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On semilinear biharmonic equations with concave-convex nonlinearities involving weight functions

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Abstract

In this paper, we consider semilinear biharmonic equations with concave-convex nonlinearities involving weight functions, where the concave nonlinear term is $\lambda f(x)|u|^{q-1}u$ and the convex nonlinear term is $h(x)|u|^{p-1}u$ with $\lambda \in \mathbb{R}^+$. By use of the Nehari manifold and the direct variational methods, the existence of multiple positive solutions is established as $\lambda \in (0, \lambda_*)$, here the explicit expression of $\lambda_* = \lambda_*(f, h, p, q, S)$ is provided.

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Keywords: biharmonic equations; concave-convex nonlinearities; weight functions

1 Introduction

In recent years, there has been extensive attention on semilinear second-order elliptic equations,

$$\begin{cases} -\Delta u = g_\lambda(x, u), & \text{in } \Omega, \\ u = 0, & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

here Ω is a bounded smooth domain in $\mathbb{R}^N$ $(N \geq 3)$, $g_\lambda : \Omega \times \mathbb{R} \to \mathbb{R}$ and λ is a positive parameter; see [1–8] and the references therein. As g_λ is sublinear, say, $g_\lambda = \lambda u^q$, $0 < q < 1$, the monotone iteration scheme or the method of sub-solutions and super-solutions are effective; see [9]. As g_λ is superlinear, for example, $g_\lambda = \lambda u + |u|^{p-1}u$, $1 < p < \frac{N+2}{N-2}$, variational methods are applicable; see [10]. In contrast with the pure sublinear case and the pure superlinear case, in [2] Ambrosetti *et al.* considered problem (1.1) when g_λ is, roughly, the sum of a sublinear and a superlinear term. To be precise, they considered the following problem:

$$\begin{cases} -\Delta u = \lambda u^q + u^p, & \text{in } \Omega, \\ 0 \leq u \in H_0^1(\Omega), \end{cases} \tag{1.2}$$

with $0 < q < 1 < p \leq \frac{N+2}{N-2}$. They proved that problem (1.2) admits at least two positive solutions for λ sufficiently small. In [6], Sun and Li considered a similar problem:

$$\begin{cases} -\Delta u = u^q + \lambda u^p, & \text{in } \Omega, \\ 0 \leq u \in H_0^1(\Omega), \end{cases}$$

with $0 < q < 1 < p = \frac{N+2}{N-2}$, the authors studied the value of Λ, the supremum of the set λ, related to the existence and multiplicity of positive solutions and established uniform lower bounds for Λ. In [8], Wu considered the subcritical case of problem (1.2) with λu^q replaced by $\lambda f(x)u^q$, here $f(x) \in C(\overline{\Omega})$ is a sign-changing function, and he showed that problem (1.2) has at least two positive solutions as λ is small enough.

Some interesting generalizations of (1.2) have been provided in the framework of quasilinear elliptic equations or systems, semilinear second-order elliptic systems or fourth-order elliptic equations. More recently, the semilinear fourth-order elliptic equations have been studied by many authors, we refer the reader to [11–13] and the references therein. Motivated by some work in [6, 8, 13], we deal with the following semilinear biharmonic elliptic equation:

$$\begin{cases} \Delta^2 u = \lambda f(x)|u|^{q-1}u + h(x)|u|^{p-1}u, & \text{in } \Omega, \\ u = \Delta u = 0, & \text{on } \partial\Omega, \end{cases} \tag{1.3}$$

where Ω is a bounded smooth domain in $\mathbb{R}^N$ ($N \geq 4$), $0 < q < 1 < p < 2^{**}$ ($2^{**} = \frac{N+4}{N-4}$ for $N > 4$ and $2^{**} = \infty$ for $N = 4$), $\lambda > 0$ is a parameter, $f \in C(\overline{\Omega})$ is a positive or sign-changing weight function and $h \in C(\overline{\Omega})$ is a positive weight function.

For convenience and simplicity, we introduce some notations. The norm of u in $L^r(\Omega)$ is denoted by $|u|_r = (\int_\Omega |u(x)|^r)^{1/r}$, the norm of u in $C(\overline{\Omega})$ is denoted by $|u|_\infty = \max_{x\in\overline{\Omega}} |u(x)|$; $H_0^1(\Omega) \cap H^2(\Omega)$ is denoted by $H(\Omega)$, endowed with the norm $\|u\| = |\Delta u|_2$; S denotes the best Sobolev constant for the embedding of $H(\Omega)$ in $L^{p+1}(\Omega)$ (see [14]); to be precise, $|u|_{p+1} \leq S\|u\|$ for all $u \in H(\Omega)$.

Now we define

$$J_\lambda(u) = \frac{1}{2}\|u\|^2 - \frac{\lambda}{q+1}\int_\Omega f(x)|u|^{q+1}\,\mathrm{d}x - \frac{1}{p+1}\int_\Omega h(x)|u|^{p+1}\,\mathrm{d}x, \quad u \in H(\Omega).$$

It is well known that the weak solutions of problem (1.3) are the critical points of the energy functional J_λ (see Rabinowitz [15]).

Next, we consider the Nehari minimization problem: for $\lambda > 0$,

$$\alpha_\lambda(\Omega) = \inf\{J_\lambda(u) \mid u \in M_\lambda(\Omega)\},$$

where $M_\lambda(\Omega) = \{u \in H(\Omega)\backslash\{0\} \mid \langle J_\lambda'(u), u\rangle = 0\}$. Define

$$\psi_\lambda(u) = \langle J_\lambda'(u), u\rangle = \|u\|^2 - \lambda\int_\Omega f(x)|u|^{q+1}\,\mathrm{d}x - \int_\Omega h(x)|u|^{p+1}\,\mathrm{d}x.$$

Then for $u \in M_\lambda(\Omega)$,

$$\langle \psi_\lambda'(u), u\rangle = 2\|u\|^2 - \lambda(q+1)\int_\Omega f(x)|u|^{q+1}\,\mathrm{d}x - (p+1)\int_\Omega h(x)|u|^{p+1}\,\mathrm{d}x.$$

Similarly to the method used in Tarantello [16], we split $M_\lambda(\Omega)$ into three parts:

$$M_\lambda^+(\Omega) = \{u \in M_\lambda(\Omega) \mid \langle \psi_\lambda'(u), u\rangle > 0\},$$
$$M_\lambda^0(\Omega) = \{u \in M_\lambda(\Omega) \mid \langle \psi_\lambda'(u), u\rangle = 0\},$$
$$M_\lambda^-(\Omega) = \{u \in M_\lambda(\Omega) \mid \langle \psi_\lambda'(u), u\rangle < 0\}.$$

Note that all solutions of (1.3) are clearly in the Nehari manifold, $M_\lambda(\Omega)$. Hence, our approach to solve problem (1.3) is to analyze the structure of $M_\lambda(\Omega)$, and then to deal with the minimization problems for J_λ on $M_\lambda^+(\Omega)$ and $M_\lambda^-(\Omega)$ applying the direct variational method.

