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Positive solutions for classes of multi-parameter fourth-order impulsive differential equations with one-dimensional singular p-Laplacian

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Abstract

The authors consider the following impulsive differential equations involving the one-dimensional singular p-Laplacian: $(\phi_p(y''(t)))'' = \lambda\omega(t)f(t,y(t))$, $t \in J$, $t \neq t_k$, $k = 1,2,\ldots,m$, $\Delta y'|_{t=t_k} = -\mu I_k(t_k,y(t_k))$, $k = 1,2,\ldots,m$, $ay(0) - by'(0) = \int_0^1 h(s)y(s)\,ds$, $ay(1) + by'(1) = \int_0^1 h(s)y(s)\,ds$, $\phi_p(y''(0)) = \phi_p(y''(1)) = \int_0^1 h(t)\phi_p(y''(t))\,dt$, where $\lambda > 0$ and $\mu > 0$ are two parameters. Several new and more general existence and multiplicity results are derived in terms of different values of $\lambda > 0$ and $\mu > 0$. In this case, our results cover equations without impulsive effects and are compared with some recent results.

Keywords: multi-parameter; impulsive differential equations; one-dimensional singular p-Laplacian; positive solution; cone and partial ordering

1 Introduction

The theory and applications of the fourth-order ordinary differential equation are emerging as an important area of investigation; it is often referred to as the beam equation. In [1], Sun and Wang pointed out that it is necessary and important to consider various fourth-order boundary value problems (BVPs for short) according to different forms of supporting. Owing to its importance in engineering, physics, and material mechanics, fourth-order BVPs have attracted much attention from many authors; see, for example [2–29] and the references therein.

Very recently, Zhang and Liu [30] studied the following fourth-order four-point boundary value problem without impulsive effect:

$$\begin{cases} (\phi_p(x''(t)))'' = w(t)f(t,x(t)), & t \in [0,1], \\ x(0) = 0, & x(1) = ax(\xi), \\ x''(0) = 0, & x''(1) = bx''(\eta), \end{cases}$$

where $0 < \xi, \eta < 1$, $0 \le a < b < 1$. By using the upper and lower solution method, fixed point theorems, and the properties of the Green's function $G(t,s)$ and $H(t,s)$, the authors give sufficient conditions for the existence of one positive solution.

In this paper, we investigate the existence of positive solutions of fourth-order impulsive differential equations with two parameters

$$\begin{cases} (\phi_p(y''(t)))'' = \lambda\omega(t)f(t,y(t)), \quad t \in J, t \neq t_k, k = 1,2,\ldots,m, \\ \Delta y'|_{t=t_k} = -\mu I_k(t_k, y(t_k)), \quad k = 1,2,\ldots,m, \\ ay(0) - by'(0) = \int_0^1 g(s)y(s)\,ds, \\ ay(1) + by'(1) = \int_0^1 g(s)y(s)\,ds, \\ \phi_p(y''(0)) = \phi_p(y''(1)) = \int_0^1 h(s)\phi_p(y''(s))\,ds, \end{cases} \tag{1.1}$$

where $\lambda > 0$ and $\mu > 0$ are two parameters, $a, b > 0$, $J = [0,1]$, $\phi_p(s)$ is a p-Laplace operator, *i.e.*, $\phi_p(s) = |s|^{p-2}s$, $p > 1$, $(\phi_p)^{-1} = \phi_q$, $\frac{1}{p} + \frac{1}{q} = 1$, ω is a nonnegative measurable function on $(0,1)$, $\omega \not\equiv 0$ on any open subinterval in $(0,1)$ which may be singular at $t = 0$ and/or $t = 1$, t_k $(k = 1,2,\ldots,m)$ (where m is fixed positive integer) are fixed points with $0 = t_0 < t_1 < t_2 < \cdots < t_k < \cdots < t_m < t_{m+1} = 1$, $\Delta y'|_{t=t_k} = y'(t_k^+) - x'(t_k^-)$, where $y'(t_k^+)$ and $y'(t_k^-)$ represent the right-hand limit and left-hand limit of $y'(t)$ at $t = t_k$, respectively. In addition, ω, f, I_k, g, and h satisfy

(H_1) $\omega \in L^1_{\mathrm{loc}}(0,1)$;
(H_2) $f \in C([0,1] \times [0,+\infty), [0,+\infty))$ with $f(t,y) > 0$ for all t and $y > 0$;
(H_3) $I_k \in C([0,1] \times [0,+\infty), [0,+\infty))$ with $I_k(t,y) > 0$ $(k = 1,2,\ldots,n)$ for all t and $y > 0$;
(H_4) $g, h \in L^1[0,1]$ are nonnegative and $\xi \in [0,a)$, $\nu \in [0,1)$, where

$$\xi = \int_0^1 g(t)\,dt, \qquad \nu = \int_0^1 h(t)\,dt. \tag{1.2}$$

Some special cases of (1.1) have been investigated. For example, Bai and Wang [14] studied the existence of multiple solutions of problem (1.1) with $p = 2$, $I_k = 0$, $k = 1,2,\ldots,m$ and $\omega \equiv 1$ for $t \in J$. By using a fixed point theorem and degree theory, the authors proved the existence of one or two positive solutions of problem (1.1).

Feng [31] considered problem (1.1) with $\lambda = 1$, $I_k(t_k, y(t_k)) = I_k(y(t_k))$, $\omega \equiv 1$ for $t \in J$ and $\mu = 1$. By using a suitably constructed cone and fixed point theory for cones, the author proved the existence results of multiple positive solutions of problem (1.1).

Motivated by the papers mentioned above, we will extend the results of [14, 30, 31] to problem (1.1). We remark that on impulsive differential equations with a parameter only a few results have been obtained, not to mention impulsive differential equations with two parameters; see, for instance, [32–34]. However, these results only dealt with the case that $p = 2$ and $\mu = 1$.

The rest of the paper is organized as follows: in Section 2, we state the main results of problem (1.1). In Section 3, we provide some preliminary results, and the proofs of the main results together with several technical lemmas are given in Section 4.

2 Main results

In this section, we state the main results, including existence and multiplicity of positive solutions for problem (1.1).

We begin by introducing the notation

$$f^0 = \limsup_{y\to 0^+} \max_{t\in J} \frac{f(t,y)}{\phi_p(y)}, \qquad f^\infty = \limsup_{y\to \infty} \max_{t\in J} \frac{f(t,y)}{\phi_p(y)},$$

$$f_0 = \liminf_{y\to 0^+} \min_{t\in J} \frac{f(t,y)}{\phi_p(y)}, \qquad f_\infty = \liminf_{y\to \infty} \min_{t\in J} \frac{f(t,y)}{\phi_p(y)},$$
$$I^0(k) = \limsup_{y\to 0^+} \max_{t\in J} \frac{I_k(t,y)}{y}, \qquad I^\infty(k) = \limsup_{y\to \infty} \max_{t\in J} \frac{I_k(t,y)}{y},$$
$$I_0(k) = \liminf_{y\to 0^+} \min_{t\in J} \frac{I_k(t,y)}{y}, \qquad I_\infty(k) = \liminf_{y\to \infty} \min_{t\in J} \frac{I_k(t,y)}{y}, \quad k = 1,2,\ldots,m.$$

We also choose four numbers r, r_1, r_2, and R satisfying

$$0 < r < r_1 < \delta r_2 < r_2 < R < +\infty, \tag{2.1}$$

where δ is defined in (3.20).

