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Solvability for a coupled system of fractional differential equations with impulses at resonance

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Abstract
In this paper, some Banach spaces are introduced. Based on these spaces and the coincidence degree theory, a 2m-point boundary value problem for a coupled system of impulsive fractional differential equations at resonance is considered, and the new criterion on existence is obtained. Finally, an example is also given to illustrate the availability of our main results.

MSC: 34A08; 34B10; 34B37

Keywords: coupled system; impulsive fractional differential equations; at resonance; coincidence degree

1 Introduction

Recently, Wang *et al.* [1] presented a counterexample to show an error formula of solutions to the traditional boundary value problem for impulsive differential equations with fractional derivative in [2–5]. Meanwhile, they introduced the correct formula of solutions for an impulsive Cauchy problem with the Caputo fractional derivative. Shortly afterwards, many works on the better formula of solutions to the Cauchy problem for impulsive fractional differential equations have been reported by Li *et al.* [6], Wang *et al.* [7], Fečkan [8], *etc.*

Fractional differential equations have been paid much attention to in recent years due to their wide applications such as nonlinear oscillations of earthquakes, Nutting's law, charge transport in amorphous semiconductors, fluid dynamic traffic model, non-Markovian diffusion process with memory *etc.* [9–11]. For more details, see the monographs of Hilfer [12], Miller and Ross [13], Podlubny [14], Lakshmikantham *et al.* [15], Samko *et al.* [16], and the papers of [2, 17–19] and the references therein.

In recent years, many researchers paid much attention to the coupled system of fractional differential equations due to its applications in different fields [20–25]. Zhang *et al.* [25] investigated a three-point boundary value problem at resonance for a coupled system of nonlinear fractional differential equations given by

$$\begin{cases} D^{\alpha}_{0^+}u(t) = f(t, v(t), D^{\beta-1}_{0^+}v(t)), & 0 < t < 1; \\ D^{\beta}_{0^+}v(t) = g(t, u(t), D^{\alpha-1}_{0^+}u(t)), & 0 < t < 1; \\ u(0) = v(0) = 0, \qquad u(1) = \sigma_1 u(\eta_1), \qquad v(1) = \sigma_2 v(\eta_2), \end{cases}$$

where $1 < \alpha, \beta \le 2$, $0 < \eta_1, \eta_2 < 1$, $\sigma_1, \sigma_2 > 0$, $\sigma_1 \eta_1^{\alpha-1} = \sigma_2 \eta_2^{\beta-1} = 1$, $D_{0^+}^{\alpha}$ is the standard Riemann-Liouville fractional derivative and $f, g : [0,1] \times \mathbb{R}^2 \to \mathbb{R}$ are continuous. And Wang *et al.* [23] considered a 2m-point boundary value problem (BVP) at resonance for a coupled system as follows:

$$\begin{cases} D_{0^+}^{\alpha} u(t) = f(t, v(t), D_{0^+}^{\beta-1} v(t), D_{0^+}^{\beta-2} v(t)), \quad 0 < t < 1; \\ D_{0^+}^{\beta} v(t) = g(t, u(t), D_{0^+}^{\alpha-1} u(t), D_{0^+}^{\alpha-2} u(t)), \quad 0 < t < 1; \\ I_{0^+}^{3-\alpha} u(0) = 0, \qquad D_{0^+}^{\alpha-2} u(1) = \sum_{i=1}^{m} a_i D_{0^+}^{\alpha-2} u(\xi_i), \qquad u(1) = \sum_{i=1}^{m} b_i u(\eta_i); \\ I_{0^+}^{3-\beta} v(0) = 0, \qquad D_{0^+}^{\beta-2} v(1) = \sum_{i=1}^{m} c_i D_{0^+}^{\beta-2} v(\gamma_i), \qquad v(1) = \sum_{i=1}^{m} d_i v(\delta_i), \end{cases}$$

where $1 < \alpha, \beta \le 2$. With the help of the coincidence degree theory, many existence results have been given in the above literatures. It is worth mentioning that the orders of derivative in the nonlinear function on the right-hand of equal signs are all fixed in the above works, but the opposite case is more difficult and complicated, then this work attempts to deal exactly with this case. What is more, this case of arbitrary order derivative included in the nonlinear functions is very important in many aspects [20, 22].

There are significant developments in the theory of impulses especially in the area of impulsive differential equations with fixed moments, which provided a natural description of observed evolution processes, regarding as important tools for better understanding several real word phenomena in applied sciences [1, 7, 26–29]. In addition, motivated by the better formula of solutions cited by the work of Zhou *et al.* [1, 7, 8], the aim of this work is to discuss a boundary value problem for a coupled system of impulsive fractional differential equation. Exactly, this paper deals with the 2m-point boundary value problem of the following coupled system of impulsive fractional differential equations at resonance:

$$\begin{cases} D_{0^+}^{\alpha} u(t) = f(t, v(t), D_{0^+}^{p} v(t)), \qquad D_{0^+}^{\beta} v(t) = g(t, u(t), D_{0^+}^{q} u(t)), \quad 0 < t < 1; \\ \Delta u(t_i) = A_i(v(t_i), D_{0^+}^{p} v(t_i)), \qquad \Delta D_{0^+}^{q} u(t_i) = B_i(v(t_i), D_{0^+}^{p} v(t_i)), \\ \quad i = 1, 2, \ldots, k; \\ \Delta v(t_i) = C_i(u(t_i), D_{0^+}^{q} u(t_i)), \qquad \Delta D_{0^+}^{p} v(t_i) = D_i(u(t_i), D_{0^+}^{q} u(t_i)), \\ \quad i = 1, 2, \ldots, k; \\ D_{0^+}^{\alpha-1} u(0) = \sum_{i=1}^{m} a_i D_{0^+}^{\alpha-1} u(\xi_i), \qquad u(1) = \sum_{i=1}^{m} b_i \eta_i^{2-\alpha} u(\eta_i); \\ D_{0^+}^{\beta-1} v(0) = \sum_{i=1}^{m} c_i D_{0^+}^{\beta-1} v(\zeta_i), \qquad v(1) = \sum_{i=1}^{m} d_i \theta_i^{2-\beta} v(\theta_i), \end{cases} \tag{1.1}$$

where $1 < \alpha, \beta < 2$, $\alpha - q \ge 1$, $\beta - p \ge 1$ and $0 < \xi_1 < \xi_2 < \cdots < \xi_m < 1$, $0 < \eta_1 < \eta_2 < \cdots < \eta_m < 1$, $0 < \zeta_1 < \zeta_2 < \cdots < \zeta_m < 1$, $0 < \theta_1 < \theta_2 < \cdots < \theta_m < 1$. $f, g : [0,1] \times \mathbb{R}^2 \to \mathbb{R}$ satisfy Carathéodory conditions, $A_i, B_i, C_i, D_i : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$. $\Delta w(t_i) = w(t_i^+) - w(t_i^-)$, $\Delta D_{0^+}^{r} w(t_i) = D_{0^+}^{r} w(t_i^+) - D_{0^+}^{r} w(t_i^-)$, here $w \in \{u, v\}$, $r \in \{p, q\}$, $w(t_i^+)$ and $w(t_i^-)$ denote the right and left limits of $w(t)$ at $t = t_i$, respectively, and the fractional derivative is understood in the Riemann-Liouville sense. k, m, a_i, b_i, c_i, d_i $(i = 1, 2, \ldots, m)$ are fixed constant satisfying $\sum_{i=1}^{m} a_i = \sum_{i=1}^{m} b_i = \sum_{i=1}^{m} c_i = \sum_{i=1}^{m} d_i = 1$ and $\sum_{i=1}^{m} b_i \eta_i = \sum_{i=1}^{m} d_i \theta_i = 1$.

The coupled system (1.1) happens to be at resonance in the sense that the associated linear homogeneous coupled system

$$\begin{cases} D_{0^+}^{\alpha}u(t) = 0, \qquad D_{0^+}^{\beta}v(t) = 0, \quad 0 < t < 1; \\ D_{0^+}^{\alpha-1}u(0) = \sum_{i=1}^{m} a_i D_{0^+}^{\alpha-1}u(\xi_i), \qquad u(1) = \sum_{i=1}^{m} b_i \eta_i^{2-\alpha}u(\eta_i); \\ D_{0^+}^{\beta-1}v(0) = \sum_{i=1}^{m} c_i D_{0^+}^{\beta-1}v(\zeta_i), \qquad v(1) = \sum_{i=1}^{m} d_i \theta_i^{2-\beta}v(\theta_i) \end{cases}$$

has $(u(t), v(t)) = (h_1 t^{\alpha-1} + h_2 t^{\alpha-2}, h_3 t^{\beta-1} + h_4 t^{\beta-2})$, $c_i \in \mathbb{R}$, $i = 1, 2, 3, 4$ as a nontrivial solution. To solve this interesting and important problem and to overcome the difficulties caused by the impulses, we will construct some Banach spaces, then we shall obtain the new solvability results for the coupled system (1.1) with the help of a coincidence degree continuation theorem. The main contributions of this work are Lemma 2.1 and Lemma 3.1 in Section 3 since the calculations are disposed well.

The plan of this work is organized as follows. Section 2 contains some necessary notations, definitions and lemmas that will be used in the sequel. In Section 3, we establish a theorem on the existence of solutions for the coupled system (1.1) based on the coincidence degree theory due to Mawhin [30, 31].

2 Background materials and preliminaries

For the convenience of the readers, we recall some notations and an abstract existence theorem [30, 31].