The following is our main result.

Theorem 1.1 *Let* $\lambda_* = \frac{p-1}{p-q}\cdot[\frac{1-q}{(p-q)|h|_\infty}]^{\frac{1-q}{p-1}} S^{\frac{2(p-q)}{1-p}} |f|_{p^*}^{-1}$ *with* $p^* = \frac{p+1}{p-q}$, *then problem* (1.3) *has at least two positive solutions for any* $\lambda \in (0, \lambda_*)$.

The paper is organized as follows: in Section 2, we give some lemmas; in Section 3, we prove Theorem 1.1.

2 Preliminaries

In this section, we prove several lemmas.

Lemma 2.1 *For* $\lambda \in (0,\lambda_*)$ (*where* λ_* *is given in Theorem* 1.1), *we have* $M_\lambda^0(\Omega) = \phi$.

Proof Suppose that $M_\lambda^0(\Omega) \neq \phi$ for all $\lambda > 0$. If $u \in M_\lambda^0(\Omega)$, then we have

$$\|u\|^2 = \lambda \int_\Omega f(x)|u|^{q+1}\,\mathrm{d}x + \int_\Omega h(x)|u|^{p+1}\,\mathrm{d}x \tag{2.1}$$

and

$$2\|u\|^2 = \lambda(q+1)\int_\Omega f(x)|u|^{q+1}\,\mathrm{d}x + (p+1)\int_\Omega h(x)|u|^{p+1}\,\mathrm{d}x. \tag{2.2}$$

By (2.1)-(2.2), the Sobolev inequality, and the Hölder inequality, we get

$$\|u\|^2 = \frac{p-q}{1-q}\int_\Omega h(x)|u|^{p+1}\,\mathrm{d}x \le \frac{p-q}{1-q}|h|_\infty S^{p+1}\|u\|^{p+1} \tag{2.3}$$

and

$$\|u\|^2 = \lambda\cdot\frac{p-q}{p-1}\int_\Omega f(x)|u|^{q+1}\,\mathrm{d}x \le \lambda\cdot\frac{p-q}{p-1}|f|_{p^*}S^{q+1}\|u\|^{q+1}, \tag{2.4}$$

where $p^* = \frac{p+1}{p-q}$. Thus, using (2.3) and (2.4), we have

$$\begin{aligned}\lambda &\ge \frac{p-1}{p-q}\cdot|f|_{p^*}^{-1}S^{-(q+1)}\left[\frac{1-q}{p-q}|h|_\infty^{-1}S^{-(p+1)}\right]^{\frac{1-q}{p-1}}\\ &= \frac{p-1}{p-q}\cdot\left[\frac{1-q}{(p-q)|h|_\infty}\right]^{\frac{1-q}{p-1}}S^{\frac{2(p-q)}{1-p}}|f|_{p^*}^{-1} = \lambda_*.\end{aligned} \tag{2.5}$$

Hence, by (2.5) the desired conclusion yields. □

Lemma 2.2 *If* $u \in M_\lambda^-(\Omega)$, *then*

$$\|u\| > S^{\frac{1+p}{1-p}}\left[\frac{1-q}{(p-q)|h|_\infty}\right]^{\frac{1}{p-1}} \quad \text{and} \quad \int_\Omega h(x)|u|^{p+1}\,\mathrm{d}x > |h|_\infty^{\frac{2}{1-p}}\left[\frac{(p-q)S^2}{1-q}\right]^{\frac{1+p}{1-p}}.$$

Proof From $u \in M_\lambda^-(\Omega)$, it is easy to see that

$$\|u\|^2 < \frac{p-q}{1-q} \int_\Omega h(x)|u|^{p+1}\,\mathrm{d}x.$$

By the Sobolev inequality, we get

$$\|u\| > S^{\frac{1+p}{1-p}} \left[\frac{1-q}{(p-q)|h|_\infty} \right]^{\frac{1}{p-1}}.$$

In addition,

$$\int_\Omega h(x)|u|^{p+1}\,\mathrm{d}x > |h|_\infty^{\frac{2}{1-p}} \left[\frac{(p-q)S^2}{1-q} \right]^{\frac{1+p}{1-p}}.$$

The proof is completed. □

By Lemma 2.1, for $\lambda \in (0, \lambda_*)$ we write $M_\lambda(\Omega) = M_\lambda^+(\Omega) \cup M_\lambda^-(\Omega)$ and define

$$\alpha_\lambda^+(\Omega) = \inf_{u \in M_\lambda^+(\Omega)} J_\lambda(u), \qquad \alpha_\lambda^-(\Omega) = \inf_{u \in M_\lambda^-(\Omega)} J_\lambda(u).$$

The following lemma shows that the minimizers on $M_\lambda(\Omega)$ are 'usually' critical points for J_λ.

Lemma 2.3 *For $\lambda \in (0, \lambda_*)$, if u_0 is a local minimizer for J_λ on $M_\lambda(\Omega)$, then $J_\lambda'(u_0) = 0$ in $[H(\Omega)]^*$.*

Proof If u_0 is a local minimizer for J_λ on $M_\lambda(\Omega)$, then u_0 is a solution of the optimization problem

$$\text{minimize} \quad J_\lambda(u) \quad \text{subject to} \quad \psi_\lambda(u) = 0.$$

Hence, by the theory of Lagrange multipliers, there exists $\theta \in \mathbb{R}$ such that

$$J_\lambda'(u_0) = \theta \psi_\lambda'(u_0) \quad \text{in } \left[H(\Omega)\right]^*. \tag{2.6}$$

Thus,

$$\left\langle J_\lambda'(u_0), u_0 \right\rangle = \theta \left\langle \psi_\lambda'(u_0), u_0 \right\rangle. \tag{2.7}$$

From $u_0 \in M_\lambda(\Omega)$ and Lemma 2.1, we have $\langle J_\lambda'(u_0), u_0 \rangle = 0$ and $\langle \psi_\lambda'(u_0), u_0 \rangle \neq 0$. So, by (2.6)-(2.7) we get $J_\lambda'(u_0) = 0$ in $[H(\Omega)]^*$. □

For each $u \in H(\Omega) \backslash \{0\}$, we write

$$t_{\max} = \left(\frac{(1-q)\|u\|^2}{(p-q)\int_\Omega h(x)|u|^{p+1}\,\mathrm{d}x} \right)^{\frac{1}{p-1}} > 0.$$

Then we have the following lemma.