Theorem 2.1 *Assume that* (H_1)-(H_4) *hold.*

(i) *If* $f^\infty = 0$ *and* $I^\infty = 0$, *then there exist* $\lambda_0 > 0$ *and* $\mu_0 > 0$ *such that, for any* $\lambda > \lambda_0$ *and* $\mu > \mu_0$, *problem* (1.1) *has a positive solution* $u(t)$, $t \in J$ *with*

$$\delta r \le u(t) \le \frac{1}{\delta} R, \quad t \in J. \tag{2.2}$$

(ii) *If* $f^0 = 0$ *and* $I^0 = 0$, *then there exist* $\lambda_0 > 0$ *and* $\mu_0 > 0$ *such that, for any* $\lambda > \lambda_0$ *and* $\mu > \mu_0$, *problem* (1.1) *has a positive solution* $u(t)$ *with*

$$\delta r \le u(t) \le R, \quad t \in J. \tag{2.3}$$

(iii) *If* $f^0 = f^\infty = I^\infty = I^0 = 0$, *then there exist* $\lambda_0 > 0$ *and* $\mu_0 > 0$ *such that, for any* $\lambda > \lambda_0$ *and* $\mu > \mu_0$, *problem* (1.1) *has at least two positive solutions* $u_1(t)$ *and* $u_2(t)$ *with*

$$\delta r \le u(t) \le r_1 < \delta r_2 \le u_2(t) \le R, \quad t \in J. \tag{2.4}$$

Theorem 2.2 *Assume that* (H_1)-(H_4) *hold.*

(i) *If* $f_\infty = +\infty$ *and* $I_\infty = +\infty$, *then there exist* $\bar\lambda_0 > 0$ *and* $\bar\mu_0 > 0$ *such that, for any* $0 < \lambda < \bar\lambda_0$ *and* $0 < \mu < \bar\mu_0$, *problem* (1.1) *has a positive solution* $u(t)$, $t \in J$ *with property* (2.2).

(ii) *If* $f_0 = +\infty$ *and* $I_0 = +\infty$, *then there exist* $\bar\lambda_0 > 0$ *and* $\bar\mu_0 > 0$ *such that, for any* $0 < \lambda < \bar\lambda_0$ *and* $0 < \mu < \bar\mu_0$, *problem* (1.1) *has a positive solution* $u(t)$, $t \in J$ *with property* (2.3).

(iii) *If* $f_0 = f_\infty = I_\infty = I_0 = +\infty$, *then there exist* $\bar\lambda_0 > 0$ *and* $\bar\mu_0 > 0$ *such that, for any* $0 < \lambda < \bar\lambda_0$ *and* $0 < \mu < \bar\mu_0$, *problem* (1.1) *has at least two positive solutions* $u_1(t)$ *and* $u_2(t)$ *with*

$$\delta r \le u(t) \le r_1 < \delta r_2 \le u_2(t) \le \frac{1}{\delta} R, \quad t \in J. \tag{2.5}$$

3 Preliminaries

Let $J' = J \setminus \{t_1, t_2, \ldots, t_m\}$, and

$$PC^1[0,1] = \{ y \in C[0,1] : y'|_{(t_k,t_{k+1})} \in C(t_k,t_{k+1}), y'(t_k^-), y'(t_k^+) \text{ exists}, k = 1,2,\ldots,m \}.$$

Then $PC^1[0,1]$ is a real Banach space with norm

$$\|y\|_{PC^1} = \max\{\|y\|_\infty, \|y'\|_\infty\}, \tag{3.1}$$

where $\|y\|_\infty = \sup_{t\in J}|y(t)|$, $\|y'\|_\infty = \sup_{t\in J}|y'(t)|$.

A function $y \in PC^1[0,1] \cap C^4(J')$ with $\varphi_p(y'') \in C^2(0,1)$ is called a solution of problem (1.1) if it satisfies (1.1).

We shall reduce problem (1.1) to an integral equation. To this goal, firstly by means of the transformation

$$\phi_p\big(y''(t)\big) = -x(t), \tag{3.2}$$

we convert problem (1.1) into

$$\begin{cases} x''(t) + \lambda\omega(t)f(t,y(t)) = 0, & t \in J, \\ x(0) = x(1) = \int_0^1 h(t)x(t)\,dt \end{cases} \tag{3.3}$$

and

$$\begin{cases} y''(t) = -\phi_q(x(t)), & t \in J, t \neq t_k, \\ \Delta y'|_{t=t_k} = -\mu I_k(t_k, y(t_k)), & k = 1,2,\ldots,m, \\ ay(0) - by'(0) = \int_0^1 g(s)y(s)\,ds, \\ ay(1) + by'(1) = \int_0^1 g(s)y(s)\,ds. \end{cases} \tag{3.4}$$

Lemma 3.1 *If* (H_1), (H_2), *and* (H_4) *hold, then problem* (3.3) *has a unique solution x given by*

$$x(t) = \lambda\int_0^1 H(t,s)\omega(s)f\big(s,y(s)\big)\,ds, \tag{3.5}$$

where

$$H(t,s) = G(t,s) + \frac{1}{1-\nu}\int_0^1 G(s,\tau)h(\tau)\,d\tau, \tag{3.6}$$

$$G(t,s) = \begin{cases} t(1-s), & 0 \le t \le s \le 1, \\ s(1-t), & 0 \le s \le t \le 1. \end{cases} \tag{3.7}$$

Proof The proof of Lemma 3.1 is similar to that of Lemma 2.1 in [31]. □

Write $e(t) = t(1-t)$. Then from (3.6) and (3.7), we can prove that $H(t,s)$ and $G(t,s)$ have the following properties.

Proposition 3.1 *If* (H_4) *holds, then we have*

$$H(t,s) > 0, \qquad G(t,s) > 0, \quad \forall t,s \in (0,1), \tag{3.8}$$

$$H(t,s) \ge 0, \qquad G(t,s) \ge 0, \quad \forall t,s \in J, \tag{3.9}$$

$$e(t)e(s) \le G(t,s) \le G(t,t) = t(1-t) = e(t) \le \bar{e} = \max_{t\in[0,1]} e(t) = \frac{1}{4}, \quad \forall t,s \in J, \tag{3.10}$$

$$\rho e(s) \le H(t,s) \le \gamma s(1-s) = \gamma e(s) \le \frac{1}{4}\gamma, \quad \forall t,s \in J, \tag{3.11}$$

where

$$\gamma = \frac{1}{1-\nu}, \qquad \rho = \frac{\int_0^1 e(\tau)h(\tau)\,d\tau}{1-\nu}. \tag{3.12}$$

Remark 3.1 From (3.6) and (3.11), we obtain

$$\rho e(s) \le H(s,s) \le \gamma s(1-s) = \gamma e(s) \le \frac{1}{4}\gamma, \quad \forall s \in J.$$

Lemma 3.2 *If* (H_1), (H_3), *and* (H_4) *hold, then problem* (3.4) *has a unique solution* y *and* y *can be expressed in the form*

$$y(t) = \int_0^1 H_1(t,s)\phi_q\big(x(s)\big)\,ds + \mu \sum_{k=1}^{m} H_1(t,t_k)I_k\big(t_k,y(t_k)\big), \tag{3.13}$$

where

$$H_1(t,s) = G_1(t,s) + \frac{1}{a-\xi}\int_0^1 G_1(s,\tau)g(\tau)\,d\tau, \tag{3.14}$$

$$G_1(t,s) = \frac{1}{d}\begin{cases} (b+as)(b+a(1-t)), & \text{if } 0 \le s \le t \le 1, \\ (b+at)(b+a(1-s)), & \text{if } 0 \le t \le s \le 1, \end{cases} \tag{3.15}$$

$d = a(2b+a)$.

