |x| \leq r.$$

Thus

$$\left|(Au)(t)\right| \leq (1-\varepsilon_2)\lambda_1\bigl(K|u|\bigr)(t), \quad \forall t \in [0,1], u \in E \text{ with } \|u\| \leq r. \tag{3.4}$$

Next we will prove that

$$u \neq \mu Au \quad \text{for all } u \in \partial B_r \text{ and } \mu \in [0,1]. \tag{3.5}$$

If there exist $u_1 \in \partial B_r$ and $\mu_1 \in [0,1]$ such that $u_1 = \mu_1 Au_1$. Let $z(t) = |u_1(t)|$. Then $z \in P$ and by (3.4), $z \leq (1-\varepsilon_2)\lambda_1 Kz$. The nth iteration of this inequality shows that $z \leq (1-\varepsilon_2)^n\lambda_1^n K^n z$ ($n = 1,2,\ldots$), so $\|z\| \leq (1-\varepsilon_2)^n\lambda_1^n\|K^n\| \cdot \|z\|$, that is, $1 \leq (1-\varepsilon_2)^n\lambda_1^n\|K^n\|$. This yields $1-\varepsilon_2 = (1-\varepsilon_2)\lambda_1 r(K) = (1-\varepsilon_2)\lambda_1 \lim_{n\to\infty}\sqrt[n]{\|K^n\|} \geq 1$, which is a contradictory inequality. Hence, (3.5) holds.

It follows from (3.5) and Lemma 2.3 that

$$\deg(I-A, B_r, 0) = 1. \tag{3.6}$$

By (3.3), (3.6) and the additivity of Leray-Schauder degree, we obtain

$$\deg(I-A, B_R \setminus \overline{B}_r, 0) = \deg(I-A, B_R, 0) - \deg(I-A, B_r, 0) = -1.$$

So A has at least one fixed point on $B_R \setminus \overline{B}_r$, namely, BVP (1.1) has at least one nontrivial solution. □

Corollary 3.1 *Using* (A_1^*) *instead of* (A_1), *the conclusion of Theorem* 3.1 *remains true*.

(A_1^*) *There exist three constants* $b > 0$, $c > 0$ *and* $\alpha \in (0,1)$ *such that*

$$f(x) \geq -b - c|x|^\alpha \quad \text{for any } x \in \mathbf{R}.$$

Competing interests
The authors declare that no conflict of interest exists.

Authors' contributions
All authors participated in drafting, revising and commenting on the manuscript. All authors read and approved the final manuscript.

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