Lemma 2.4 *For each $u \in H(\Omega)\backslash\{0\}$ and $\lambda \in (0, \lambda_*)$, we have*

(i) *there is a unique $t^- = t^-(u) > t_{\max} > 0$ such that $t^-(u)u \in M_\lambda^-(\Omega)$ and $J_\lambda(t^-(u)u) = \max_{t\geq 0} J_\lambda(tu)$;*

(ii) *$t^-(u)$ is a continuous function for nonzero u;*

(iii) *$M_\lambda^-(\Omega) = \{u \in H(\Omega)\backslash\{0\} \mid \frac{1}{\|u\|}t^-(\frac{u}{\|u\|}) = 1\}$;*

(iv) *if $\int_\Omega f(x)|u|^{q+1}\,\mathrm{d}x > 0$, then there is a unique $0 < t^+ = t^+(u) < t_{\max}$ such that $t^+(u)u \in M_\lambda^+(\Omega)$ and $J_\lambda(t^+(u)u) = \min_{0\leq t\leq t^-} J_\lambda(tu)$.*

Proof (i) Fix $u \in H(\Omega)\backslash\{0\}$. Let

$$s(t) = t^{1-q}\|u\|^2 - t^{p-q}\int_\Omega h(x)|u|^{p+1}\,\mathrm{d}x, \quad t \geq 0.$$

Then we have $s(0) = 0$, $s(t) \to -\infty$ as $t \to \infty$, $s(t)$ is concave and reaches its maximum at $t_{\max}$. Moreover,

$$\begin{aligned} s(t_{\max}) &= t_{\max}^{1-q}\|u\|^2 - t_{\max}^{p-q}\int_\Omega h(x)|u|^{p+1}\,\mathrm{d}x \\ &= \|u\|^{q+1}\left[\left(\frac{1-q}{p-q}\right)^{\frac{1-q}{p-1}} - \left(\frac{1-q}{p-q}\right)^{\frac{p-q}{p-1}}\right]\left(\frac{\|u\|^{p+1}}{\int_\Omega h(x)|u|^{p+1}\,\mathrm{d}x}\right)^{\frac{1-q}{p-1}} \\ &\geq \|u\|^{q+1}\left(\frac{p-1}{p-q}\right)\left(\frac{1-q}{p-q}\right)^{\frac{1-q}{p-1}}\left(\frac{1}{|h|_\infty S^{p+1}}\right)^{\frac{1-q}{p-1}}. \end{aligned} \tag{2.8}$$

Case I. $\int_\Omega f(x)|u|^{q+1}\,\mathrm{d}x \leq 0$.

There is a unique $t^- > t_{\max}$ such that $s(t^-) = \lambda\int_\Omega f(x)|u|^{q+1}\,\mathrm{d}x$ and $s'(t^-) < 0$. Now,

$$\begin{aligned} \langle J_\lambda'(t^-u), t^-u\rangle &= \|t^-u\|^2 - \lambda\int_\Omega f(x)|t^-u|^{q+1}\,\mathrm{d}x - \int_\Omega h(x)|t^-u|^{p+1}\,\mathrm{d}x \\ &= (t^-)^{q+1}\left[s(t^-) - \lambda\int_\Omega f(x)|u|^{q+1}\,\mathrm{d}x\right] \\ &= 0 \end{aligned}$$

and

$$\begin{aligned} \langle \psi_\lambda'(t^-u), t^-u\rangle &= (1-q)\|t^-u\|^2 - (p-q)\int_\Omega h(x)|t^-u|^{p+1}\,\mathrm{d}x \\ &= (t^-)^{2+q}\left[(1-q)(t^-)^{-q}\|u\|^2 - (p-q)(t^-)^{p-q-1}\int_\Omega h(x)|u|^{p+1}\,\mathrm{d}x\right] \\ &= (t^-)^{2+q}s'(t^-) < 0. \end{aligned}$$

Thus, $t^-u \in M_\lambda^-(\Omega)$. In addition,

$$\begin{aligned} \frac{\mathrm{d}J_\lambda(tu)}{\mathrm{d}t} &= t\|u\|^2 - \lambda t^q\int_\Omega f(x)|u|^{q+1}\,\mathrm{d}x - t^p\int_\Omega h(x)|u|^{p+1}\,\mathrm{d}x \\ &= t^{-1}\langle J_\lambda'(tu), tu\rangle = 0 \quad \text{if and only if} \quad t = t^- \end{aligned}$$

and

$$\left.\frac{\mathrm{d}^2 J_\lambda(tu)}{\mathrm{d}t^2}\right|_{t=t^-} = \|u\|^2 - \lambda q\left(t^-\right)^{q-1}\int_\Omega f(x)|u|^{q+1}\,\mathrm{d}x - p\left(t^-\right)^{p-1}\int_\Omega h(x)|u|^{p+1}\,\mathrm{d}x$$
$$= \left(t^-\right)^{-2}\left\langle \psi_\lambda'\left(t^- u\right), t^- u\right\rangle < 0.$$

Hence, $J_\lambda(t^- u) = \max_{t\geq 0} J_\lambda(tu)$.

Case II. $\int_\Omega f(x)|u|^{q+1}\,\mathrm{d}x > 0$.

From (2.8) and

$$\begin{aligned} s(0) = 0 < \lambda \int_\Omega f(x)|u|^{q+1}\,\mathrm{d}x &\leq \lambda |f|_{p^*} S^{q+1}\|u\|^{q+1} \\ &< \|u\|^{q+1}\left(\frac{p-1}{p-q}\right)\left(\frac{1-q}{p-q}\right)^{\frac{1-q}{p-1}}\left(\frac{1}{|h|_\infty S^{p+1}}\right)^{\frac{1-q}{p-1}} \\ &\leq s(t_{\max}) \quad \text{for } \lambda \in (0, \lambda_*), \end{aligned}$$

there exist unique t^+ and t^- such that $0 < t^+ < t_{\max} < t^-$,

$$s\left(t^+\right) = \lambda \int_\Omega f(x)|u|^{q+1}\,\mathrm{d}x = s\left(t^-\right)$$

and

$$s'\left(t^+\right) > 0 > s'\left(t^-\right).$$

Similar to the argument in Case I above, we have $t^+ u \in M_\lambda^+(\Omega)$, $t^- u \in M_\lambda^-(\Omega)$, and

$$J_\lambda\left(t^- u\right) = \max_{t\geq 0} J_\lambda(tu), \qquad J_\lambda\left(t^+ u\right) = \min_{0\leq t\leq t^-} J_\lambda(tu).$$

(ii) By the uniqueness of $t^-(u)$ and the external property of $t^-(u)$, we find that $t^-(u)$ is continuous function of $u \neq 0$.