Proof The proof of Lemma 3.2 is similar to that of Lemma 2.2 in [31]. □

From (3.14) and (3.15), we can prove that $H_1(t,s)$ and $G_1(t,s)$ have the following properties.

Proposition 3.2 *If* (H_4) *holds, then we have*

$$H_1(t,s) > 0, \qquad G_1(t,s) > 0, \quad \forall t,s \in J; \tag{3.16}$$

$$\frac{1}{d}b^2 \le G_1(t,s) \le G_1(s,s) \le \frac{1}{d}(b+a)^2, \quad \forall t,s \in J, \tag{3.17}$$

$$\rho_1 \le H_1(t,s) \le H_1(s,s) \le \rho_2, \quad \forall t,s \in J, \tag{3.18}$$

where

$$\rho_1 = \frac{b^2\gamma_1}{a+2b}, \qquad \rho_2 = \frac{\gamma_1(b+a)^2}{a+2b}, \qquad \gamma_1 = \frac{1}{a-\xi}.$$

Suppose that y is a solution of problem (1.1). Then from Lemma 3.1 and Lemma 3.2, we have

$$y(t) = \int_0^1 H_1(t,s)\phi_q\left(\lambda \int_0^1 H(s,\tau)\omega(\tau)f\big(\tau,y(\tau)\big)\,d\tau\right)ds + \mu \sum_{k=1}^{m} H_1(t,t_k)I_k\big(t_k,y(t_k)\big).$$

Define a cone in $PC^1[0,1]$ by

$$K = \{y \in PC^1[0,1] : y \geq 0, y(t) \geq \delta \|y\|_{PC^1}, t \in J\}, \tag{3.19}$$

where

$$\delta = \frac{\rho_1 \rho^{q-1}}{\rho_2 \gamma^{q-1}}. \tag{3.20}$$

It is easy to see K is a closed convex cone of $PC^1[0,1]$.

Define an operator $T_\lambda^\mu : K \to PC^1[0,1]$ by

$$\begin{aligned}(T_\lambda^\mu y)(t) = {} & \int_0^1 H_1(t,s)\phi_q\left(\lambda \int_0^1 H(s,\tau)\omega(\tau)f(\tau, y(\tau))\, d\tau\right) ds \\ & + \mu \sum_{k=1}^m H_1(t,t_k) I_k(t_k, y(t_k)).\end{aligned} \tag{3.21}$$

From (3.21), we know that $y \in PC^1[0,1]$ is a solution of problem (1.1) if and only if y is a fixed point of operator T_λ^μ.

Lemma 3.3 *Suppose that* (H_1)-(H_4) *hold. Then* $T_\lambda^\mu(K) \subset K$ *and* $T_\lambda^\mu : K \to K$ *is completely continuous.*

Proof The proof of Lemma 3.3 is similar to that of Lemma 2.4 in [31]. □

To obtain positive solutions of problem (1.1), the following fixed point theorem in cones is fundamental, which can be found in [35, p.94].

Lemma 3.4 *Let P be a cone in a real Banach space E. Assume Ω_1, Ω_2 are bounded open sets in E with* $0 \in \Omega_1$, $\bar{\Omega}_1 \subset \Omega_2$. *If*

$$A : P \cap (\bar{\Omega}_2 \setminus \Omega_1) \to P$$

is completely continuous such that either

(a) $\|Ax\| \leq \|x\|, \forall x \in P \cap \partial\Omega_1$ *and* $\|Ax\| \geq \|x\|, \forall x \in P \cap \partial\Omega_2$, *or*

(b) $\|Ax\| \geq \|x\|, \forall x \in P \cap \partial\Omega_1$ *and* $\|Ax\| \leq \|x\|, \forall x \in P \cap \partial\Omega_2$,

then A has at least one fixed point in $P \cap (\bar{\Omega}_2 \setminus \Omega_1)$.

Remark 3.2 To make the reader clear what $\bar{\Omega}_2$, $\partial\Omega_2$, $\partial\Omega_1$, and $\Omega_2 \setminus \bar{\Omega}_1$ mean, we give typical examples of Ω_1 and Ω_2, *e.g.*,

$$\Omega_1 = \{x \in C[a,b] : \|x\|_\infty < r\}, \qquad \Omega_2 = \{x \in C[a,b] : \|x\|_\infty < R\}$$

with $0 < r < R$, where $\|x\|_\infty = \sup_{t\in[a,b]} |x(t)|$.

4 Proofs of the main results

For convenience we introduce the following notation:

$$\eta = \varphi_q\left(\int_0^1 \omega(s)\, ds\right), \qquad \eta^* = \varphi_q\left(\int_{t_1}^{t_m} \omega(s)\, ds\right)$$

and

$$\Omega_r = \{y \in K : \|y\|_{PC^1} < r\}, \qquad \partial\Omega_r = \{y \in K : \|y\|_{PC^1} = r\},$$

where $r > 0$ is a constant.

Proof of Theorem 2.1 Part (i). Noticing that $f(t,y) > 0$, $I_k(t,y) > 0$ $(k = 1,2,\ldots,m)$ for all t and $y > 0$, we can define

$$m_r = \min_{t\in J,\delta r\leq y\leq r}\{f(t,y)\} > 0, \qquad m^* = \min\{m_k, k = 1,2,\ldots,m\} > 0,$$

where $r > 0$, and

$$m_k = \min_{t\in J,\delta r\leq y\leq r}\{I_k(t,y)\}, \quad k = 1,2,\ldots,m.$$

Let

$$\lambda_0 \geq \left(\frac{1}{2\rho_1\eta^*}r\right)^{p-1}\left[\rho m_r t_1(1-t_m)\right]^{-1}, \qquad \mu_0 \geq \frac{1}{2m\rho_1 m^*}r.$$