Let Y, Z be real Banach spaces, $L : \mathrm{dom}(L) \subset Y \to Z$ be a Fredholm map of index zero and $P : Y \to Y$, $Q : Z \to Z$ be continuous projectors such that $\mathrm{Im}(P) = \mathrm{Ker}(L)$, $\mathrm{Ker}(Q) = \mathrm{Im}(L)$ and $Y = \mathrm{Ker}(L) \oplus \mathrm{Ker}(P)$, $Z = \mathrm{Im}(L) \oplus \mathrm{Im}(Q)$. It follows that $L|\,\mathrm{dom}(L) \cap \mathrm{Ker}(P) : \mathrm{dom}(L) \cap \mathrm{Ker}(P) \to \mathrm{Im}(L)$ is invertible. We denote the inverse of the map by K_P. If Ω is an open bounded subset of Y such that $\mathrm{dom}(L) \cap \Omega \neq \emptyset$, the map $N : Y \to Z$ will be called L-compact on $\overline{\Omega}$ if $QN(\overline{\Omega})$ is bounded and $K_P(I - Q)N : \overline{\Omega} \to Y$ is compact.

The main tool we used is Theorem 2.4 of [30].

Theorem 2.1 *Let L be a Fredholm operator of index zero, and let N be L-compact on $\overline{\Omega}$. Assume that the following conditions are satisfied*:

(i) $Lx \neq \lambda Nx$ *for every* $(x, \lambda) \in [(\mathrm{dom}(L) \setminus \mathrm{Ker}(L)) \cap \partial\Omega] \times (0, 1)$;

(ii) $Nx \notin \mathrm{Im}(L)$ *for every* $x \in \mathrm{Ker}(L) \cap \partial\Omega$;

(iii) $\deg(QN|_{\mathrm{Ker}(L)}, \Omega \cap \mathrm{Ker}(L), 0) \neq 0$, *where $Q : Z \to Z$ is a projection as above with* $\mathrm{Im}(L) = \mathrm{Ker}(Q)$.

Then the equation $Lx = Nx$ has at least one solution in $\mathrm{dom}(L) \cap \overline{\Omega}$.

Now, we present some basic knowledge and definitions about fractional calculus theory, which can be found in the recent works [13, 16, 32].

Definition 2.1 The fractional integral of order $\alpha > 0$ of a function $y : (0, \infty) \to \mathbb{R}$ is defined by

$$I_{0^+}^{\alpha}y(t) = \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} y(s)\,ds,$$

provided the right-hand side is pointwise defined on $(0, \infty)$.

Definition 2.2 The fractional derivative of order $\alpha > 0$ of a function $y : (0, \infty) \to \mathbb{R}$ is defined by

$$D_{0^+}^{\alpha} y(t) = \frac{1}{\Gamma(n-\alpha)} \left(\frac{d}{dt}\right)^n \int_0^t (t-s)^{n-\alpha-1} y(s)\, ds,$$

where $n = [\alpha] + 1$, provided the right-hand side is pointwise defined on $(0, \infty)$.

Remark 2.1 It can be directly verified that the Riemann-Liouville fractional integration and fractional differentiation operators of the power functions t^{μ} yield power functions of the same form. For $\alpha \geq 0$, $\mu \geq -1$, we have

$$I_{0^+}^{\alpha} t^{\mu} = \frac{\Gamma(\mu+1)}{\Gamma(\mu+\alpha+1)} t^{\mu+\alpha}, \qquad D_{0^+}^{\alpha} t^{\mu} = \frac{\Gamma(\mu+1)}{\Gamma(\mu-\alpha+1)} t^{\mu-\alpha} \quad (\mu \geq \alpha). \tag{2.1}$$

Proposition 2.1 [17] *Assume that* $y \in C(0,1) \cap L[0,1]$ *with a fractional derivative of order* $\alpha > 0$ *that belongs to* $C(0,1) \cap L[0,1]$. *Then*

$$I_{0^+}^{\alpha} D_{0^+}^{\alpha} y(t) = y(t) + c_1 t^{\alpha-1} + c_2 t^{\alpha-2} + \cdots + c_N t^{\alpha-N} \tag{2.2}$$

for some $c_i \to \mathbb{R}$, $i = 1, 2, \ldots, N$, *where* N *is the smallest integer grater than or equal to* α.

Proposition 2.2 [32] *If* $\alpha > 0$, $\beta > 0$, *then the equation*

$$\left(I_{0^+}^{\alpha} I_{0^+}^{\beta} y\right)(t) = \left(I_{0^+}^{\alpha+\beta} y\right)(t)$$

is satisfied for a continuous function y.

If $\alpha > 0$, $m \in N$ *and* $D = d/dt$, *the fractional derivatives* $(D_{0^+}^{\alpha} y)(t)$ *and* $(D_{0^+}^{\alpha+m} y)(t)$ *exist, then*

$$\left(D^m D_{0^+}^{\alpha} y\right)(t) = \left(D_{0^+}^{\alpha+m} y\right)(t).$$

If $\alpha > 0$, *then the equation*

$$\left(D_{0^+}^{\alpha} I_{0^+}^{\alpha} y\right)(t) = y(t)$$

is satisfied for a continuous function y.

If $\alpha > \beta > 0$, *then the relation*

$$\left(D_{0^+}^{\beta} I_{0^+}^{\alpha} y\right)(t) = \left(I_{0^+}^{\alpha-\beta} y\right)(t)$$

holds for a continuous function y.

Let $C[0,1] = \{u | u \text{ is continuous in } [0,1]\}$ with the norm $\|u\|_{\infty} = \max_{t \in [0,1]} |u(t)|$ and

$$PC[0,1] = \bigl\{x : x \in C(t_i, t_{i+1}], \text{ there exist } x(t_i^-) \text{ and } x(t_i^+) \text{ with } x(t_i^-) = x(t_i), i = 1, 2, \ldots, k-1\bigr\}$$

with the norm $\|x\|_{PC} = \sup_{t\in[0,1]} |x(t)|$. Denote

$$Y_1 = \{u|u_\alpha \in PC[0,1], D_{0^+}^{\alpha-1}u \in PC[0,1]\},$$
$$Y_2 = \{v|v_\beta \in PC[0,1], D_{0^+}^{\beta-1}v \in PC[0,1]\},$$

where $u_\alpha(t) = t^{2-\alpha}u(t)$, $v_\beta(t) = t^{2-\beta}v(t)$ with the norm

$$\|u\|_{Y_1} = \max\{\|u_\alpha\|_{PC}, \|D_{0^+}^{\alpha-1}u\|_{PC}\},$$
$$\|v\|_{Y_2} = \max\{\|v_\beta\|_{PC}, \|D_{0^+}^{\beta-1}v\|_{PC}\}.$$

Thus, $Y = Y_1 \times Y_2$ is a Banach space with the norm defined by $\|(u,v)\|_Y = \max\{\|u\|_{Y_1}, \|v\|_{Y_2}\}$.

Set $Z_1 = Z_2 = PC[0,1] \times \mathbb{R}^{2k}$ equipped with the norm

$$\|x\|_{Z_1} = \max\{\|y\|_{PC}, |c|\}, \quad \forall x = (y,c) \in Z_1,$$

thus $Z = Z_1 \times Z_2$ is a Banach space with the norm defined by $\|(x,y)\|_Z = \max\{\|x\|_{Z_1}, \|y\|_{Z_2}\}$.

Define the operator $L : Y \to Z$, $L(u,v) = (L_1u, L_2v)$, $\operatorname{dom}(L) = \operatorname{dom}(L_1) \times \operatorname{dom}(L_2)$, where

$$L_1u = (D_{0^+}^{\alpha}u, \Delta u(t_1), \dots, \Delta u(t_k), \qquad \Delta D_{0^+}^{q}u(t_1), \dots, \Delta D_{0^+}^{q}u(t_k)),$$
$$L_2v = (D_{0^+}^{\beta}v, \Delta v(t_1), \dots, \Delta v(t_k), \qquad \Delta D_{0^+}^{p}v(t_1), \dots, \Delta D_{0^+}^{p}v(t_k)),$$

with

$$\operatorname{dom}(L_1) = \left\{u \in Y_1 \middle| D_{0^+}^{\alpha-1}u(0) = \sum_{i=1}^{m} a_i D_{0^+}^{\alpha-1}u(\xi_i), u(1) = \sum_{i=1}^{m} b_i \eta_i^{2-\alpha}u(\eta_i)\right\},$$
$$\operatorname{dom}(L_2) = \left\{v \in Y_2 \middle| D_{0^+}^{\beta-1}v(0) = \sum_{i=1}^{m} c_i D_{0^+}^{\beta-1}v(\zeta_i), v(1) = \sum_{i=1}^{m} d_i \theta_i^{2-\beta}v(\theta_i)\right\}.$$

Let $N : Y \to Z$ be defined as $N(u,v) = (N_1v, N_2u)$, where

$$\begin{aligned}N_1v = \big(&f(t, v(t), D_{0^+}^{p}v(t)),\ A_1(v(t_1), D_{0^+}^{p}v(t_1)), \dots, A_k(v(t_k), D_{0^+}^{p}v(t_k)),\\ &B_1(v(t_1), D_{0^+}^{p}v(t_1)), \dots, B_k(v(t_k), D_{0^+}^{p}v(t_k))\big),\\ N_2u = \big(&g(t, u(t), D_{0^+}^{q}u(t)), C_1(u(t_1), D_{0^+}^{q}u(t_1)), \dots, C_k(u(t_k), D_{0^+}^{q}u(t_k)),\\ &D_1(u(t_1), D_{0^+}^{q}u(t_1)), \dots, D_k(u(t_k), D_{0^+}^{q}u(t_k))\big).\end{aligned}$$