(iii) For $u \in M_\lambda^-(\Omega)$, let $v = \frac{u}{\|u\|}$. By item (i), there is a unique $t^-(v) > 0$ such that $t^-(v)v \in M_\lambda^-(\Omega)$, that is, $t^-(\frac{u}{\|u\|})\frac{1}{\|u\|}u \in M_\lambda^-(\Omega)$. Since $u \in M_\lambda^-(\Omega)$, we have $t^-(\frac{u}{\|u\|})\frac{1}{\|u\|} = 1$, which implies

$$M_\lambda^-(\Omega) \subset \left\{u \in H(\Omega)\backslash\{0\} \;\middle|\; \frac{1}{\|u\|}t^-\left(\frac{u}{\|u\|}\right) = 1\right\}.$$

Conversely, let $u \in H(\Omega)\backslash\{0\}$ such that $\frac{1}{\|u\|}t^-(\frac{u}{\|u\|}) = 1$. Then $t^-(\frac{u}{\|u\|})\frac{u}{\|u\|} \in M_\lambda^-(\Omega)$. Therefore,

$$M_\lambda^-(\Omega) = \left\{u \in H(\Omega)\backslash\{0\} \;\middle|\; \frac{1}{\|u\|}t^-\left(\frac{u}{\|u\|}\right) = 1\right\}.$$

(iv) By Case II of item (i). □

By $f \in C(\overline{\Omega})$ and changes sign in Ω, we have $\Theta = \{x \in \Omega \mid f(x) > 0\}$ is an open set in R^N. Without loss of generality, we may assume that Θ is a domain in R^N. Consider the following biharmonic equation:

$$\begin{cases} \Delta^2 u = h(x)|u|^{p-1}u, & \text{in } \Theta, \\ u = \Delta u = 0, & \text{on } \partial\Theta. \end{cases} \tag{2.9}$$

Associated with (2.9), we consider the energy functional

$$K(u) = \frac{1}{2}\|u\|^2 - \frac{1}{p+1}\int_\Theta h(x)|u|^{p+1}\,\mathrm{d}x, \quad u \in H(\Theta)$$

and the minimization problem

$$\beta(\Theta) = \inf\bigl\{K(u) \mid u \in N(\Theta)\bigr\},$$

where $N(\Theta) = \{u \in H(\Theta)\backslash\{0\} \mid \langle K'(u), u\rangle = 0\}$. Now we prove that problem (2.9) has a positive solution w_0 such that $K(w_0) = \beta(\Theta) > 0$.

Lemma 2.5 *For any $u \in H(\Theta)\backslash\{0\}$, there exists a unique $t(u) > 0$ such that $t(u)u \in N(\Theta)$. The maximum of $K(tu)$ for $t \geq 0$ is reached at $t = t(u)$, the map*

$$t : H(\Theta)\backslash\{0\} \to (0, +\infty); \qquad u \mapsto t(u)$$

is continuous and the induced continuous map $u \to t(u)u$ defines a homeomorphism of the unit sphere of $H(\Theta)$ with $N(\Theta)$.

Proof For any given $u \in H(\Theta)\backslash\{0\}$, consider the function $g(t) = K(tu)$, $t \geq 0$. Clearly,

$$g'(t) = 0 \quad \Leftrightarrow \quad tu \in N(\Theta) \quad \Leftrightarrow \quad \|u\|^2 = t^{p-1}\int_\Omega h(x)|u|^{p+1}\,\mathrm{d}x. \tag{2.10}$$

It is easy to verify that $g(0) = 0$, $g(t) > 0$ for $t > 0$ small and $g(t) < 0$ for $t > 0$ large. Hence, $\max_{t\geq 0} g(t)$ is reached at a unique $t = t(u)$ such that $g'(t(u)) = 0$ and $t(u)u \in N(\Theta)$. To prove the continuity of $t(u)$, assume that $u_n \to u$ in $H(\Theta)\backslash\{0\}$. It is easy to verify that $\{t(u_n)\}$ is bounded. If a subsequence of $\{t(u_n)\}$ converges to t_0, it follows from (2.10) that $t_0 = t(u)$ and then $t(u_n) \to t(u)$. Finally the continuous map from the unit sphere of $H(\Theta)$ to $N(\Theta)$, $u \to t(u)u$, is inverse to the retraction $u \to \frac{u}{\|u\|}$. □

Define

$$c_* = \inf_{u\in H(\Theta)\backslash\{0\}} \max_{t\geq 0} K(tu), \qquad c = \inf_{\gamma\in\Gamma} \max_{t\in[0,1]} K\bigl(\gamma(t)\bigr),$$

where $\Gamma = \{\gamma \in C([0,1], H(\Theta)) \mid \gamma(0) = 0, K(\gamma(1)) < 0\}$.

Lemma 2.6 *$\beta(\Theta) = c_* = c > 0$ is a critical value of K.*

Proof From Lemma 2.5, we know that $\beta(\Theta) = c_*$. Since $K(tu) < 0$ for $u \in H(\Theta)\backslash\{0\}$ and t large, we obtain $c \leq c_*$. The manifold $N(\Theta)$ separates $H(\Theta)$ into two components. The component containing the origin also contains a small ball around the origin. Moreover,

$K(u) \geq 0$ for all u in this component, because $\langle K'(tu), u\rangle \geq 0$, $\forall t \in [0, t(u)]$. Then each $\gamma \in \Gamma$ has to cross $N(\Theta)$ and $\beta(\Theta) \leq c$. Since the embedding $H(\Theta) \hookrightarrow L^{p+1}(\Theta)$ is compact (see [14]), it is easy to prove that $c > 0$ is a critical value of K and w_0 a positive solution corresponding to c. □

With the help of Lemma 2.6, we have the following result.

Lemma 2.7 (i) *For* $\lambda \in (0, \lambda_*)$, *there exists* $t_\lambda > 0$ *such that*

$$\alpha_\lambda(\Omega) \leq \alpha_\lambda^+(\Omega) < -\frac{1-q}{q+1} t_\lambda^2 \beta_\lambda(\Theta) < 0;$$

(ii) J_λ *is coercive and bounded below on* $M_\lambda(\Omega)$ *for all* $\lambda > 0$.

Proof (i) Let w_0 be a positive solution of problem (2.9) such that $K(w_0) = \beta(\Theta)$. Then

$$\int_\Omega f(x) w_0^{q+1}\,\mathrm{d}x = \int_\Theta f(x) w_0^{q+1}\,\mathrm{d}x > 0.$$

Set $t_\lambda = t^+(w_0)$ as defined by Lemma 2.4(iv). Hence, $t_\lambda w_0 \in M_\lambda^+(\Omega)$ and

$$\begin{aligned} J_\lambda(t_\lambda w_0) &= \frac{1}{2}\|t_\lambda w_0\|^2 - \frac{\lambda}{q+1}\int_\Omega f(x)|t_\lambda w_0|^{q+1}\,\mathrm{d}x - \frac{1}{p+1}\int_\Omega h(x)|t_\lambda w_0|^{p+1}\,\mathrm{d}x \\ &= \left(\frac{1}{2} - \frac{1}{q+1}\right)\|t_\lambda w_0\|^2 + \left(\frac{1}{q+1} - \frac{1}{p+1}\right)\int_\Omega h(x)|t_\lambda w_0|^{p+1}\,\mathrm{d}x \\ &< -\frac{1-q}{q+1} t_\lambda^2 \beta(\Theta) < 0. \end{aligned}$$