Then, for $u \in K \cap \partial\Omega_r$ and $\lambda > \lambda_0$, $\mu > \mu_0$, we have

$$\begin{aligned}
\left(T_\lambda^\mu y\right)(t) &= \int_0^1 H_1(t,s)\phi_q\left(\lambda\int_0^1 H(s,\tau)\omega(\tau)f\left(\tau,y(\tau)\right)d\tau\right)ds \\
&\quad + \mu\sum_{k=1}^m H_1(t,t_k)I_k\left(t_k,y(t_k)\right) \\
&\geq \rho_1\rho^{q-1}\varphi_q\left(\lambda\int_0^1 e(\tau)\omega(\tau)f\left(\tau,y(\tau)\right)d\tau\right) + \mu\rho_1\sum_{k=1}^m I_k\left(t_k,y(t_k)\right) \\
&\geq \rho_1\rho^{q-1}\varphi_q\left(\lambda\int_0^1 e(\tau)\omega(\tau)m_r\,d\tau\right) + \mu\rho_1\sum_{k=1}^m m^* \\
&= \rho_1\rho^{q-1}m_r^{q-1}\lambda^{q-1}\varphi_q\left(\int_0^1 e(\tau)\omega(\tau)\,d\tau\right) + \mu m\rho_1 m^* \\
&\geq \rho_1\rho^{q-1}m_r^{q-1}\lambda^{q-1}\varphi_q\left(\int_{t_1}^{t_m} e(\tau)\omega(\tau)\,d\tau\right) + \mu m\rho_1 m^* \\
&\geq \rho_1\rho^{q-1}m_r^{q-1}\lambda^{q-1}\left[t_1(1-t_m)\right]^{q-1}\varphi_q\left(\int_{t_1}^{t_m}\omega(\tau)\,d\tau\right) + \mu m\rho_1 m^* \\
&> \rho_1\rho^{q-1}m_r^{q-1}\lambda_0^{q-1}\left[t_1(1-t_m)\right]^{q-1}\varphi_q\left(\int_{t_1}^{t_m}\omega(\tau)\,d\tau\right) + \mu_0 m\rho_1 m^* \\
&= \rho_1\rho^{q-1}m_r^{q-1}\lambda_0^{q-1}\left[t_1(1-t_m)\right]^{q-1}\eta^* + \mu_0 m\rho_1 m^* \\
&\geq \frac{1}{2}r + \frac{1}{2}r = r = \|y\|_{PC^1},
\end{aligned}$$

which implies that

$$\|T_\lambda^\mu y\|_{PC^1} > \|y\|_{PC^1}, \quad \forall y \in K \cap \partial\Omega_r, \lambda > \lambda_0 \text{ and } \mu > \mu_0. \tag{4.1}$$

If $f^\infty = 0$, $I^\infty = 0$, then there exist $l_1 > 0$, $l_2 > 0$, and $R > r > 0$ such that

$$f(t,y) < l_1\varphi_p(y), \qquad I_k(t,y) < l_2 y, \quad \forall t \in J, y \geq R, k = 1,2,\ldots,m,$$

where l_1 satisfies

$$2\max\{\rho_2, a(a+b)\}\eta\varphi_q\left(\frac{1}{4}\gamma\lambda l_1\right) \leq 1, \tag{4.2}$$

l_2 satisfies

$$2\max\{\rho_2, a(a+b)\}m\mu l_2 \leq 1. \tag{4.3}$$

Let $\alpha = \frac{R}{\delta}$. Thus, when $y \in K \cap \partial\Omega_\alpha$ we have

$$y(t) \geq \delta\|y\|_{PC^1} = \delta\alpha = R, \quad t \in J,$$

and then we get

$$\begin{aligned}
\left(T_\lambda^\mu y\right)(t) &= \int_0^1 H_1(t,s)\phi_q\left(\lambda\int_0^1 H(s,\tau)\omega(\tau)f\big(\tau,y(\tau)\big)\,d\tau\right)ds \\
&\quad + \mu\sum_{k=1}^m H_1(t,t_k)I_k\big(t_k,y(t_k)\big) \\
&\leq \rho_2\left(\frac{1}{4}\lambda\gamma\right)^{q-1}\varphi_q\left(\int_0^1 \omega(\tau)f\big(\tau,y(\tau)\big)\,d\tau\right) + \mu\rho_2\sum_{k=1}^m I_k\big(t_k,y(t_k)\big) \\
&\leq \rho_2\left(\frac{1}{4}\lambda\gamma\right)^{q-1}\varphi_q\left(\int_0^1 \omega(\tau)l_1\phi_p\big(y(\tau)\big)\,d\tau\right) + \mu\rho_2\sum_{k=1}^m l_2 y(t_k) \\
&\leq \rho_2\left(\frac{1}{4}\lambda\gamma\right)^{q-1}\varphi_q\left(\int_0^1 \omega(\tau)l_1\phi_p\big(\|y\|_{PC^1}\big)\,d\tau\right) + \mu\rho_2\sum_{k=1}^m l_2\|y\|_{PC^1} \\
&\leq \rho_2\left(\frac{1}{4}\lambda\gamma\right)^{q-1} l_1^{q-1}\|y\|_{PC^1}\varphi_q\left(\int_0^1 \omega(\tau)\,d\tau\right) + \mu\rho_2 m l_2\|y\|_{PC^1} \\
&= \rho_2\left(\frac{1}{4}\lambda\gamma\right)^{q-1} l_1^{q-1}\|y\|_{PC^1}\eta + \mu\rho_2 m l_2\|y\|_{PC^1} \\
&\leq \frac{1}{2}\|y\|_{PC^1} + \frac{1}{2}\|y\|_{PC^1} = \|y\|_{PC^1},
\end{aligned} \tag{4.4}$$

$$\begin{aligned}
\left|\left(T_\lambda^\mu y\right)'(t)\right| &\leq \int_0^1 \left|H'_{1t}(t,s)\right|\phi_q\left(\lambda\int_0^1 H(s,\tau)\omega(\tau)f\big(\tau,y(\tau)\big)\,d\tau\right)ds \\
&\quad + \mu\sum_{k=1}^m \left|H'_{1t}(t,t_k)\right| I_k\big(t_k,y(t_k)\big) \\
&\leq a(b+a)\left(\frac{1}{4}\lambda\gamma\right)^{q-1}\varphi_q\left(\int_0^1 \omega(\tau)f\big(\tau,y(\tau)\big)\,d\tau\right) + \mu a(b+a)\sum_{k=1}^m I_k\big(t_k,y(t_k)\big) \\
&\leq a(b+a)\left(\frac{1}{4}\lambda\gamma\right)^{q-1}\varphi_q\left(\int_0^1 \omega(\tau)l_1\phi_p\big(y(\tau)\big)\,d\tau\right) + \mu a(b+a)\sum_{k=1}^m l_2 y(t_k)
\end{aligned}$$

$$
\begin{aligned}
&\le a(b+a)\left(\frac{1}{4}\lambda\gamma\right)^{q-1}\varphi_q\left(\int_0^1 \omega(\tau)l_1\phi_p\left(\|y\|_{PC^1}\right)d\tau\right) \\
&\quad + \mu a(b+a)\sum_{k=1}^{m} l_2\|y\|_{PC^1} \\
&\le a(b+a)\left(\frac{1}{4}\lambda\gamma\right)^{q-1} l_1^{q-1}\|y\|_{PC^1}\eta + \mu a(b+a)ml_2\|y\|_{PC^1} \\
&\le \frac{1}{2}\|y\|_{PC^1} + \frac{1}{2}\|y\|_{PC^1} = \|y\|_{PC^1}, \qquad (4.5)
\end{aligned}
$$

where

$$
H'_{1t}(t,s) = G'_{1t}(t,s) = \begin{cases} -a(b+as), & \text{if } 0 \le s \le t \le 1, \\ a(b+a(1-s)), & \text{if } 0 \le t \le s \le 1 \end{cases}
$$

and

$$
\max_{t,s\in J, t\neq s}\left|H'_{1t}(t,s)\right| = \max_{t,s\in J, t\neq s}\left|G'_{1t}(t,s)\right| = a(b+a).
$$

It follows from (4.4) and (4.5) that

$$
\left\|T_\lambda^\mu y\right\|_{PC^1} \le \|y\|_{PC^1}, \quad \forall y \in K\cap\partial\Omega_\alpha. \qquad (4.6)
$$

Applying (b) of Lemma 3.4 to (4.1) and (4.6) shows that T_λ^μ has a fixed point $y \in K \cap (\bar{\Omega}_\alpha \setminus \Omega_r)$ with $r \le \|y\|_{PC^1} \le \alpha = \frac{1}{\delta}R$. Hence, since for $y \in K$ we have $y(t) \ge \delta\|y\|_{PC^1}$, $t \in J$, it follows that (2.2) holds. This gives the proof of part (i).