Then the coupled system of boundary value problem (1.1) can be written as

$$L(u,v) = N(u,v).$$

For the sake of simplicity, we define the operators $T_1, T_2 : Z_1 \to Z_1$ for $X = (x, \delta_1, \dots, \delta_k, \omega_1, \dots, \omega_k)$ as follows:

$$T_1X = \left(\sum_{i=1}^{m} a_i\left(\int_0^{\xi_i} x(s)\,ds + \Gamma(\alpha-q)\sum_{t_i<\xi_i}\omega_i t_i^{q+1-\alpha}\right), 0, \dots, 0,\right), \tag{2.3}$$

$$\begin{aligned} T_2X = \Biggl(&\int_0^1 (1-s)^{\alpha-1}x(s)\,ds - \sum_{i=1}^m b_i\eta_i^{2-\alpha}\int_0^{\eta_i}(\eta_i-s)^{\alpha-1}x(s)\,ds \\ &+ \Gamma(\alpha)\sum_{i=1}^m b_i \sum_{\eta_i<t_i<1}\delta_i t_i^{2-\alpha} + \Gamma(\alpha-q)\sum_{i=1}^m b_i\eta_i \sum_{\eta_i<t_i<1}\omega_i t_i^{q+1-\alpha} \\ &- \Gamma(\alpha-q)\sum_{i=1}^m b_i \sum_{\eta_i<t_i<1}\omega_i t_i^{q+2-\alpha}, 0,\ldots,0\Biggr). \end{aligned} \tag{2.4}$$

By the same way, we define the operators $T_3, T_4 : Z_2 \to Z_2$ for $Y = (y, \rho_1, \ldots, \rho_k, \tau_1, \ldots, \tau_k)$ as follows:

$$T_3Y = \Biggl(\sum_{i=1}^m c_i\biggl(\int_0^{\zeta_i} y(s)\,ds + \Gamma(\beta-p)\sum_{t_i<\zeta_i}\tau_i t_i^{p+1-\beta}\biggr), 0,\ldots,0,\Biggr), \tag{2.5}$$

$$\begin{aligned} T_4Y = \Biggl(&\int_0^1 (1-s)^{\beta-1}y(s)\,ds - \sum_{i=1}^m d_i\theta_i^{2-\beta}\int_0^{\theta_i}(\theta_i-s)^{\beta-1}y(s)\,ds \\ &+ \Gamma(\beta)\sum_{i=1}^m d_i \sum_{\theta_i<t_i<1}\rho_i t_i^{2-\beta} + \Gamma(\beta-p)\sum_{i=1}^m d_i\theta_i \sum_{\theta_i<t_i<1}\tau_i t_i^{p+1-\beta} \\ &- \Gamma(\beta-p)\sum_{i=1}^m d_i \sum_{\theta_i<t_i<1}\tau_i t_i^{p+2-\beta}, 0,\ldots,0\Biggr). \end{aligned} \tag{2.6}$$

In what follows, we present the following lemmas which will be used to prove our main results.

Lemma 2.1 *If the following condition is satisfied*:
(H_1) $\sigma_1 = \begin{vmatrix} \sigma_{11} & \sigma_{12} \\ \sigma_{13} & \sigma_{14} \end{vmatrix} \neq 0$, $\sigma_2 = \begin{vmatrix} \sigma_{21} & \sigma_{22} \\ \sigma_{23} & \sigma_{24} \end{vmatrix} \neq 0$, *where*

$$\begin{aligned} &\sigma_{11} = \frac{1}{\alpha(\alpha+1)}\Biggl(1-\sum_{i=1}^m b_i\eta_i^3\Biggr), \qquad \sigma_{12} = \frac{1}{2}\sum_{i=1}^m a_i\xi_i^2, \\ &\sigma_{13} = \frac{1}{\alpha}\Biggl(1-\sum_{i=1}^m b_i\eta_i^2\Biggr), \qquad \sigma_{14} = \sum_{i=1}^m a_i\xi_i; \\ &\sigma_{21} = \frac{1}{\beta(\beta+1)}\Biggl(1-\sum_{i=1}^m d_i\theta_i^3\Biggr), \qquad \sigma_{22} = \frac{1}{2}\sum_{i=1}^m c_i\zeta_i^2, \\ &\sigma_{23} = \frac{1}{\beta}\Biggl(1-\sum_{i=1}^m d_i\theta_i^2\Biggr), \qquad \sigma_{24} = \sum_{i=1}^m c_i\zeta_i, \end{aligned}$$

then $L : \mathrm{dom}(L) \subset Y \to Z$ *is a Fredholm operator of index zero. Moreover,* $\mathrm{Ker}(L) = \mathrm{Ker}(L_1) \times \mathrm{Ker}(L_2)$, *where*

$$\begin{aligned} &\mathrm{Ker}(L_1) = \bigl\{h_1t^{\alpha-1} + h_2t^{\alpha-2}, h_1, h_2 \in \mathbb{R}\bigr\}, \\ &\mathrm{Ker}(L_2) = \bigl\{h_3t^{\beta-1} + h_4t^{\beta-2}, h_3, h_4 \in \mathbb{R}\bigr\} \end{aligned} \tag{2.7}$$

and $\operatorname{Im}(L) = \operatorname{Im}(L_1) \times \operatorname{Im}(L_2)$, *here*

$$\begin{aligned}\operatorname{Im}(L_1) &= \{X = (x, \delta_1, \ldots, \delta_k, \omega_1, \ldots, \omega_k) | D_{0^+}^{\alpha} u(t) = x(t), \Delta u(t_i) = \delta_i, \Delta D_{0^+}^{q} u(t_i) = \omega_i\} \\ &= \{X = (x, \delta_1, \ldots, \delta_k, \omega_1, \ldots, \omega_k) | T_1 X = T_2 X = (0, 0, \ldots, 0)\}, \qquad (2.8)\\ \operatorname{Im}(L_2) &= \{Y = (y, \rho_1, \ldots, \rho_k, \tau_1, \ldots, \tau_k) | D_{0^+}^{\beta} v(t) = y(t), \Delta v(t_i) = \rho_i, \Delta D_{0^+}^{p} v(t_i) = \tau_i\} \\ &= \{Y = (y, \rho_1, \ldots, \rho_k, \tau_1, \ldots, \tau_k) | T_3 Y = T_4 Y = (0, 0, \ldots, 0)\}. \qquad (2.9)\end{aligned}$$

Proof It is clear that (2.7) holds. For $(u, v) \in \operatorname{Ker}(L)$, we have $L(u, v) = (L_1 u, L_2 v) = (0, 0)$, *i.e.*, $L_1 u = 0$, $L_2 v = 0$, then $u \in \operatorname{Ker}(L_1)$, $v \in \operatorname{Ker}(L_2)$, so $\operatorname{Ker}(L) = \operatorname{Ker}(L_1) \times \operatorname{Ker}(L_2)$. Similarly, it is not difficult to see that $\operatorname{Im}(L) = \operatorname{Im}(L_1) \times \operatorname{Im}(L_2)$. Next, we will show that (2.8) and (2.9) hold.

If $Z_1 = (z_1, \delta_1, \ldots, \delta_k, \omega_1, \ldots, \omega_k) \in \operatorname{Im}(L_1)$, $Z_2 = (z_2, \rho_1, \ldots, \rho_k, \tau_1, \ldots, \tau_k) \in \operatorname{Im}(L_2)$, then there exist $u \in \operatorname{dom}(L_1)$ and $v \in \operatorname{dom}(L_2)$ such that

$$\begin{cases} D_{0^+}^{\alpha} u(t) = z_1(t), \\ \Delta u(t_i) = \delta_i, \\ \Delta D_{0^+}^{q} u(t_i) = \omega_i, \end{cases} \qquad \begin{cases} D_{0^+}^{\beta} v(t) = z_2(t), \\ \Delta v(t_i) = \rho_i, \\ \Delta D_{0^+}^{p} v(t_i) = \tau_i \end{cases} \qquad (2.10)$$

and

$$D_{0^+}^{\alpha-1} u(0) = \sum_{i=1}^{m} a_i D_{0^+}^{\alpha-1} u(\xi_i), \qquad u(1) = \sum_{i=1}^{m} b_i \eta_i^{2-\alpha} u(\eta_i), \qquad (2.11)$$

$$D_{0^+}^{\beta-1} v(0) = \sum_{i=1}^{m} c_i D_{0^+}^{\beta-1} v(\zeta_i), \qquad v(1) = \sum_{i=1}^{m} d_i \theta_i^{2-\beta} v(\theta_i). \qquad (2.12)$$

Proposition 2.1 together with (2.10)-(2.12) gives that

$$\begin{aligned} u(t) &= \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} z_1(s)\, ds + \left(h_1 + \frac{\Gamma(\alpha - q)}{\Gamma(\alpha)} \sum_{t_i < t} \omega_i t_i^{q+1-\alpha}\right) t^{\alpha-1} \\ &\quad + \left(h_2 + \sum_{t_i < t} \delta_i t_i^{2-\alpha} - \frac{\Gamma(\alpha - q)}{\Gamma(\alpha)} \sum_{t_i < t} \omega_i t_i^{2+q-\alpha}\right) t^{\alpha-2}, \qquad (2.13) \\ v(t) &= \frac{1}{\Gamma(\beta)} \int_0^t (t-s)^{\beta-1} z_2(s)\, ds + \left(h_3 + \frac{\Gamma(\beta - p)}{\Gamma(\beta)} \sum_{t_i < t} \tau_i t_i^{p+1-\beta}\right) t^{\beta-1} \\ &\quad + \left(h_4 + \sum_{t_i < t} \rho_i t_i^{2-\beta} - \frac{\Gamma(\beta - p)}{\Gamma(\beta)} \sum_{t_i < t} \tau_i t_i^{2+p-\beta}\right) t^{\beta-2}. \qquad (2.14) \end{aligned}$$