This implies

$$\alpha_\lambda(\Omega) \leq \alpha_\lambda^+(\Omega) < -\frac{1-q}{q+1} t_\lambda^2 \beta(\Theta) < 0.$$

(ii) For $u \in M_\lambda(\Omega)$, we have $\|u\|^2 = \lambda\int_\Omega f(x)|u|^{q+1}\,\mathrm{d}x + \int_\Omega h(x)|u|^{p+1}\,\mathrm{d}x$. Then by the Hölder, Sobolev, and Young inequalities,

$$\begin{aligned} J_\lambda(u) &= \frac{p-1}{2(p+1)}\|u\|^2 - \frac{\lambda(p-q)}{(p+1)(q+1)}\int_\Omega f(x)|u|^{q+1}\,\mathrm{d}x \\ &\geq \frac{p-1}{2(p+1)}\|u\|^2 - \frac{\lambda(p-q)}{(p+1)(q+1)}|f|_{p^*} S^{q+1}\|u\|^{q+1} \\ &\geq \frac{p-1}{4(p+1)}\|u\|^2 - \lambda^{\frac{2}{1-q}} C(p,q)\big(|f|_{p^*} S^{q+1}\big)^{\frac{2}{1-q}}, \end{aligned}$$

here $C(p,q) = [\frac{p-q}{(p+1)(q+1)}]^{\frac{2}{1-q}} \cdot [\frac{4(p+1)}{p-1}]^{\frac{1+q}{1-q}}$.

Thus, J_λ is coercive on $M_\lambda(\Omega)$ and

$$J_\lambda(u) \geq -\lambda^{\frac{2}{1-q}} C(p,q)\big(|f|_{p^*} S^{q+1}\big)^{\frac{2}{1-q}}$$

for all $\lambda > 0$. □

Next, we will use the idea of Tarantello [16] to get the following results.

Lemma 2.8 *For $\lambda \in (0,\lambda_*)$ and any given $u \in M_\lambda(\Omega)$, there exist $\epsilon > 0$ and a differentiable functional $\xi : B(0;\epsilon) \subset H(\Omega) \to \mathbb{R}^+$ such that $\xi(0) = 1$, the function $\xi(v)(u+v) \in M_\lambda(\Omega)$ and*

$$\langle \xi'(0), v\rangle = \frac{2\int_\Omega \Delta u \Delta v - \lambda(q+1)\int_\Omega f|u|^{q-1}uv - (p+1)\int_\Omega h|u|^{p-1}uv}{(1-q)\|u\|^2 - (p-q)\int_\Omega h(x)|u|^{p+1}\,\mathrm{d}x} \tag{2.11}$$

for all $v \in H(\Omega)$.

Proof Define $F : \mathbb{R} \times H(\Omega) \to \mathbb{R}$ as follows:

$$F(\xi, w) = \xi^2\|u+w\|^2 - \lambda\xi^{q+1}\int_\Omega f(x)|u+w|^{q+1}\,\mathrm{d}x - \xi^{p+1}\int_\Omega h(x)|u+w|^{p+1}\,\mathrm{d}x.$$

Since $F(1,0) = \langle J'_\lambda(u), u\rangle = 0$ and by Lemma 2.1, we obtain

$$\begin{aligned} F'_\xi(1,0) &= 2\|u\|^2 - \lambda(q+1)\int_\Omega f(x)|u|^{q+1}\,\mathrm{d}x - (p+1)\int_\Omega h(x)|u|^{p+1}\,\mathrm{d}x \\ &= \langle \psi'_\lambda(u), u\rangle \neq 0, \end{aligned}$$

we can get the desired results applying the implicit function theorem at the point $(1,0)$. □

Lemma 2.9 *For $\lambda \in (0,\lambda_*)$ and any given $u \in M^-_\lambda(\Omega)$, there exist $\epsilon > 0$ and a differentiable functional $\xi^- : B(0;\epsilon) \subset H(\Omega) \to \mathbb{R}^+$ such that $\xi^-(0) = 1$, the function $\xi^-(v)(u+v) \in M^-_\lambda(\Omega)$ and*

$$\langle (\xi^-)'(0), v\rangle = \frac{2\int_\Omega \Delta u \Delta v - \lambda(q+1)\int_\Omega f|u|^{q-1}uv - (p+1)\int_\Omega h|u|^{p-1}uv}{(1-q)\|u\|^2 - (p-q)\int_\Omega h(x)|u|^{p+1}\,\mathrm{d}x} \tag{2.12}$$

for all $v \in H(\Omega)$.

Proof In view of Lemma 2.8, there exist $\epsilon > 0$ and a differentiable functional ξ^- such that $\xi^-(0) = 1$, $\xi^-(v)(u+v) \in M_\lambda(\Omega)$ for all $v \in B(0;\epsilon) \subset H(\Omega)$ and we have (2.12). By use of $u \in M^-_\lambda(\Omega)$, we have $\langle \psi'_\lambda(u), u\rangle < 0$. In combination with the continuity of the functions ψ'_λ and ξ^-, we get $\langle \psi'_\lambda(\xi^-(v)(u+v)), \xi^-(v)(u+v)\rangle < 0$ as ϵ sufficiently small, this implies that $\xi^-(v)(u+v) \in M^-_\lambda(\Omega)$. □

3 Proof of Theorem 1.1

Firstly, we provide the existence of minimizing sequences for J_λ on $M_\lambda(\Omega)$ and $M^-_\lambda(\Omega)$ as λ is sufficiently small.

Proposition 3.1 *Let $\lambda \in (0,\lambda_*)$, then*

(i) *there exists a minimizing sequence $\{u_n\} \subset M_\lambda(\Omega)$ such that*

$$J_\lambda(u_n) = \alpha_\lambda(\Omega) + o(1) \quad \textit{and} \quad J'_\lambda(u_n) = o(1) \quad \textit{in } \big[H(\Omega)\big]^*;$$

(ii) *there exists a minimizing sequence* $\{u_n\} \subset M_\lambda^-(\Omega)$ *such that*

$$J_\lambda(u_n) = \alpha_\lambda^-(\Omega) + o(1) \quad \textit{and} \quad J_\lambda'(u_n) = o(1) \quad \textit{in } \big[H(\Omega)\big]^*.$$

Proof (i) By Lemma 2.7(ii) and the Ekeland variational principle [17], there exists a minimizing sequence $\{u_n\} \subset M_\lambda(\Omega)$ such that

$$J_\lambda(u_n) < \alpha_\lambda(\Omega) + \frac{1}{n} \tag{3.1}$$

and

$$J_\lambda(u_n) < J_\lambda(w) + \frac{1}{n}\|w - u_n\| \quad \text{for each } w \in M_\lambda(\Omega). \tag{3.2}$$