Part (ii). Noticing that $f(t,y) > 0$, $I_k(t,y) > 0$ $(k = 1,2,\ldots,m)$ for all t and $y > 0$, we can define

$$
m_R = \min_{t\in J, \delta R\le y\le R}\left\{f(t,y)\right\} > 0, \qquad m^{**} = \min\left\{m_k^*, k = 1,2,\ldots,m\right\} > 0,
$$

where $R > 0$, and

$$
m_k^* = \min_{t\in J, \delta R\le y\le R}\left\{I_k(t,y)\right\}, \quad k = 1,2,\ldots,m.
$$

Let

$$
\lambda_0 \ge \left(\frac{1}{2\rho_1\eta^*}R\right)^{p-1}\left[\rho m_R t_1(1-t_m)\right]^{-1}, \qquad \mu_0 \ge \frac{1}{2m\rho_1 m^{**}}R.
$$

Then, for $y \in K\cap\partial\Omega_R$ and $\lambda > \lambda_0$, $\mu > \mu_0$, we have

$$
\begin{aligned}
\left(T_\lambda^\mu y\right)(t) &= \int_0^1 H_1(t,s)\phi_q\left(\lambda\int_0^1 H(s,\tau)\omega(\tau)f\left(\tau,y(\tau)\right)d\tau\right)ds \\
&\quad + \mu\sum_{k=1}^{m} H_1(t,t_k)I_k\left(t_k,y(t_k)\right)
\end{aligned}
$$

$$
\begin{aligned}
&\geq \rho_1\rho^{q-1}\varphi_q\left(\lambda\int_0^1 e(\tau)\omega(\tau)f\big(\tau,y(\tau)\big)\,d\tau\right)+\mu\rho_1\sum_{k=1}^m I_k\big(t_k,y(t_k)\big)\\
&\geq \rho_1\rho^{q-1}\varphi_q\left(\lambda\int_0^1 e(\tau)\omega(\tau)m_R\,d\tau\right)+\mu\rho_1\sum_{k=1}^m m^{**}\\
&= \rho_1\rho^{q-1}m_R^{q-1}\lambda^{q-1}\varphi_q\left(\int_0^1 e(\tau)\omega(\tau)\,d\tau\right)+\mu m\rho_1 m^{**}\\
&\geq \rho_1\rho^{q-1}m_R^{q-1}\lambda^{q-1}\varphi_q\left(\int_{t_1}^{t_m} e(\tau)\omega(\tau)\,d\tau\right)+\mu m\rho_1 m^{**}\\
&\geq \rho_1\rho^{q-1}m_R^{q-1}\lambda^{q-1}\big[t_1(1-t_m)\big]^{q-1}\varphi_q\left(\int_{t_1}^{t_m}\omega(\tau)\,d\tau\right)+\mu m\rho_1 m^{**}\\
&> \rho_1\rho^{q-1}m_R^{q-1}\lambda_0^{q-1}\big[t_1(1-t_m)\big]^{q-1}\varphi_q\left(\int_{t_1}^{t_m}\omega(\tau)\,d\tau\right)+\mu_0 m\rho_1 m^{**}\\
&= \rho_1\rho^{q-1}m_R^{q-1}\lambda_0^{q-1}\big[t_1(1-t_m)\big]^{q-1}\eta^*+\mu_0 m\rho_1 m^{**}\\
&\geq \frac{1}{2}R+\frac{1}{2}R=\|y\|_{PC^1},
\end{aligned}
$$

which implies that

$$\big\|T_\lambda^\mu y\big\|_{PC^1}>\|y\|_{PC^1},\quad \forall y\in K\cap\partial\Omega_R, \lambda>\lambda_0 \text{ and } \mu>\mu_0. \tag{4.7}$$

If $f^0=0$, $I^0=0$, then there exist $l_1>0$, $l_2>0$, and $0<r<R$ such that

$$f(t,y)<l_1\varphi_p(y),\qquad I_k(t,y)<l_2 y\quad (\forall t\in J, 0\leq y\leq r, k=1,2,\ldots,m),$$

where l_1 and l_2 satisfy (4.2) and (4.3), respectively.

Similar to the proof of (4.6), we can prove that

$$\big\|T_\lambda^\mu y\big\|_{PC^1}\leq\|y\|_{PC^1},\quad \forall y\in K\cap\partial\Omega_r. \tag{4.8}$$

Applying (a) of Lemma 3.4 to (4.7) and (4.8) shows that T_λ^μ has a fixed point $y\in K\cap(\bar{\Omega}_R\setminus\Omega_r)$ with $r\leq\|y\|_{PC^1}\leq R$. Hence, since for $y\in K$ we have $y(t)\geq\delta\|y\|_{PC^1}$ for $t\in J$, it follows that (2.3) holds. This gives the proof of part (ii).

Consider part (iii). Choose two numbers r_1 and r_2 satisfying (2.1). By part (i) and part (ii), there exist $\lambda_0>0$ and $\mu_0>0$ such that

$$\big\|T_\lambda^\mu y\big\|_{PC^1}>\|y\|_{PC^1},\quad \forall y\in K\cap\partial\Omega_{r_i}, i=1,2. \tag{4.9}$$

Since $f^0=f^\infty=I^\infty=I^0=0$, from the proof of part (i) and part (ii), it follows that

$$\big\|T_\lambda^\mu y\big\|_{PC^1}<\|y\|_{PC^1},\quad \forall y\in K\cap\partial\Omega_r \tag{4.10}$$

and

$$\big\|T_\lambda^\mu y\big\|_{PC^1}<\|y\|_{PC^1},\quad \forall y\in K\cap\partial\Omega_R. \tag{4.11}$$

Applying Lemma 3.4 to (4.9)-(4.11) shows that T_λ^μ has two fixed points y_1 and y_2 such that $y_1 \in K \cap (\bar{\Omega}_{r_1} \setminus \Omega_r)$ and $y_2 \in K \cap (\bar{\Omega}_R \setminus \Omega_{r_2})$. These are the desired distinct positive solutions of problem (1.1) for $\lambda_0 > 0$ and $\mu_0 > 0$ satisfying (2.4). Then the result of part (iii) follows. □

Proof of Theorem 2.2 Part (i). Noticing that $f(t,y) > 0$, $I_k(t,y) > 0$ $(k = 1,2,\dots,m)$ for all t and $y > 0$, we can define

$$M_r = \max_{t\in J,\delta r\leq y\leq r}\{f(t,y)\} > 0, \qquad M^* = \max\{M_k, k = 1,2,\dots,m\} > 0,$$

where $r > 0$, and

$$M_k = \max_{t\in J,\delta r\leq y\leq r}\{I_k(t,y)\}, \quad k = 1,2,\dots,m.$$

Let

$$\bar{\lambda}_0 \leq 4\left(\frac{1}{2\max\{\rho_2, a(a+b)\}\eta}r\right)^{p-1}(M_r\gamma)^{-1},$$
$$\bar{\mu}_0 \leq \frac{1}{2\max\{\rho_2, a(a+b)\}mM^*}r.$$