Substituting the boundary condition $D_{0^+}^{\alpha-1} u(0) = \sum_{i=1}^{m} a_i D_{0^+}^{\alpha-1} u(\xi_i)$ into (2.13), one has

$$\sum_{i=1}^{m} a_i \left(\int_0^{\xi_i} z_1(s)\, ds + \Gamma(\alpha - q) \sum_{t_i < \xi_i} \omega_i t_i^{q+1-\alpha}\right) = 0, \qquad (2.15)$$

and substituting the boundary condition $u(1) = \sum_{i=1}^{m} b_i \eta_i^{2-\alpha} u(\eta_i)$ into (2.13), one has

$$\begin{aligned}&\int_0^1 (1-s)^{\alpha-1} z_1(s)\,ds - \sum_{i=1}^{m} b_i \eta_i^{2-\alpha} \int_0^{\eta_i} (\eta_i - s)^{\alpha-1} z_1(s)\,ds + \Gamma(\alpha) \sum_{i=1}^{m} b_i \sum_{\eta_i<t_i<1} \delta_i t_i^{2-\alpha} \\ &\quad + \Gamma(\alpha-q) \sum_{i=1}^{m} b_i \eta_i \sum_{\eta_i<t_i<1} \omega_i t_i^{q+1-\alpha} - \Gamma(\alpha-q) \sum_{i=1}^{m} b_i \sum_{\eta_i<t_i<1} \omega_i t_i^{q+2-\alpha} = 0. \qquad (2.16)\end{aligned}$$

By the same way, if we substitute the condition (2.12) into (2.14), then we can obtain that

$$\sum_{i=1}^{m} c_i \left(\int_0^{\zeta_i} z_2(s)\,ds + \Gamma(\beta-p) \sum_{t_i<\zeta_i} \tau_i t_i^{p+1-\beta} \right) = 0, \qquad (2.17)$$

and

$$\begin{aligned}&\int_0^1 (1-s)^{\beta-1} z_2(s)\,ds - \sum_{i=1}^{m} d_i \theta_i^{2-\beta} \int_0^{\theta_i} (\theta_i - s)^{\beta-1} z_2(s)\,ds + \Gamma(\beta) \sum_{i=1}^{m} d_i \sum_{\theta_i<t_i<1} \rho_i t_i^{2-\beta} \\ &\quad + \Gamma(\beta-p) \sum_{i=1}^{m} d_i \theta_i \sum_{\theta_i<t_i<1} \tau_i t_i^{p+1-\beta} - \Gamma(\beta-p) \sum_{i=1}^{m} d_i \sum_{\theta_i<t_i<1} \tau_i t_i^{p+2-\beta} = 0. \qquad (2.18)\end{aligned}$$

Conversely, if (2.15)-(2.18) hold, set

$$\begin{aligned}u(t) &= \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} z_1(s)\,ds + \left(\frac{\Gamma(\alpha-q)}{\Gamma(\alpha)} \sum_{t_i<t} \omega_i t_i^{q+1-\alpha} \right) t^{\alpha-1} \\ &\quad \times \left(\sum_{t_i<t} \delta_i t_i^{2-\alpha} - \frac{\Gamma(\alpha-q)}{\Gamma(\alpha)} \sum_{t_i<t} \omega_i t_i^{2+q-\alpha} \right) t^{\alpha-2}, \\ v(t) &= \frac{1}{\Gamma(\beta)} \int_0^t (t-s)^{\beta-1} z_2(s)\,ds + \left(\frac{\Gamma(\beta-p)}{\Gamma(\beta)} \sum_{t_i<t} \tau_i t_i^{p+1-\beta} \right) t^{\beta-1} \\ &\quad \times \left(\sum_{t_i<t} \rho_i t_i^{2-\beta} - \frac{\Gamma(\beta-p)}{\Gamma(\beta)} \sum_{t_i<t} \tau_i t_i^{2+p-\beta} \right) t^{\beta-2}.\end{aligned}$$

It is easy to check that the above u, v satisfy equation (2.10)-(2.12). Thus, (2.8) and (2.9) hold.

Define the operator $Q : Z \to Z$, $Q(x,y) = (Q_1 x, Q_2 y)$ with $Q_1 X = Q_{11} X + Q_{12} X \cdot t$, $Q_2 Y = Q_{21} Y + Q_{22} Y \cdot t$, here

$$\begin{aligned}Q_{11} X &= \frac{1}{\sigma_1} (\sigma_{11} T_1 X - \sigma_{12} T_2 X) \triangleq (\bar{x}, 0, \dots, 0), \\ Q_{12} X &= -\frac{1}{\sigma_1} (\sigma_{13} T_1 X - \sigma_{14} T_2 X) \triangleq -\big(x^*, 0, \dots, 0\big), \\ Q_{21} Y &= \frac{1}{\sigma_2} (\sigma_{21} T_3 Y - \sigma_{22} T_4 Y) \triangleq (\bar{y}, 0, \dots, 0), \\ Q_{22} Y &= -\frac{1}{\sigma_2} (\sigma_{23} T_3 Y - \sigma_{24} T_4 Y) \triangleq -\big(y^*, 0, \dots, 0\big).\end{aligned}$$

In what follows, we will show that Q_1 and Q_2 are linear projectors. By some direct computations, we have

$$\begin{aligned}
T_1(Q_{11}X) &= \left(\sum_{i=1}^m a_i\xi_i \cdot \overline{x}, 0, \ldots, 0\right) = \sum_{i=1}^m a_i\xi_i \cdot Q_{11}X = \sigma_{14} \cdot Q_{11}X, \\
T_2(Q_{11}X) &= \left(\overline{x}\left[\int_0^1 (1-s)^{\alpha-1}\,ds - \sum_{i=1}^m b_i\eta_i^{2-\alpha}\int_0^{\eta_i}(\eta_i - s)^{\alpha-1}\,ds\right], 0, \ldots, 0\right) \\
&= \frac{1}{\alpha}\left(1 - \sum_{i=1}^m b_i\eta_i^2\right)(\overline{x}, 0, \ldots, 0) = \sigma_{13} \cdot Q_{11}X, \\
T_1(Q_{12}X \cdot t) &= -\left(x^*\sum_{i=1}^m a_i\int_0^{\xi_i} s\,ds, 0, \ldots, 0\right) = \frac{1}{2}\sum_{i=1}^m a_i\xi_i^2 \cdot Q_{12}X = \sigma_{12} \cdot Q_{12}X, \\
T_2(Q_{12}X \cdot t) &= -\left(x^*\left[\int_0^1 (1-s)^{\alpha-1}s\,ds - \sum_{i=1}^m b_i\eta_i^{2-\alpha}\int_0^{\eta_i}(\eta_i - s)^{\alpha-1}s\,ds\right], 0, \ldots, 0\right) \\
&= \frac{1}{\alpha(\alpha+1)}\left(1 - \sum_{i=1}^m b_i\eta_i^3\right) \cdot Q_{12}X = \sigma_{11} \cdot Q_{12}X, \\
Q_{11}(Q_{11}X) &= \frac{1}{\sigma_1}\big(\sigma_{11}T_1(Q_{11}X) - \sigma_{12}T_2(Q_{11}X)\big) = \frac{\sigma_{11}\sigma_{14} - \sigma_{12}\sigma_{13}}{\sigma_1}Q_{11}X = Q_{11}X, \\
Q_{12}(Q_{11}X) &= -\frac{1}{\sigma_1}\big(\sigma_{13}T_1(Q_{11}X) - \sigma_{14}T_2(Q_{11}X)\big) = -\frac{\sigma_{13}\sigma_{14} - \sigma_{14}\sigma_{13}}{\sigma_1}Q_{11}X = 0, \\
Q_{11}(Q_{12}X \cdot t) &= \frac{1}{\sigma_1}\big(\sigma_{11}T_1(Q_{12}X \cdot t) - \sigma_{12}T_2(Q_{12}X \cdot t)\big) = -\frac{\sigma_{11}\sigma_{12} - \sigma_{12}\sigma_{11}}{\sigma_1}Q_{12}X = 0, \\
Q_{12}(Q_{12}X \cdot t) &= -\frac{1}{\sigma_1}\big(\sigma_{13}T_1(Q_{12}X \cdot t) - \sigma_{14}T_2(Q_{12}X \cdot t)\big) \\
&= -\frac{\sigma_{13}\sigma_{12} - \sigma_{14}\sigma_{11}}{\sigma_1}Q_{12}X = Q_{12}X.
\end{aligned}$$

As a result,

$$\begin{aligned}
Q_1(Q_1X) &= Q_1(Q_{11}X + Q_{12}X \cdot t) \\
&= Q_{11}(Q_{11}X + Q_{12}X \cdot t) + Q_{12}(Q_{11}X + Q_{12}X \cdot t) \cdot t \\
&= Q_{11}^2X + Q_{11}(Q_{12}X \cdot t) + \big[Q_{12}(Q_{11}X) + Q_{12}(Q_{12}X \cdot t)\big] \cdot t \\
&= Q_{11}X + Q_{12}X \cdot t = Q_1X.
\end{aligned}$$

Similarly, we can see that $Q_2(Q_2Y) = Q_2Y$. Then for $(X, Y) \in Z$, we have $Q^2(X, Y) = Q(Q_1X, Q_2Y) = (Q_1^2X, Q_2^2Y) = (Q_1X, Q_2Y) = Q(X, Y)$. It means that the operator $Q : Z \to Z$ is a projector.