Taking n large, from Lemma 2.7(i) and (3.1), we have

$$\begin{aligned} J_\lambda(u_n) &= \left(\frac{1}{2} - \frac{1}{p+1}\right)\|u_n\|^2 - \left(\frac{1}{q+1} - \frac{1}{p+1}\right)\lambda \int_\Omega f(x)|u_n|^{q+1}\,\mathrm{d}x \\ &< \alpha_\lambda(\Omega) + \frac{1}{n} < -\frac{1-q}{q+1}t_\lambda^2\beta(\Theta). \end{aligned} \tag{3.3}$$

This implies

$$|f|_{p^*}S^{q+1}\|u_n\|^{q+1} \geq \int_\Omega f(x)|u_n|^{q+1}\,\mathrm{d}x > \frac{(p+1)(1-q)}{\lambda(p-q)}t_\lambda^2\beta(\Theta) > 0, \tag{3.4}$$

that is,

$$\|u_n\| > \left[\frac{(p+1)(1-q)}{\lambda(p-q)}t_\lambda^2\beta(\Theta)S^{-(q+1)}|f|_{p^*}^{-1}\right]^{\frac{1}{q+1}}. \tag{3.5}$$

Now, we will show that

$$\big\langle J_\lambda'(u_n), \varphi\big\rangle \to 0 \quad \text{as } n \to \infty, \forall \varphi \in H(\Omega).$$

Exactly as in Lemma 2.8 we may apply suitable functionals $\xi_n(v) > 0$ to u_n and obtain

$$\xi_n(v)(u_n + v) \in M_\lambda(\Omega), \quad \forall v \in H(\Omega), \|v\| < \epsilon_n. \tag{3.6}$$

Hence, if $\varphi \in H(\Omega)$ and $s > 0$ small, substituting in (3.6) $v = s\varphi$ and applying (3.2), we have

$$\begin{aligned} &\frac{1}{n}\big[\big|\xi_n(s\varphi) - 1\big| \cdot \|u_n\| + \xi_n(s\varphi)\|s\varphi\|\big] \\ &\quad \geq J_\lambda(u_n) - J_\lambda\big(\xi_n(s\varphi)(u_n + s\varphi)\big) \\ &\quad = \frac{1}{2}\|u_n\|^2 - \frac{\lambda}{q+1}\int_\Omega f(x)|u_n|^{q+1}\,\mathrm{d}x - \frac{1}{p+1}\int_\Omega h(x)|u_n|^{p+1}\,\mathrm{d}x \\ &\qquad - \frac{1}{2}\xi_n^2(s\varphi)\|u_n + s\varphi\|^2 + \frac{\lambda}{q+1}\xi_n^{q+1}(s\varphi)\int_\Omega f(x)|u_n + s\varphi|^{q+1}\,\mathrm{d}x \end{aligned}$$

$$
\begin{aligned}
&+\frac{1}{p+1}\xi_n^{p+1}(s\varphi)\int_\Omega h(x)|u_n+s\varphi|^{p+1}\,\mathrm{d}x\\
&=-\frac{\xi_n^2(s\varphi)-1}{2}\|u_n+s\varphi\|^2-\frac{1}{2}\big(\|u_n+s\varphi\|^2-\|u_n\|^2\big)\\
&\quad+\lambda\frac{\xi_n^{q+1}(s\varphi)-1}{q+1}\int_\Omega f(x)|u_n+s\varphi|^{q+1}\,\mathrm{d}x\\
&\quad+\frac{\lambda}{q+1}\int_\Omega f(x)\big(|u_n+s\varphi|^{q+1}-|u_n|^{q+1}\big)\,\mathrm{d}x\\
&\quad+\frac{\xi_n^{p+1}(s\varphi)-1}{p+1}\int_\Omega h(x)|u_n+s\varphi|^{p+1}\,\mathrm{d}x+\frac{1}{p+1}\int_\Omega h(x)\big(|u_n+s\varphi|^{p+1}-|u_n|^{p+1}\big)\,\mathrm{d}x.
\end{aligned}
$$

Dividing by $s > 0$ and passing to the limit as $s \to 0$ we derive

$$
\begin{aligned}
&\frac{1}{n}\big[\big|\xi_n'(0)\varphi\big|\|u_n\|+\|\varphi\|\big]\\
&\quad\geq-\big[\xi_n'(0)\varphi\big]\bigg[\|u_n\|^2-\lambda\int_\Omega f(x)|u_n|^{q+1}\,\mathrm{d}x-\int_\Omega h(x)|u_n|^{p+1}\,\mathrm{d}x\bigg]\\
&\qquad-\int_\Omega \Delta u_n\Delta\varphi\,\mathrm{d}x+\lambda\int_\Omega f(x)|u_n|^{q-1}u_n\varphi\,\mathrm{d}x+\int_\Omega h(x)|u_n|^{p-1}u_n\varphi\,\mathrm{d}x\\
&\quad=-\int_\Omega \Delta u_n\Delta\varphi\,\mathrm{d}x+\lambda\int_\Omega f(x)|u_n|^{q-1}u_n\varphi\,\mathrm{d}x+\int_\Omega h(x)|u_n|^{p-1}u_n\varphi\,\mathrm{d}x. \qquad (3.7)
\end{aligned}
$$

Since

$$
\xi_n'(0)\varphi=\frac{2\int_\Omega \Delta u_n\Delta\varphi-\lambda(q+1)\int_\Omega f|u_n|^{q-1}u_n\varphi-(p+1)\int_\Omega h|u_n|^{p-1}u_n\varphi}{(1-q)\|u_n\|^2-(p-q)\int_\Omega h(x)|u_n|^{p+1}\,\mathrm{d}x},
$$

by the boundedness of u_n we get

$$
\big\|\xi_n'(0)\big\|\leq\frac{C_1}{|(1-q)\|u_n\|^2-(p-q)\int_\Omega h(x)|u_n|^{p+1}\,\mathrm{d}x|} \qquad (3.8)
$$

for a suitable positive constant C_1.

Next, we show that $|(1-q)\|u_n\|^2-(p-q)\int_\Omega h(x)|u_n|^{p+1}\,\mathrm{d}x|$ is bounded away from zero. Arguing by contradiction, assume that

$$
(1-q)\|u_n\|^2-(p-q)\int_\Omega h(x)|u_n|^{p+1}\,\mathrm{d}x=o(1),\quad n\to\infty. \qquad (3.9)
$$

Since $u_n \in M_\lambda(\Omega)$, we have

$$
\|u_n\|^2=\lambda\int_\Omega f(x)|u_n|^{q+1}\,\mathrm{d}x+\int_\Omega h(x)|u_n|^{p+1}\,\mathrm{d}x,
$$

and consequently by (3.9),

$$
\frac{p-1}{1-q}\int_\Omega h(x)|u_n|^{p+1}\,\mathrm{d}x=\lambda\int_\Omega f(x)|u_n|^{q+1}\,\mathrm{d}x+o(1),\quad n\to\infty. \qquad (3.10)
$$

Then by (3.4), the Hölder inequality, Sobolev inequality and (3.9)-(3.10), we obtain

$$\begin{aligned}
0 &< (\lambda_* - \lambda) \int_\Omega f(x)|u_n|^{q+1}\,\mathrm{d}x \\
&\le \frac{p-1}{1-q} \int_\Omega h(x)|u_n|^{p+1}\,\mathrm{d}x \left[\frac{(p-q)\int_\Omega h(x)|u_n|^{p+1}\,\mathrm{d}x}{(1-q)\|u_n\|^2}\right]^{\frac{q-p}{p-1}} - \lambda \int_\Omega f(x)|u_n|^{q+1}\,\mathrm{d}x \\
&= o(1),
\end{aligned}$$

moreover, $\|u_n\| = o(1)$, which contradicts (3.5).