Then, for $y \in K \cap \partial\Omega_r$ and $\lambda < \bar{\lambda}_0$, $\mu < \bar{\mu}_0$, we have

$$\begin{aligned}
\left(T_\lambda^\mu y\right)(t) &= \int_0^1 H_1(t,s)\phi_q\left(\lambda\int_0^1 H(s,\tau)\omega(\tau)f\left(\tau,y(\tau)\right)d\tau\right)ds \\
&\quad + \mu\sum_{k=1}^m H_1(t,t_k)I_k\left(t_k,y(t_k)\right) \\
&\leq \rho_2\left(\frac{1}{4}\gamma\right)^{q-1}\varphi_q\left(\lambda\int_0^1\omega(\tau)f\left(\tau,y(\tau)\right)d\tau\right) + \mu\rho_2\sum_{k=1}^m I_k\left(t_k,y(t_k)\right) \\
&\leq \rho_2\left(\frac{1}{4}\gamma\lambda\right)^{q-1}\varphi_q\left(\int_0^1\omega(\tau)M_r\,d\tau\right) + \mu\rho_2\sum_{k=1}^m M^* \\
&= \rho_2\left(\frac{1}{4}\gamma\lambda M_r\right)^{q-1}\varphi_q\left(\int_0^1\omega(\tau)\,d\tau\right) + \mu\rho_2 mM^* \\
&< \rho_2\left(\frac{1}{4}\gamma\bar{\lambda}_0 M_r\right)^{q-1}\eta + \bar{\mu}_0\rho_2 mM^* \\
&\leq \frac{1}{2}r + \frac{1}{2}r = \|y\|_{PC^1}.
\end{aligned} \tag{4.12}$$

Similar to the proof of (4.5), we can prove

$$\left|\left(T_\lambda^\mu y\right)'(t)\right| < \|y\|_{PC^1}. \tag{4.13}$$

It follows from (4.12) and (4.13) that

$$\left\|T_\lambda^\mu y\right\|_{PC^1} < \|y\|_{PC^1}, \quad \forall y \in K \cap \partial\Omega_r. \tag{4.14}$$

If $f_\infty = \infty$, $I_\infty = \infty$, then there exist $l_3 > 0$, $l_4 > 0$, and $R > r > 0$ such that

$$f(t,y) > l_3\varphi_p(y), \qquad I_k(t,y) > l_4 y \quad (\forall t \in J, y \geq R, k = 1,2,\ldots,m),$$

where l_3 satisfies

$$2\rho_1\rho^{q-1}\lambda^{q-1}l_3^{q-1}\delta\big[t_1(1-t_m)\big]^{q-1}\eta^* \geq 1, \tag{4.15}$$

l_4 satisfies

$$2\mu\rho_1 m l_4\delta \geq 1. \tag{4.16}$$

Let $\alpha = \frac{R}{\delta}$. Thus, when $y \in K \cap \partial\Omega_\alpha$ we have

$$y(t) \geq \delta\|y\|_{PC^1} = \delta\alpha = R, \quad t \in J,$$

and then we get

$$\begin{aligned}
\big(T_\lambda^\mu y\big)(t) &= \int_0^1 H_1(t,s)\phi_q\left(\lambda\int_0^1 H(s,\tau)\omega(\tau)f\big(\tau,y(\tau)\big)\,d\tau\right)ds \\
&\quad + \mu\sum_{k=1}^m H_1(t,t_k)I_k\big(t_k,y(t_k)\big) \\
&\geq \rho_1\rho^{q-1}\varphi_q\left(\lambda\int_0^1 e(\tau)\omega(\tau)f\big(\tau,y(\tau)\big)\,d\tau\right) + \mu\rho_1\sum_{k=1}^m I_k\big(t_k,y(t_k)\big) \\
&\geq \rho_1\rho^{q-1}\lambda^{q-1}\varphi_q\left(\int_0^1 e(\tau)\omega(\tau)l_3\phi_p\big(y(\tau)\big)\,d\tau\right) + \mu\rho_1\sum_{k=1}^m l_4 y(t_k) \\
&\geq \rho_1\rho^{q-1}\lambda^{q-1}\varphi_q\left(\int_0^1 e(\tau)\omega(\tau)l_3\phi_p\big(\delta\|y\|_{PC^1}\big)\,d\tau\right) + \mu\rho_1\sum_{k=1}^m l_4\delta\|y\|_{PC^1} \\
&= \rho_1\rho^{q-1}\lambda^{q-1}l_3^{q-1}\delta\|y\|_{PC^1}\varphi_q\left(\int_0^1 e(\tau)\omega(\tau)\,d\tau\right) + \mu\rho_1 m l_4\delta\|y\|_{PC^1} \\
&\geq \rho_1\rho^{q-1}\lambda^{q-1}l_3^{q-1}\delta\|y\|_{PC^1}\varphi_q\left(\int_{t_1}^{t_m} e(\tau)\omega(\tau)\,d\tau\right) + \mu\rho_1 m l_4\delta\|y\|_{PC^1} \\
&\geq \rho_1\rho^{q-1}\lambda^{q-1}l_3^{q-1}\delta\|y\|_{PC^1}\big[t_1(1-t_m)\big]^{q-1}\varphi_q\left(\int_{t_1}^{t_m}\omega(\tau)\,d\tau\right) + \mu\rho_1 m l_4\delta\|y\|_{PC^1} \\
&> \rho_1\rho^{q-1}\lambda^{q-1}l_3^{q-1}\delta\|y\|_{PC^1}\big[t_1(1-t_m)\big]^{q-1}\eta^* + \mu\rho_1 m l_4\delta\|y\|_{PC^1} \\
&\geq \frac{1}{2}\alpha + \frac{1}{2}\alpha = \alpha.
\end{aligned}$$

This yields

$$\big\|T_\lambda^\mu y\big\|_{PC^1} \geq \|y\|_{PC^1}, \quad \forall y \in K \cap \partial\Omega_\alpha. \tag{4.17}$$

Applying (b) of Lemma 3.4 to (4.14) and (4.17) shows that T_λ^μ has a fixed point $y \in K \cap (\bar{\Omega}_\alpha \setminus \Omega_r)$ with $r \leq \|y\|_{PC^1} \leq \alpha = \frac{1}{\delta}R$. Hence, since for $y \in K$ we have $y(t) \geq \delta\|y\|_{PC^1}$, $t \in J$, it follows that (2.2) holds. This gives the proof of part (i).