Now, we show that $\operatorname{Ker}(Q) = \operatorname{Im}(L)$. Obviously, $\operatorname{Im}(L) \subseteq \operatorname{Ker}(Q)$. On the other hand, for $(X, Y) \in \operatorname{Ker}(Q)$, then $Q(X, Y) = (0, 0)$ implies that

$$\begin{cases} \sigma_{11}T_1X - \sigma_{12}T_2X = (0, 0, \ldots, 0), \\ \sigma_{13}T_1X - \sigma_{14}T_2X = (0, 0, \ldots, 0), \end{cases} \qquad \begin{cases} \sigma_{21}T_1Y - \sigma_{22}T_2Y = (0, 0, \ldots, 0), \\ \sigma_{23}T_1Y - \sigma_{24}T_2Y = (0, 0, \ldots, 0). \end{cases}$$

The condition (H_1) guarantees that $T_1X = T_2X = (0,0,\dots,0)$, $T_3Y = T_4Y = (0,0,\dots,0)$, then $(X,Y) \in \operatorname{Im}(L)$. Hence, $\operatorname{Ker}(Q) = \operatorname{Im}(L)$.

For $W \in Z$, let $W = (W - QW) + QW$. Then $W - QW \in \operatorname{Ker}(Q) = \operatorname{Im}(L)$, $QW \in \operatorname{Im}(Q)$, it means that $Z = \operatorname{Im}(L) + \operatorname{Im}(Q)$. Moreover, $\operatorname{Ker}(Q) = \operatorname{Im}(L)$ gives that $\operatorname{Im}(L) \cap \operatorname{Im}(Q) = (0,0)$. Thus, $Z = \operatorname{Im}(L) \oplus \operatorname{Im}(Q)$. Then $\dim \operatorname{Ker}(L) = \dim \operatorname{Im}(Q) = \operatorname{codim} \operatorname{Im}(L) = 4$, L is a Fredholm map of index zero. □

Define the operator $P : Y \to Y$ with $P(u,v) = (P_1u, P_2v)$, here $P_1 : Y_1 \to Y_1$, $P_2 : Y_2 \to Y_2$ are defined as follows:

$$P_1u = \frac{1}{\Gamma(\alpha)} D_{0^+}^{\alpha-1}u(0) \cdot t^{\alpha-1} + \lim_{t\to 0} t^{2-\alpha}u(t) \cdot t^{\alpha-2},$$
$$P_2v = \frac{1}{\Gamma(\beta)} D_{0^+}^{\beta-1}v(0) \cdot t^{\beta-1} + \lim_{t\to 0} t^{2-\beta}v(t) \cdot t^{\beta-2}.$$

Moreover, we define $K_P : \operatorname{Im}(L) \to \operatorname{dom}(L) \cap \operatorname{Ker}(P)$ as $K_P(X,Y) = (K_{P_1}X, K_{P_2}Y)$, where $K_{P_i} : \operatorname{Im}(L_i) \to \operatorname{dom}(L_i) \cap \operatorname{Ker}(P_i)$, $i = 1,2$ is defined as follows:

$$\begin{aligned} K_{P_1}X &= K_{P_1}(x, \delta_1, \dots, \delta_k, \omega_1, \dots, \omega_k) \\ &= \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1}x(s)\,ds + \frac{\Gamma(\alpha-q)}{\Gamma(\alpha)} \sum_{t_i<t} \omega_i t_i^{q+1-\alpha} \cdot t^{\alpha-1} \\ &\quad + \left(\sum_{t_i<t} \delta_i t_i^{2-\alpha} - \frac{\Gamma(\alpha-q)}{\Gamma(\alpha)} \sum_{t_i<t} \omega_i t_i^{q+2-\alpha} \right) \cdot t^{\alpha-2}, \\ K_{P_2}Y &= K_{P_2}(y, \rho_1, \dots, \rho_k, \tau_1, \dots, \tau_k) \\ &= \frac{1}{\Gamma(\beta)} \int_0^t (t-s)^{\beta-1}y(s)\,ds + \frac{\Gamma(\beta-p)}{\Gamma(\beta)} \sum_{t_i<t} \tau_i t_i^{p+1-\beta} \cdot t^{\beta-1} \\ &\quad + \left(\sum_{t_i<t} \rho_i t_i^{2-\beta} - \frac{\Gamma(\beta-p)}{\Gamma(\beta)} \sum_{t_i<t} \tau_i t_i^{p+2-\beta} \right) \cdot t^{\beta-2}. \end{aligned}$$

Lemma 2.2 *Assume that $\Omega \subset Y$ is an open bounded subset with $\operatorname{dom}(L) \cap \overline{\Omega} \neq \emptyset$, then N is L-compact on $\overline{\Omega}$.*

Proof Obviously, $\operatorname{Im}(P) = \operatorname{Ker}(L)$. By a direct computation, we have that

$$\begin{aligned} P_1^2u &= \frac{1}{\Gamma(\alpha)} D_{0^+}^{\alpha-1}P_1u(0) \cdot t^{\alpha-1} + \lim_{t\to 0} t^{2-\alpha}P_1u(t) \cdot t^{\alpha-2} \\ &= \frac{1}{\Gamma(\alpha)} D_{0^+}^{\alpha-1}u(0) \cdot t^{\alpha-1} + \lim_{t\to 0} t^{2-\alpha}u(t) \cdot t^{\alpha-2} = P_1u. \end{aligned}$$

Similarly, $P_2^2v = P_2v$. This gives that $P^2(u,v) = P(P_1u, P_2v) = (P_1^2u, P_2^2v) = (P_1u, P_2v) = P(u,v)$, that is to say, the operator P is a linear projector. It is easy to check from $w = (w - Pw) + Pw$ that $Y = \operatorname{Ker}(P) + \operatorname{Ker}(L)$. Moreover, we can see that $\operatorname{Ker}(P) \cap \operatorname{Ker}(L) = (0,0)$. Thus, $Y = \operatorname{Ker}(P) \oplus \operatorname{Ker}(L)$.

In what follows, we will show that K_P defined above is the inverse of $L|_{\operatorname{dom}(L)\cap\operatorname{Ker}(P)}$.

If $(X,Y) \in \operatorname{Im}(L)$, then $L_1K_{P_1}X = X$, $L_2K_{P_2}Y = Y$, which gives that

$$LK_P(X,Y) = (L_1K_{P_1}X, L_2K_{P_2}Y) = (X,Y).$$

On the other hand, for $(u,v) \in \operatorname{dom}(L) \cap \operatorname{Ker}(P)$, we have

$$\begin{aligned}(K_{P_1}L_1)u(t) &= K_{P_1}\big(D_{0^+}^{\alpha}u(t), \delta_1, \dots, \delta_k, \omega_1, \dots, \omega_k\big)\\ &= u(t) + \left(h_1 + \frac{\Gamma(\alpha-q)}{\Gamma(\alpha)}\sum_{t_i<t}\omega_i t_i^{q+1-\alpha}\right)\cdot t^{\alpha-1}\\ &\quad + \left(h_2 + \sum_{t_i<t}\delta_i t_i^{2-\alpha} - \frac{\Gamma(\alpha-q)}{\Gamma(\alpha)}\sum_{t_i<t}\omega_i t_i^{q+2-\alpha}\right)\cdot t^{\alpha-2}.\end{aligned}$$

Since $u \in K_{P_1}$ and $K_{P_1}L_1u \in \operatorname{Ker}(P_1)$, then

$$\lim_{t\to 0} t^{2-\alpha}u(t) = D_{0^+}^{\alpha-1}u(0) = 0, \tag{2.19}$$

$$\lim_{t\to 0} t^{2-\alpha}K_{P_1}L_1u(t) = D_{0^+}^{\alpha-1}K_{P_1}L_1u(0) = 0. \tag{2.20}$$

By some calculations, (2.19) and (2.20) imply that

$$\begin{aligned}&h_1 + \frac{\Gamma(\alpha-q)}{\Gamma(\alpha)}\sum_{t_i<t}\omega_i t_i^{q+1-\alpha} = 0,\\ &h_2 + \sum_{t_i<t}\delta_i t_i^{2-\alpha} - \frac{\Gamma(\alpha-q)}{\Gamma(\alpha)}\sum_{t_i<t}\omega_i t_i^{q+2-\alpha} = 0.\end{aligned}$$

It means that $K_{P_1}L_1u = u$. Analogously, $K_{P_2}L_2v = v$. Thus, $K_PL(u,v) = (K_{P_1}L_1u, K_{P_2}L_2v) = (u,v)$. So, K_P is the inverse of $L|_{\operatorname{dom}(L)\cap\operatorname{Ker}(P)}$.