Thus, we get from (3.8) that

$$\left\|\xi_n'(0)\right\| \le C_2, \quad \text{independent of } n.$$

Hence, by (3.7) it follows that

$$\int_\Omega \Delta u_n \Delta\varphi\,\mathrm{d}x - \lambda \int_\Omega f(x)|u_n|^{q-1}u_n\varphi\,\mathrm{d}x - \int_\Omega h(x)|u_n|^{p-1}u_n\varphi\,\mathrm{d}x \ge -\frac{C_3}{n},$$

which implies that $\langle J_\lambda'(u_n), \varphi\rangle \to 0$, as $n \to \infty$.

(ii) Similar to the arguments in (i), by Lemma 2.9 and Lemma 2.2, we can prove (ii). □

Now, we establish the existence of a local minimum for J_λ on $M_\lambda^+(\Omega)$.

Theorem 3.1 *Let $\lambda \in (0, \lambda_*)$, then the functional J_λ has a minimizer u_0^+ in $M_\lambda^+(\Omega)$ and it satisfies*

(i) $J_\lambda(u_0^+) = \alpha_\lambda(\Omega) = \alpha_\lambda^+(\Omega)$;

(ii) u_0^+ *is a positive solution of problem* (1.3);

(iii) $J_\lambda(u_0^+) \to 0$ *as* $\lambda \to 0$.

Proof By Proposition 3.1(i), there is a minimizing sequence $\{u_n\}$ for J_λ on $M_\lambda(\Omega)$ such that

$$J_\lambda(u_n) = \alpha_\lambda(\Omega) + o(1) \quad \text{and} \quad J_\lambda'(u_n) = o(1) \quad \text{in } \left[H(\Omega)\right]^*. \tag{3.11}$$

Then by Lemma 2.7 and the compact imbedding theorem, there exist a subsequence $\{u_n\}$ and $u_0^+ \in H(\Omega)$ such that

$$u_n \rightharpoonup u_0^+ \quad \text{weakly in } H(\Omega), \tag{3.12}$$

$$u_n \to u_0^+ \quad \text{strongly in } L^{p+1}(\Omega) \tag{3.13}$$

and

$$u_n \to u_0^+ \quad \text{strongly in } L^{q+1}(\Omega). \tag{3.14}$$

First, we claim that

$$\int_\Omega f(x)\left|u_0^+\right|^{q+1}\,\mathrm{d}x > 0.$$

If not, by (3.14) we conclude that

$$\int_\Omega f(x)|u_n|^{q+1}\,\mathrm{d}x \to \int_\Omega f(x)\left|u_0^+\right|^{q+1}\,\mathrm{d}x \le 0 \quad \text{as } n \to \infty.$$

Therefore, as $n \to \infty$,

$$\begin{aligned} J_\lambda(u_n) &= \frac{1}{2}\|u_n\|^2 - \frac{\lambda}{q+1}\int_\Omega f(x)|u_n|^{q+1}\,\mathrm{d}x - \frac{1}{p+1}\int_\Omega h(x)|u_n|^{p+1}\,\mathrm{d}x \\ &= \left(\frac{1}{2} - \frac{1}{q+1}\right)\lambda\int_\Omega f(x)|u_n|^{q+1}\,\mathrm{d}x + \left(\frac{1}{2} - \frac{1}{p+1}\right)\int_\Omega h(x)|u_n|^{p+1}\,\mathrm{d}x \\ &= \left(\frac{1}{2} - \frac{1}{q+1}\right)\lambda\int_\Omega f(x)\left|u_0^+\right|^{q+1}\,\mathrm{d}x + \left(\frac{1}{2} - \frac{1}{p+1}\right)\int_\Omega h(x)\left|u_0^+\right|^{p+1}\,\mathrm{d}x + o(1), \end{aligned}$$

this contradicts $J_\lambda(u_n) \to \alpha_\lambda(\Omega) < 0$ as $n \to \infty$.

In combination with (3.11)-(3.14), it is easy to verify that $u_0^+ \in M_\lambda(\Omega)$ is a nontrivial weak solution of problem (1.3).

Now we prove that $u_n \to u_0^+$ strongly in $H(\Omega)$. Supposing the contrary, then $\|u_0^+\| < \liminf_{n\to\infty}\|u_n\|$ and so

$$\begin{aligned} &\left\|u_0^+\right\|^2 - \lambda\int_\Omega f(x)\left|u_0^+\right|^{q+1}\,\mathrm{d}x - \int_\Omega h(x)\left|u_0^+\right|^{p+1}\,\mathrm{d}x \\ &\quad < \liminf_{n\to\infty}\left(\|u_n\|^2 - \lambda\int_\Omega f(x)|u_n|^{q+1}\,\mathrm{d}x - \int_\Omega h(x)|u_n|^{p+1}\,\mathrm{d}x\right) = 0, \end{aligned}$$

this contradicts $u_0^+ \in M_\lambda(\Omega)$. Hence, $u_n \to u_0^+$ strongly in $H(\Omega)$. This implies

$$J_\lambda(u_n) \to J_\lambda\left(u_0^+\right) = \alpha_\lambda(\Omega) \quad \text{as } n \to \infty.$$

Moreover, we have $u_0^+ \in M_\lambda^+(\Omega)$. In fact, if $u_0^+ \in M_\lambda^-(\Omega)$, by Lemma 2.4, there exist unique t_0^+ and t_0^- such that $t_0^+u_0^+ \in M_\lambda^+(\Omega)$ and $t_0^-u_0^+ \in M_\lambda^-(\Omega)$, we get $t_0^+ < t_0^- = 1$. Since

$$\frac{\mathrm{d}J_\lambda(tu)}{\mathrm{d}t} = 0 \quad \text{if and only if} \quad t = t_0^+ \text{ and } t_0^-$$

and

$$\left.\frac{\mathrm{d}^2J_\lambda(tu)}{\mathrm{d}t^2}\right|_{t=t_0^+} > 0, \qquad \left.\frac{\mathrm{d}^2J_\lambda(tu)}{\mathrm{d}t^2}\right|_{t=t_0^-} < 0,$$

there exists $\tilde{t} \in (t_0^+, t_0^-]$ such that $J_\lambda(t_0^+u_0^+) < J_\lambda(\tilde{t}u_0^+)$. By Lemma 2.4,