Part (ii). Noticing that $f(t,y) > 0$, $I_k(t,y) > 0$ $(k = 1,2,\dots,m)$ for all t and $y > 0$, we can define

$$M_R = \max_{t\in J, 0\le y\le R}\{f(t,y)\} > 0, \qquad M^{**} = \max\{M_k^*, k = 1,2,\dots,m\} > 0,$$

where $R > 0$, and

$$M_k^* = \max_{t\in J, 0\le y\le R}\{I_k(t,y)\}, \quad k = 1,2,\dots,m.$$

Let

$$\bar{\lambda}_0 \le 4\left(\frac{R}{2\rho_2\eta}\right)^{p-1}(\gamma M_R)^{-1}, \qquad \bar{\mu}_0 \le \frac{R}{2\rho_2 m M^{**}}.$$

Then, for $y \in K \cap \partial\Omega_R$ and $\lambda < \bar{\lambda}_0$, $\mu < \bar{\mu}_0$, we have

$$\begin{aligned}
(T_\lambda^\mu y)(t) &= \int_0^1 H_1(t,s)\phi_q\left(\lambda\int_0^1 H(s,\tau)\omega(\tau)f(\tau,y(\tau))\,d\tau\right)ds \\
&\quad + \mu\sum_{k=1}^m H_1(t,t_k)I_k(t_k,y(t_k)) \\
&\le \rho_2\left(\frac{1}{4}\gamma\right)^{q-1}\varphi_q\left(\lambda\int_0^1 \omega(\tau)f(\tau,y(\tau))\,d\tau\right) + \mu\rho_2\sum_{k=1}^m I_k(t_k,y(t_k)) \\
&\le \rho_2\left(\frac{1}{4}\gamma\lambda\right)^{q-1}\varphi_q\left(\int_0^1 \omega(\tau)M_R\,d\tau\right) + \mu\rho_2\sum_{k=1}^m M^{**} \\
&= \rho_2\left(\frac{1}{4}\gamma\lambda M_R\right)^{q-1}\varphi_q\left(\int_0^1 \omega(\tau)\,d\tau\right) + \mu\rho_2 m M^{**} \\
&< \rho_2\left(\frac{1}{4}\gamma\bar{\lambda}_0 M_R\right)^{q-1}\eta + \bar{\mu}_0\rho_2 m M^{**} \\
&\le \frac{1}{2}R + \frac{1}{2}R = \|y\|_{PC^1}.
\end{aligned} \tag{4.18}$$

Similar to the proof of (4.5), we can prove

$$\left|(T_\lambda^\mu y)'(t)\right| \le \|y\|_{PC^1}, \quad \forall y \in K \cap \partial\Omega_R. \tag{4.19}$$

It follows from (4.18) and (4.19) that

$$\left\|T_\lambda^\mu y\right\|_{PC^1} < \|y\|_{PC^1}, \quad \forall y \in K \cap \partial\Omega_R. \tag{4.20}$$

If $f_0 = \infty$, $I_0 = \infty$, then there exist $l_3 > 0$, $l_4 > 0$, and $0 < r < R$ such that

$$f(t,y) > l_3\varphi_p(y), \qquad I_k(t,y) > l_4 y \quad (\forall t \in J, 0 \le y \le r, k = 1,2,\dots,m),$$

where l_3 and l_4 satisfy (4.15) and (4.16), respectively.

Therefore, for $y \in K \cap \partial\Omega_r$, we obtain

$$\begin{aligned}
(T_\lambda^\mu y)(t) &= \int_0^1 H_1(t,s)\phi_q\left(\lambda\int_0^1 H(s,\tau)\omega(\tau)f(\tau,y(\tau))\,d\tau\right)ds \\
&\quad + \mu\sum_{k=1}^m H_1(t,t_k)I_k(t_k,y(t_k)) \\
&\geq \rho_1\rho^{q-1}\varphi_q\left(\lambda\int_0^1 e(\tau)\omega(\tau)f(\tau,y(\tau))\,d\tau\right) + \mu\rho_1\sum_{k=1}^m I_k(t_k,y(t_k)) \\
&\geq \rho_1\rho^{q-1}\lambda^{q-1}\varphi_q\left(\int_0^1 e(\tau)\omega(\tau)l_3\phi_p(y(\tau))\,d\tau\right) + \mu\rho_1\sum_{k=1}^m l_4 y(t_k) \\
&\geq \rho_1\rho^{q-1}\lambda^{q-1}\varphi_q\left(\int_0^1 e(\tau)\omega(\tau)l_3\phi_p(\delta\|y\|_{PC^1})\,d\tau\right) + \mu\rho_1\sum_{k=1}^m l_4\delta\|y\|_{PC^1} \\
&= \rho_1\rho^{q-1}\lambda^{q-1}l_3^{q-1}\delta\|y\|_{PC^1}\varphi_q\left(\int_0^1 e(\tau)\omega(\tau)\,d\tau\right) + \mu\rho_1 m l_4\delta\|y\|_{PC^1} \\
&\geq \rho_1\rho^{q-1}\lambda^{q-1}l_3^{q-1}\delta\|y\|_{PC^1}\varphi_q\left(\int_{t_1}^{t_m} e(\tau)\omega(\tau)\,d\tau\right) + \mu\rho_1 m l_4\delta\|y\|_{PC^1} \\
&\geq \rho_1\rho^{q-1}\lambda^{q-1}l_3^{q-1}\delta\|y\|_{PC^1}\left[t_1(1-t_m)\right]^{q-1}\varphi_q\left(\int_{t_1}^{t_m}\omega(\tau)\,d\tau\right) + \mu\rho_1 m l_4\delta\|y\|_{PC^1} \\
&> \rho_1\rho^{q-1}\lambda^{q-1}l_3^{q-1}\delta\|y\|_{PC^1}\left[t_1(1-t_m)\right]^{q-1}\eta^* + \mu\rho_1 m l_4\delta\|y\|_{PC^1} \\
&\geq \frac{1}{2}\|y\|_{PC^1} + \frac{1}{2}\|y\|_{PC^1} = \|y\|_{PC^1}.
\end{aligned}$$

This yields

$$\left\|T_\lambda^\mu y\right\|_{PC^1} > \|y\|_{PC^1}, \quad \forall y \in K \cap \partial\Omega_r. \tag{4.21}$$

Applying (a) of Lemma 3.4 to (4.20) and (4.21) shows that T_λ^μ has a fixed point $y \in K \cap (\bar{\Omega}_R \setminus \Omega_r)$ with $r \leq \|y\|_{PC^1} \leq R$. Hence, since for $y \in K$ we have $y(t) \geq \delta\|y\|_{PC^1}$, $t \in J$, it follows that (2.3) holds. This gives the proof of part (ii).

Consider part (iii). Choose two numbers r_1 and r_2 satisfying (2.1). By part (i) and part (ii), there exist $\bar{\lambda}_0 > 0$ and $\bar{\mu}_0 > 0$ such that

$$\left\|T_\lambda^\mu y\right\|_{PC^1} < \|y\|_{PC^1}, \quad \forall 0 < \lambda < \bar{\lambda}_0, 0 < \mu < \bar{\mu}_0, y \in K \cap \partial\Omega_{r_i}, i = 1,2. \tag{4.22}$$

Since $f_0 = f_\infty = I_\infty = I_0 = \infty$, from the proof of part (i) and part (ii), it follows that

$$\left\|T_\lambda^\mu y\right\|_{PC^1} > \|y\|_{PC^1}, \quad \forall y \in K \cap \partial\Omega_r \tag{4.23}$$

and

$$\left\|T_\lambda^\mu y\right\|_{PC^1} > \|y\|_{PC^1}, \quad \forall y \in K \cap \partial\Omega_R. \tag{4.24}$$

Applying Lemma 3.4 to (4.22)-(4.24) shows that T_λ^μ has two fixed points y_1 and y_2 such that $y_1 \in K \cap (\bar{\Omega}_{r_1} \setminus \Omega_r)$ and $y_2 \in K \cap (\bar{\Omega}_R \setminus \Omega_{r_2})$. These are the desired distinct positive solutions of problem (1.1) for $0 < \lambda < \bar{\lambda}_0$ and $0 < \mu < \bar{\mu}_0$ satisfying (2.5). Then the proof of part (iii) is complete. □

Remark 4.1 Comparing with Feng [31], the main features of this paper are as follows.