Finally, we show that N is L-compact on $\overline{\Omega}$. Denote $Q_1N_1v = (v^*, 0, \dots, 0)$, $Q_2N_2u = (u^*, 0, \dots, 0)$, where

$$\begin{aligned}Q_1N_1v &= \frac{\sigma_{11}-\sigma_{13}t}{\sigma_1}T_1(N_1v) - \frac{\sigma_{12}-\sigma_{14}t}{\sigma_1}T_2(N_1v),\\ Q_2N_2u &= \frac{\sigma_{21}-\sigma_{23}t}{\sigma_2}T_3(N_2u) - \frac{\sigma_{22}-\sigma_{24}t}{\sigma_2}T_4(N_2u).\end{aligned}$$

Then we can see that

$$K_P(I-Q)N(u,v) = K_P(I-Q)(N_1v, N_2u) = \big(K_{P_1}(I-Q_1)N_1v, K_{P_2}(I-Q_2)N_2u\big),$$

where

$$\begin{aligned}&K_{P_1}(I-Q_1)N_1v\\ &\quad= I^{\alpha}\big(f\big(t,v,D_{0^+}^{p}v\big) - v^*\big) + \frac{\Gamma(\alpha-q)}{\Gamma(\alpha)}\sum_{t_i<t}B_i t_i^{q+1-\alpha}\cdot t^{\alpha-1}\\ &\qquad + \left(\sum_{t_i<t}A_i t_i^{2-\alpha} - \frac{\Gamma(\alpha-q)}{\Gamma(\alpha)}\sum_{t_i<t}B_i t_i^{q+2-\alpha}\right)\cdot t^{\alpha-2}\\ &\quad= \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}f\big(s,v(s),D_{0^+}^{p}v(s)\big)\,ds\\ &\qquad - \frac{t^{\alpha}(\sigma_{11}(1+\alpha)-\sigma_{13}t)}{\sigma_1\Gamma(2+\alpha)}\sum_{i=1}^{m}a_i\left(\int_0^{\xi_i} f\big(s,v(s),D_{0^+}^{p}v(s)\big)\,ds + \Gamma(\alpha-q)\sum_{t_i<\xi_i}B_i t_i^{q+1-\alpha}\right)\end{aligned}$$

$$
\begin{aligned}
&+ \frac{t^{\alpha}(\sigma_{12}(1+\alpha)-\sigma_{14}t)}{\sigma_1\Gamma(2+\alpha)}\bigg[\int_0^1 (1-s)^{\alpha-1} f\big(s,v(s),D_{0^+}^p v(s)\big)\,ds \\
&- \sum_{i=1}^m b_i\eta_i^{2-\alpha}\int_0^{\eta_i}(\eta_i-s)^{\alpha-1} f\big(t,v(s),D_{0^+}^p v(s)\big)\,ds\bigg] \\
&+ \frac{t^{\alpha}(\sigma_{12}(1+\alpha)-\sigma_{14}t)}{\sigma_1\Gamma(2+\alpha)}\bigg[\Gamma(\alpha-q)\sum_{i=1}^m b_i\eta_i\sum_{\eta_i<t_i<1}B_i t_i^{q+1-\alpha} + \Gamma(\alpha)\sum_{i=1}^m b_i\sum_{\eta_i<t_i<1}A_i t_i^{2-\alpha} \\
&- \Gamma(\alpha-q)\sum_{i=1}^m b_i\sum_{\eta_i<t_i<1}B_i t_i^{q+2-\alpha}\bigg] \\
&+ \frac{\Gamma(\alpha-q)}{\Gamma(\alpha)}\sum_{t_i<t}B_i t_i^{q+1-\alpha}\cdot t^{\alpha-1} + \bigg(\sum_{t_i<t}A_i t_i^{2-\alpha} - \frac{\Gamma(\alpha-q)}{\Gamma(\alpha)}\sum_{t_i<t}B_i t_i^{q+2-\alpha}\bigg)\cdot t^{\alpha-2}.
\end{aligned}
$$

So, we can see that Q_1N_1 is bounded and $K_{P_1}(I-Q_1)N_1$ is uniformly bounded.

For $0 \le t_1 < t_2 \le 1$, we have

$$
\begin{aligned}
&\bigg|\frac{1}{\Gamma(\alpha)}\int_0^{t_2}(t_2-s)^{\alpha-1} f\big(s,v(s),D_{0^+}^p v(s)\big)\,ds - \frac{1}{\Gamma(\alpha)}\int_0^{t_1}(t_1-s)^{\alpha-1} f\big(s,v(s),D_{0^+}^p v(s)\big)\,ds\bigg| \\
&\quad\le \frac{\sup_{t\in[0,1]}|f(t,v(t),D_{0^+}^p v(t))|}{\Gamma(\alpha)}\bigg|\int_0^{t_1}\big[(t_2-s)^{\alpha-1}-(t_1-s)^{\alpha-1}\big]\,ds + \int_{t_1}^{t_2}(t_2-s)^{\alpha-1}\,ds\bigg| \\
&\quad= \frac{\sup_{t\in[0,1]}|f(t,v(t),D_{0^+}^p v(t))|}{\Gamma(\alpha)}\big|t_2^{\alpha}-t_1^{\alpha}\big|, \qquad (2.21)
\end{aligned}
$$

$$
\begin{aligned}
&\bigg|\sum_{t_i<t_2}B_i t_i^{q+1-\alpha}\cdot t_2^{\alpha-1} - \sum_{t_i<t_1}B_i t_i^{q+1-\alpha}\cdot t_1^{\alpha-1}\bigg| \\
&\quad= \bigg|\bigg(\sum_{t_i<t_1}B_i t_i^{q+1-\alpha} + \sum_{t_1\le t_i<t_2}B_i t_i^{q+1-\alpha}\bigg)\cdot t_2^{\alpha-1} - \sum_{t_i<t_1}B_i t_i^{q+1-\alpha}\cdot t_1^{\alpha-1}\bigg| \\
&\quad\le \bigg(\sum_{t_i<t_1}B_i t_i^{q+1-\alpha}\bigg)\cdot\big|t_2^{\alpha-1}-t_1^{\alpha-1}\big| + \bigg(\sum_{t_1\le t_i<t_2}B_i t_i^{q+1-\alpha}\bigg)\cdot t_2^{\alpha-1}. \qquad (2.22)
\end{aligned}
$$

The equicontinuity of $t^{\alpha}, t^{\alpha+1}$ together with (2.21) and (2.22) gives that $|K_{P_1}(I-Q_1)N_1v(t_2) - K_{P_1}(I-Q_1)N_1v(t_1)| \to 0$ as $t_2 \to t_1$, which yields that $K_{P_1}(I-Q_1)N_1$ is equicontinuous. By the Ascoli-Arzela theorem, we can see that $K_{P_1}(I-Q_1)N_1$ is compact. By the same way, Q_2N_2 is bounded and $K_{P_2}(I-Q_2)N_2$ is compact. Since $QN(u,v) = Q(N_1v, N_2u) = (Q_1N_1v, Q_2N_2u)$ and $K_P(I-Q)N(u,v) = (K_{P_1}(I-Q_1)N_1v, K_{P_2}(I-Q_2)N_2u)$, then QN is bounded and $K_P(I-Q)N$ is compact. This means that N is L-compact on $\overline{\Omega}$. □

3 Main results

In this section, we present the existence results of the coupled system (1.1). To do this, we need the following hypotheses.

(H_2) There exist functions $\varphi_i, \psi_i, \gamma_i \in C[0,1]$, $i = 1,2$, such that

$$
\begin{aligned}
&\big|f(t,x,y)\big| \le \big|\varphi_1(t)\big| + t^{2-\alpha}\big|\psi_1(t)\big|\cdot|x| + \big|\gamma_1(t)\big|\cdot|y|, \\
&\big|g(t,x,y)\big| \le \big|\varphi_2(t)\big| + t^{2-\beta}\big|\psi_2(t)\big|\cdot|x| + \big|\gamma_2(t)\big|\cdot|y|,
\end{aligned}
$$

where ψ_i, γ_i $(i = 1, 2)$ satisfy

$$\frac{16\|\psi_1\|_\infty\|\psi_2\|_\infty}{\Gamma(\alpha)\Gamma(\beta)} + \frac{8\|\gamma_1\|_\infty\|\psi_2\|_\infty}{\Gamma(\alpha)\Gamma(\beta-p)} < 1, \qquad \frac{16\|\psi_1\|_\infty\|\psi_2\|_\infty}{\Gamma(\alpha)\Gamma(\beta)} + \frac{8\|\gamma_2\|_\infty\|\psi_1\|_\infty}{\Gamma(\beta)\Gamma(\alpha-q)} < 1,$$
$$\frac{4}{\Gamma(\alpha-q)\Gamma(\beta-p)}\big(\|\psi_1\|_\infty A' + \|\gamma_1\|_\infty\big)\big(\|\psi_2\|_\infty A + \|\gamma_2\|_\infty\big) < 1,$$

here

$$A = \left(\frac{16\|\psi_1\|_\infty\|\gamma_2\|_\infty}{\Gamma(\alpha)\Gamma(\beta)} + \frac{8\|\gamma_1\|_\infty\|\gamma_2\|_\infty}{\Gamma(\alpha)\Gamma(\beta-p)}\right)\Big/\left(1 - \frac{16\|\psi_1\|_\infty\|\psi_2\|_\infty}{\Gamma(\alpha)\Gamma(\beta)} - \frac{8\|\gamma_1\|_\infty\|\psi_2\|_\infty}{\Gamma(\alpha)\Gamma(\beta-p)}\right),$$
$$A' = \left(\frac{16\|\psi_2\|_\infty\|\gamma_1\|_\infty}{\Gamma(\alpha)\Gamma(\beta)} + \frac{8\|\gamma_1\|_\infty\|\gamma_2\|_\infty}{\Gamma(\beta)\Gamma(\alpha-q)}\right)\Big/\left(1 - \frac{16\|\psi_1\|_\infty\|\psi_2\|_\infty}{\Gamma(\alpha)\Gamma(\beta)} - \frac{8\|\gamma_2\|_\infty\|\psi_1\|_\infty}{\Gamma(\beta)\Gamma(\alpha-q)}\right).$$

(H_3) For $(u, v) \in \operatorname{dom}(L)$, there exist constants $e_i \in (0, 1)$ $(i = 0, 1, 2)$, $M_i > 0$ $(i = 1, 2)$ such that

(1) if either $|u(t)| > M_1$ or $|v(t)| > M_1$ for $\forall t \in [e_0, e_1]$, then either $T_2N_1v(t) \neq 0$ or $T_4N_2u(t) \neq 0$;
(2) if either $|D^q_{0^+}u(t)| > M_2$ or $|D^p_{0^+}v(t)| > M_2$, $\forall t \in [e_2, 1]$, then either $T_1N_1v(t) \neq 0$ or $T_3N_2u(t) \neq 0$.