$$J_\lambda\left(t_0^+u_0^+\right) < J_\lambda\left(\tilde{t}u_0^+\right) \le J_\lambda\left(t_0^-u_0^+\right) = J_\lambda\left(u_0^+\right),$$

which is a contradiction. Since $J_\lambda(u_0^+) = J_\lambda(|u_0^+|)$ and $|u_0^+| \in M_\lambda^+(\Omega)$, by Lemma 2.3 we may assume that u_0^+ is a nonnegative weak solution to problem (1.3). Applying the regularity theory and strong maximum principle of elliptic equations, we find that u_0^+ is one positive solution of problem (1.3). In addition, by Lemma 2.7,

$$0 > J_\lambda\left(u_0^+\right) \ge -\lambda^{\frac{2}{1-q}}C(p,q)\left(|f|_{p^*}S^{q+1}\right)^{\frac{2}{1-q}},$$

which implies that $J_\lambda(u_0^+) \to 0$ as $\lambda \to 0$. □

Next, we establish the existence of a local minimum for J_λ on $M_\lambda^-(\Omega)$.

Theorem 3.2 *Let* $\lambda \in (0, \lambda_*)$, *then the functional* J_λ *has a minimizer* u_0^- *in* $M_\lambda^-(\Omega)$ *and it satisfies*

(i) $J_\lambda(u_0^-) = \alpha_\lambda^-(\Omega)$;

(ii) u_0^- *is a positive solution of problem* (1.3).

Proof By Proposition 3.1(ii), there is a minimizing sequence $\{u_n\}$ for J_λ on $M_\lambda^-(\Omega)$ such that

$$J_\lambda(u_n) = \alpha_\lambda^-(\Omega) + o(1) \quad \text{and} \quad J_\lambda'(u_n) = o(1) \quad \text{in } \big[H(\Omega)\big]^*.$$

Then by Lemma 2.7 and the compact imbedding theorem, there exist a subsequence $\{u_n\}$ and $u_0^- \in H(\Omega)$ such that

$$u_n \rightharpoonup u_0^- \quad \text{weakly in } H(\Omega),$$
$$u_n \to u_0^- \quad \text{strongly in } L^{p+1}(\Omega)$$

and

$$u_n \to u_0^- \quad \text{strongly in } L^{q+1}(\Omega).$$

Connecting with Lemma 2.2, it is easy to see that $u_0^- \in M_\lambda(\Omega)$ is a nontrivial weak solution of problem (1.3).

Next we prove that $u_n \to u_0^-$ strongly in $H(\Omega)$. Supposing the contrary, then $\|u_0^-\| < \liminf_{n\to\infty} \|u_n\|$ and so

$$\begin{aligned}
&\|u_0^-\|^2 - \lambda \int_\Omega f(x)|u_0^-|^{q+1}\,dx - \int_\Omega h(x)|u_0^-|^{p+1}\,dx \\
&\quad < \liminf_{n\to\infty}\left(\|u_n\|^2 - \lambda \int_\Omega f(x)|u_n|^{q+1}\,dx - \int_\Omega h(x)|u_n|^{p+1}\,dx\right) = 0,
\end{aligned}$$

this contradicts $u_0^- \in M_\lambda(\Omega)$. Hence, $u_n \to u_0^-$ strongly in $H(\Omega)$. This implies

$$J_\lambda(u_n) \to J_\lambda\big(u_0^-\big) = \alpha_\lambda^-(\Omega) \quad \text{as } n \to \infty.$$

In addition, from Lemma 2.4(ii)-(iii), we have $u_0^- \in M_\lambda^-(\Omega)$. Since $J_\lambda(u_0^-) = J_\lambda(|u_0^-|)$ and $|u_0^-| \in M_\lambda^-(\Omega)$, by Lemma 2.3 we may assume that u_0^- is a nonnegative weak solution to problem (1.3). Applying the regularity theory and strong maximum principle of elliptic equations, we see that u_0^- is one positive solution of problem (1.3). □

Proof of Theorem 1.1 It is an immediate consequence of Theorems 3.1 and 3.2. □

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
The authors read and approved the final manuscript.

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References
1. Alama, S, Tarantello, G: On semilinear elliptic equations with indefinite nonlinearities. Calc. Var. Partial Differ. Equ. **1**, 439-475 (1993)
2. Ambrosetti, A, Brezis, H, Cerami, G: Combined effects of concave and convex nonlinearities in some elliptic problems. J. Funct. Anal. **122**, 519-543 (1994)
3. Bartsch, T, Willem, M: On an elliptic equation with concave and convex nonlinearities. Proc. Am. Math. Soc. **123**, 3555-3561 (1995)
4. Brown, KJ, Zhang, Y: The Nehari manifold for a semilinear elliptic equation with a sign-changing weight function. J. Differ. Equ. **193**, 481-499 (2003)
5. Li, S, Wu, S, Zhou, HS: Solutions to semilinear elliptic problems with combined nonlinearities. J. Differ. Equ. **185**, 200-224 (2001)
6. Sun, Y, Li, S: A nonlinear elliptic equation with critical exponent: estimates for extremal values. Nonlinear Anal. **69**, 1856-1869 (2008)
7. Tang, M: Exact multiplicity for semilinear elliptic Dirichlet problems involving concave and convex nonlinearities. Proc. R. Soc. Edinb., Sect. A **133**, 705-717 (2003)
8. Wu, TF: On semilinear elliptic equations involving concave-convex nonlinearities and sign-changing weight function. J. Math. Anal. Appl. **318**, 253-270 (2006)
9. Guo, D, Lakshmikantham, V: Nonlinear Problems in Abstract Cones. Academic Press, New York (1988)
10. Willem, M: Minimax Theorems. Birkhäuser, Basel (1996)
11. Ebobisse, F, Ahmedou, MO: On a nonlinear fourth-order elliptic equation involving the critical Sobolev exponent. Nonlinear Anal. **52**, 1535-1552 (2003)
12. Gazzola, F, Grunau, H-C, Squassina, M: Existence and nonexistence results for critical growth biharmonic elliptic equations. Calc. Var. Partial Differ. Equ. **18**, 117-143 (2003)
13. Zhang, Y: Positive solutions of semilinear biharmonic equations with critical Sobolev exponents. Nonlinear Anal. **75**, 55-67 (2012)
14. Adams, RA: Sobolev Spaces. Academic Press, New York (1975)
15. Rabinowitz, PH: Minimax Methods in Critical Point Theory with Applications to Differential Equations. Reg. Conf. Ser. Math. Am. Math. Soc., Providence (1986)
16. Tarantello, G: On nonhomogeneous elliptic involving critical Sobolev exponent. Ann. Inst. Henri Poincaré, Anal. Non Linéaire **9**, 281-304 (1992)
17. Ekeland, I: On the variational principle. J. Math. Anal. Appl. **17**, 324-353 (1974)

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