(i) Two parameters $\lambda > 0$ and $\mu > 0$ are considered.

(ii) $\omega \in L^1_{\text{loc}}(0,1)$, not only $\omega(t) \equiv 1$ for $t \in J$.

(iii) It follows from the proof of Theorem 2.1 that the conditions of Corollary 3.2 in [31] are not the optimal conditions, which guarantee the existence of at least one positive solution for problem (1.1). In fact, if $f_0 = \infty$, or $f^\infty = 0$, $I^\infty(k) = 0$, we can prove that problem (1.1) has at least one positive solution, respectively.

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
XZ completed the main study and carried out the results of this article. MF checked the proofs and verified the calculation. All the authors read and approved the final manuscript.

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References

1. Sun, J, Wang, X: Monotone positive solutions for an elastic beam equation with nonlinear boundary conditions. Math. Probl. Eng. (2011). doi:10.1155/2011/609189
2. Yao, Q: Positive solutions of nonlinear beam equations with time and space singularities. J. Math. Anal. Appl. **374**, 681-692 (2011)
3. Yao, Q: Local existence of multiple positive solutions to a singular cantilever beam equation. J. Math. Anal. Appl. **363**, 138-154 (2010)
4. O'Regan, D: Solvability of some fourth (and higher) order singular boundary value problems. J. Math. Anal. Appl. **161**, 78-116 (1991)
5. Wei, Z: A class of fourth order singular boundary value problems. Appl. Math. Comput. **153**, 865-884 (2004)
6. Yang, B: Positive solutions for the beam equation under certain boundary conditions. Electron. J. Differ. Equ. **2005**, 78 (2005)
7. Zhang, X: Existence and iteration of monotone positive solutions for an elastic beam equation with a corner. Nonlinear Anal., Real World Appl. **10**, 2097-2103 (2009)
8. Gupta, GP: Existence and uniqueness theorems for the bending of an elastic beam equation. Appl. Anal. **26**, 289-304 (1988)
9. Gupta, GP: A nonlinear boundary value problem associated with the static equilibrium of an elastic beam supported by sliding clamps. Int. J. Math. Math. Sci. **12**, 697-711 (1989)
10. Graef, JR, Yang, B: On a nonlinear boundary value problem for fourth order equations. Appl. Anal. **72**, 439-448 (1999)
11. Agarwal, RP: On fourth-order boundary value problems arising in beam analysis. Differ. Integral Equ. **2**, 91-110 (1989)
12. Davis, J, Henderson, J: Uniqueness implies existence for fourth-order Lidstone boundary value problems. Panam. Math. J. **8**, 23-35 (1998)
13. Kosmatov, N: Countably many solutions of a fourth order boundary value problem. Electron. J. Qual. Theory Differ. Equ. **2004**, 12 (2004)
14. Bai, Z, Wang, H: On the positive solutions of some nonlinear fourth-order beam equations. J. Math. Anal. Appl. **270**, 357-368 (2002)
15. Bai, Z, Huang, B, Ge, W: The iterative solutions for some fourth-order *p*-Laplace equation boundary value problems. Appl. Math. Lett. **19**, 8-14 (2006)
16. Liu, X-L, Li, W-T: Existence and multiplicity of solutions for fourth-order boundary values problems with parameters. J. Math. Anal. Appl. **327**, 362-375 (2007)
17. Bonanno, G, Bella, B: A boundary value problem for fourth-order elastic beam equations. J. Math. Anal. Appl. **343**, 1166-1176 (2008)
18. Ma, R, Wang, H: On the existence of positive solutions of fourth-order ordinary differential equations. Appl. Anal. **59**, 225-231 (1995)
19. Han, G, Xu, Z: Multiple solutions of some nonlinear fourth-order beam equations. Nonlinear Anal. **68**, 3646-3656 (2008)

20. Zhang, X, Ge, W: Symmetric positive solutions of boundary value problems with integral boundary conditions. Appl. Math. Comput. **219**, 3553-3564 (2012)
21. Zhai, C, Song, R, Han, Q: The existence and the uniqueness of symmetric positive solutions for a fourth-order boundary value problem. Comput. Math. Appl. **62**, 2639-2647 (2011)
22. Zhang, X, Feng, M, Ge, W: Symmetric positive solutions for *p*-Laplacian fourth order differential equation with integral boundary conditions. J. Comput. Appl. Math. **222**, 561-573 (2008)
23. Zhang, X, Feng, M, Ge, W: Existence results for nonlinear boundary-value problems with integral boundary conditions in Banach spaces. Nonlinear Anal. **69**, 3310-3321 (2008)
24. Zhang, X, Liu, L: A necessary and sufficient condition of positive solutions for fourth order multi-point boundary value problem with *p*-Laplacian. Nonlinear Anal. **68**, 3127-3137 (2008)
25. Aftabizadeh, AR: Existence and uniqueness theorems for fourth-order boundary value problems. J. Math. Anal. Appl. **116**, 415-426 (1986)
26. Kang, P, Wei, Z, Xu, J: Positive solutions to fourth-order singular boundary value problems with integral boundary conditions in abstract spaces. Appl. Math. Comput. **206**, 245-256 (2008)
27. Xu, J, Yang, Z: Positive solutions for a fourth order *p*-Laplacian boundary value problem. Nonlinear Anal. **74**, 2612-2623 (2011)
28. Webb, JRL, Infante, G, Franco, D: Positive solutions of nonlinear fourth-order boundary value problems with local and non-local boundary conditions. Proc. R. Soc. Edinb. **138**, 427-446 (2008)
29. Ma, H: Symmetric positive solutions for nonlocal boundary value problems of fourth order. Nonlinear Anal. **68**, 645-651 (2008)
30. Zhang, X, Liu, L: Positive solutions of fourth-order four-point boundary value problems with *p*-Laplacian operator. J. Math. Anal. Appl. **336**, 1414-1423 (2007)
31. Feng, M: Multiple positive solutions of four-order impulsive differential equations with integral boundary conditions and one-dimensional *p*-Laplacian. Bound. Value Probl. (2011). doi:10.1155/2011/654871
32. Hao, X, Liu, L, Wu, Y: Positive solutions for second order impulsive differential equations with integral boundary conditions. Commun. Nonlinear Sci. Numer. Simul. **16**, 101-111 (2011)
33. Sun, J, Chen, H, Yang, L: The existence and multiplicity of solutions for an impulsive differential equation with two parameters via a variational method. Nonlinear Anal. TMA **73**, 440-449 (2010)
34. Ning, P, Huan, Q, Ding, W: Existence result for impulsive differential equations with integral boundary conditions. Abstr. Appl. Anal. (2013). doi:10.1155/2013/134691
35. Guo, D, Lakshmikantham, V: Nonlinear Problems in Abstract Cones. Academic Press, New York (1988)

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