(H_4) For $(u, v) \in \operatorname{Ker}(L)$, there exist constants $g_i > 0$ $(i = 1, 2)$ such that if either $|h_1| \geq g_1$ or $|h_2| \geq g_1$, either $|h_3| \geq g_2$ or $|h_4| \geq g_2$, then either (1) or (2) holds, where

(1)

$$h_1T_1N_1\big(h_3t^{\beta-1} + h_4t^{\beta-2}\big) + h_2T_2N_1\big(h_3t^{\beta-1} + h_4t^{\beta-2}\big) = (s_1, 0, \dots, 0),$$
$$h_3T_1N_2\big(h_1t^{\alpha-1} + h_2t^{\alpha-2}\big) + h_4T_2N_2\big(h_1t^{\alpha-1} + h_2t^{\alpha-2}\big) = (s_2, 0, \dots, 0),$$

here s_1, s_2 are positive constants;

(2)

$$h_1T_1N_1\big(h_3t^{\beta-1} + h_4t^{\beta-2}\big) + h_2T_2N_1\big(h_3t^{\beta-1} + h_4t^{\beta-2}\big) = (s_3, 0, \dots, 0),$$
$$h_3T_1N_2\big(h_1t^{\alpha-1} + h_2t^{\alpha-2}\big) + h_4T_2N_2\big(h_1t^{\alpha-1} + h_2t^{\alpha-2}\big) = (s_4, 0, \dots, 0),$$

here s_3, s_4 are negative constants.

Lemma 3.1 *Suppose that* (H_2)-(H_3) *hold. Then the set*

$$\Omega_1 = \big\{(u, v) \in \operatorname{dom}(L)\setminus \operatorname{Ker}(L) | L(u, v) = \lambda N(u, v), \lambda \in (0, 1)\big\}$$

is bounded in Y.

Proof For $(u, v) \in \Omega_1$, by $L(u, v) = (L_1u, L_2v) = \lambda N(u, v) = (\lambda N_1v, \lambda N_2u)$ and $(u, v) \in \operatorname{dom}(L)$, we have

$$u(t) = \frac{\lambda}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}f\big(s, v(s), D^p_{0^+}v(s)\big)\,ds + \left(h_1 + \frac{\lambda\Gamma(\alpha-q)}{\Gamma(\alpha)}\sum_{t_i<t}B_it_i^{q+1-\alpha}\right)t^{\alpha-1}$$
$$+ \left(h_2 + \lambda\sum_{t_i<t}A_it_i^{2-\alpha} - \lambda\frac{\Gamma(\alpha-q)}{\Gamma(\alpha)}\sum_{t_i<t}B_it_i^{2+q-\alpha}\right)t^{\alpha-2}, \tag{3.1}$$

$$D_{0^+}^q u(t) = \frac{\lambda}{\Gamma(\alpha-q)} \int_0^t (t-s)^{\alpha-q-1} f\big(s, v(s), D_{0^+}^p v(s)\big)\, ds + \left(\frac{\Gamma(\alpha)}{\Gamma(\alpha-q)} h_1 + \lambda \sum_{t_i<t} B_i t_i^{q+1-\alpha} \right) t^{\alpha-q-1}, \tag{3.2}$$

$$v(t) = \frac{\lambda}{\Gamma(\beta)} \int_0^t (t-s)^{\beta-1} g\big(s, u(s), D_{0^+}^q u(s)\big)\, ds + \left(h_3 + \frac{\lambda\Gamma(\beta-p)}{\Gamma(\beta)} \sum_{t_i<t} D_i t_i^{p+1-\beta} \right) t^{\beta-1} + \left(h_4 + \lambda \sum_{t_i<t} C_i t_i^{2-\beta} - \lambda \frac{\Gamma(\beta-p)}{\Gamma(\beta)} \sum_{t_i<t} D_i t_i^{2+p-\beta} \right) t^{\beta-2}, \tag{3.3}$$

$$D_{0^+}^p v(t) = \frac{\lambda}{\Gamma(\beta-p)} \int_0^t (t-s)^{\beta-p-1} g\big(s, v(s), D_{0^+}^q u(s)\big)\, ds + \left(\frac{\Gamma(\beta)}{\Gamma(\beta-p)} h_3 + \lambda \sum_{t_i<t} D_i t_i^{p+1-\beta} \right) t^{\beta-p-1}. \tag{3.4}$$

Since $N_1 v \in \operatorname{Im}(L_1)$, $N_2 u \in \operatorname{Im}(L_2)$, then $T_1(N_1 v) = T_2(N_1 v) = 0$, $T_3(N_2 u) = T_4(N_2 u) = 0$. Then we can see, from the condition (H_3), that there exist constants $e_0, e_*, e^* \in (0,1)$ such that $|u(t)| \le M_1$, $|v(t)| \le M_1$ for $t_* \in [e_0, e_*]$ and $|D_{0^+}^q u(t)| \le M_2$, $|D_{0^+}^p v(t)| \le M_2$ for $t^* \in [e^*, 1]$. So, we can see from (3.1) and (3.2) that

$$\begin{aligned} |h_1| &\le \frac{\Gamma(\alpha-q)}{\Gamma(\alpha) t^{\alpha-q-1}} M_2 + \frac{1}{\Gamma(\alpha)} \sup_{t\in[0,1]} \big| f\big(t, v(t), D_{0^+}^p v(t)\big)\big| + \frac{\Gamma(\alpha-q)}{\Gamma(\alpha)} \sum_{t_i<t^*} |B_i| t_i^{q+1-\alpha} \\ &\le \frac{\Gamma(\alpha-q)}{\Gamma(\alpha) {e^*}^{\alpha-q-1}} M_2 + \frac{1}{\Gamma(\alpha)} \sup_{t\in[0,1]} \big| f\big(t, v(t), D_{0^+}^p v(t)\big)\big| + \frac{\Gamma(\alpha-q)}{\Gamma(\alpha)} \sum_{i=1}^k |B_i| t_i^{q+1-\alpha} \end{aligned} \tag{3.5}$$

and

$$\begin{aligned} |h_2| &\le M_1 + \frac{1}{\Gamma(\alpha)} \sup_{t\in[0,1]} \big| f\big(t, v(t), D_{0^+}^p v(t)\big)\big| + |h_1| + \frac{\Gamma(\alpha-q)}{\Gamma(\alpha)} \left(\sum_{t_i<t_*} |B_i| t_i^{q+1-\alpha} \right) t_* \\ &\quad + \sum_{t_i<t_*} |A_i| t_i^{2-\alpha} + \frac{\Gamma(\alpha-q)}{\Gamma(\alpha)} \sum_{t_i<t_*} |B_i| t_i^{q+2-\alpha} \\ &\le M_1 + \frac{1}{\Gamma(\alpha)} \sup_{t\in[0,1]} \big| f\big(t, v(t), D_{0^+}^p v(t)\big)\big| + |h_1| \\ &\quad + \sum_{i=1}^k |A_i| t_i^{2-\alpha} + \frac{\Gamma(\alpha-q)}{\Gamma(\alpha)} \sum_{i=1}^k |B_i| t_i^{q+1-\alpha} (1+t_i). \end{aligned} \tag{3.6}$$

Then for $t \in [0,1]$ and $u \in \operatorname{dom}(L_2)$, we have

$$\begin{aligned} \big| D_{0^+}^{\alpha-1} u(t) \big| &= \left| \lambda \int_0^t f\big(s, v(s), D_{0^+}^p v(s)\big)\, ds + \Gamma(\alpha) h_1 + \lambda \Gamma(\alpha-q) \sum_{t_i<t} B_i t_i^{q+1-\alpha} \right| \\ &\le \big| f\big(t, v(t), D_{0^+}^p v(t)\big)\big| + \Gamma(\alpha)|h_1| + \Gamma(\alpha-q) \sum_{i=1}^k |B_i| t_i^{q+1-\alpha} \\ &\le 2\big[\|\varphi_1\|_\infty + \|\psi_1\|_\infty \cdot \|v_\beta\|_{PC} + \|\gamma_1\|_\infty \cdot \big\| D_{0^+}^p v \big\|_{PC} \big] + \frac{\Gamma(\alpha-q)}{{e^*}^{\alpha-q-1}} M_2 \end{aligned}$$

$$+ 2\Gamma(\alpha - q)\sum_{i=1}^{k}|B_i|t_i^{q+1-\alpha}$$

$$\triangleq 2\big[\|\varphi_1\|_\infty + \|\psi_1\|_\infty \cdot \|v_\beta\|_{PC} + \|\gamma_1\|_\infty \cdot \big\|D_{0^+}^{p}v\big\|_{PC}\big] + R_1, \tag{3.7}$$

$$\begin{aligned}\big|t^{2-\alpha}u(t)\big| &\le \frac{1}{\Gamma(\alpha)}\sup_{t\in[0,1]}\big|f\big(t,v(t),D_{0^+}^{p}v(t)\big)\big| + |h_1| + \frac{\Gamma(\alpha - q)}{\Gamma(\alpha)}\sum_{i=1}^{k}|B_i|t_i^{q+1-\alpha} + |h_2| \\ &\quad + \sum_{i=1}^{k}|A_i|t_i^{2-\alpha} + \frac{\Gamma(\alpha - q)}{\Gamma(\alpha)}\sum_{i=1}^{k}|B_i|t_i^{q+2-\alpha} \\ &\le M_1 + \frac{2\Gamma(\alpha - q)}{\Gamma(\alpha)e^{*\alpha-q-1}}M_2 + \frac{4}{\Gamma(\alpha)}\big[\|\varphi_1\|_\infty + \|\psi_1\|_\infty \cdot \|v_\beta\|_{PC} \\ &\quad + \|\gamma_1\|_\infty \cdot \big\|D_{0^+}^{p}v\big\|_{PC}\big] + 2\sum_{i=1}^{k}|A_i|t_i^{2-\alpha} + \frac{\Gamma(\alpha - q)}{\Gamma(\alpha)}\sum_{i=1}^{k}|B_i|t_i^{q+1-\alpha}(4 + 2t_i) \\ &\triangleq \frac{4}{\Gamma(\alpha)}\big[\|\varphi_1\|_\infty + \|\psi_1\|_\infty \cdot \|v_\beta\|_{PC} + \|\gamma_1\|_\infty \cdot \big\|D_{0^+}^{p}v\big\|_{PC}\big] + R_2.\end{aligned} \tag{3.8}$$

$$\begin{aligned}\big|D_{0^+}^{q}u(t)\big| &\le \frac{2}{\Gamma(\alpha - q)}\big[\|\varphi_1\|_\infty + \|\psi_1\|_\infty \cdot \|v_\beta\|_{PC} + \|\gamma_1\|_\infty \cdot \big\|D_{0^+}^{p}v\big\|_{PC}\big] \\ &\quad + M_2 + 2\sum_{i=1}^{k}|B_i|t_i^{q+1-\alpha} \\ &\triangleq \frac{2}{\Gamma(\alpha - q)}\big[\|\varphi_1\|_\infty + \|\psi_1\|_\infty \cdot \|v_\beta\|_{PC} + \|\gamma_1\|_\infty \cdot \big\|D_{0^+}^{p}v\big\|_{PC}\big] + R_3.\end{aligned} \tag{3.9}$$

Similarly, for $u \in \operatorname{dom}(L_2)$, we have that

$$\big|D_{0^+}^{\beta-1}v(t)\big| \triangleq 2\big[\|\varphi_2\|_\infty + \|\psi_2\|_\infty \cdot \|u_\alpha\|_{PC} + \|\gamma_2\|_\infty \cdot \big\|D_{0^+}^{q}u\big\|_{PC}\big] + R_1', \tag{3.10}$$

$$\big|t^{2-\beta}v(t)\big| \triangleq \frac{4}{\Gamma(\beta)}\big[\|\varphi_2\|_\infty + \|\psi_2\|_\infty \cdot \|u_\alpha\|_{PC} + \|\gamma_2\|_\infty \cdot \big\|D_{0^+}^{q}u\big\|_{PC}\big] + R_2', \tag{3.11}$$

$$\big|D_{0^+}^{p}v(t)\big| \le \frac{2}{\Gamma(\beta - p)}\big[\|\varphi_2\|_\infty + \|\psi_2\|_\infty \cdot \|u_\alpha\|_{PC} + \|\gamma_2\|_\infty \cdot \big\|D_{0^+}^{q}u\big\|_{PC}\big] + R_3'. \tag{3.12}$$

Substitute (3.11) and (3.12) into (3.8), then we have

$$\begin{aligned}\|u_\alpha\|_{PC} &\le \frac{4}{\Gamma(\alpha)}\|\varphi_1\|_\infty + \frac{4}{\Gamma(\alpha)}\|\psi_1\|_\infty \\ &\quad \times \left(\frac{4}{\Gamma(\beta)}\big[\|\varphi_2\|_\infty + \|\psi_2\|_\infty \cdot \|u_\alpha\|_{PC} + \|\gamma_2\|_\infty \cdot \big\|D_{0^+}^{q}u\big\|_{PC}\big] + R_2'\right) \\ &\quad + \frac{4}{\Gamma(\alpha)}\|\gamma_1\|_\infty\left(\frac{2}{\Gamma(\beta - p)}\big[\|\varphi_2\|_\infty + \|\psi_2\|_\infty \cdot \|u_\alpha\|_{PC} \right. \\ &\quad \left. + \|\gamma_2\|_\infty \cdot \big\|D_{0^+}^{q}u\big\|_{PC}\big] + R_3'\right) \\ &= \left(\frac{16\|\psi_1\|_\infty}{\Gamma(\alpha)\Gamma(\beta)} + \frac{8\|\gamma_1\|_\infty}{\Gamma(\alpha)\Gamma(\beta - p)}\right)\big[\|\varphi_2\|_\infty + \|\psi_2\|_\infty \cdot \|u_\alpha\|_{PC} \\ &\quad + \|\gamma_2\|_\infty \cdot \big\|D_{0^+}^{q}u\big\|_{PC}\big] \\ &\quad + \frac{4}{\Gamma(\alpha)}\big(\|\varphi_1\|_\infty + \|\psi_1\|_\infty R_2' + \|\gamma_1\|_\infty R_3'\big).\end{aligned} \tag{3.13}$$

It means that

$$\|u_\alpha\|_{PC} \le A\left\|D_{0^+}^q u\right\|_{PC} + B,$$

similarly,

$$\|v_\beta\|_{PC} \le A'\left\|D_{0^+}^p v\right\|_{PC} + B'.$$

Substituting the above two into (3.9) and (3.12), we can see that

$$\begin{aligned}\left\|D_{0^+}^q u\right\|_{PC} \le{}& \frac{2}{\Gamma(\alpha-q)}\left(\|\psi_1\|_\infty A' + \|\gamma_1\|_\infty\right)\cdot\left\|D_{0^+}^p v\right\|_{PC} \\ &+ \frac{2}{\Gamma(\alpha-q)}\left(\|\varphi_1\|_\infty + \|\psi_1\|_\infty B'\right)R_3 \end{aligned} \tag{3.14}$$

and

$$\begin{aligned}\left\|D_{0^+}^p v\right\|_{PC} \le{}& \frac{2}{\Gamma(\beta-p)}\left(\|\psi_2\|_\infty A + \|\gamma_2\|_\infty\right)\cdot\left\|D_{0^+}^q u\right\|_{PC} \\ &+ \frac{2}{\Gamma(\beta-p)}\left(\|\varphi_2\|_\infty + \|\psi_2\|_\infty B\right)R_3'. \end{aligned} \tag{3.15}$$

From the condition (H_2), (3.14) and (3.15) give that $\|D_{0^+}^q u\|_{PC}$ and $\|D_{0^+}^p v\|_{PC}$ are bounded, then $\|u_\alpha\|_{PC}$ and $\|v_\beta\|_{PC}$ are also bounded. Thus, by the definition of the norm on Y, $\|u\|_{Y_1}$ and $\|v\|_{Y_2}$ are bounded. That is, Ω_1 is bounded in Y. □

Lemma 3.2 *Suppose that the condition* (H_3) *holds. Then the set*

$$\Omega_2 = \left\{(u,v) | (u,v) \in \operatorname{Ker}(L), N(u,v) \in \operatorname{Im}(L)\right\}$$

is bounded in Y.

Proof For $(u,v) \in \operatorname{Ker}(L)$, we have that $(u,v) = (h_1 t^{\alpha-1} + h_2 t^{\alpha-2}, h_3 t^{\beta-1} + h_4 t^{\beta-2})$, where h_i, $i \in \{1,2,3,4\}$. Since $N(u,v) \in \operatorname{Im}(L)$, so we have

$$T_1 N_1\left(h_3 t^{\beta-1} + h_4 t^{\beta-2}\right) = T_2 N_1\left(h_3 t^{\beta-1} + h_4 t^{\beta-2}\right) = 0$$

and

$$T_3 N_2\left(h_1 t^{\alpha-1} + h_2 t^{\alpha-2}\right) = T_4 N_2\left(h_1 t^{\alpha-1} + h_2 t^{\alpha-2}\right) = 0.$$

From (H_3), there exist positive constants M', M'', e_0, e', e'' such that for $t' \in [e', 1]$,

$$\left|D_{0^+}^q u(t')\right| = \frac{\Gamma(\alpha)}{\Gamma(\alpha-q)}|h_1| t'^{\alpha-q-1} \le M',$$

which means that $|h_1| \le \frac{M'\Gamma(\alpha-q)}{\Gamma(\alpha)e'^{\alpha-q-1}}$. And for $t'' \in [e_0, e'']$,

$$\left|u(t'')\right| = \left|h_1 t''^{\alpha-1} + h_2 t''^{\alpha-2}\right| \le M'',$$

which means that $|h_2| = |u(t'')t''^{2-\alpha} + h_1 t''| \leq |u(t'')| + |h_1| \leq M'' + |h_1|$. So, we can see that for $t \in [0,1]$,

$$\left|t^{2-\alpha}u(t)\right| = |h_1 t + h_2| \leq |h_1| + |h_2|,$$
$$\left|D_{0^+}^q u(t)\right| \leq \frac{\Gamma(\alpha)}{\Gamma(\alpha - q)}|h_1|.$$

The above two arguments imply that $|u|_{Y_1}$ is bounded. In the same way, $|v|_{Y_2}$ is bounded. Thus, Ω_2 is bounded in Y. □

Lemma 3.3 *The set*

$$\Omega_3 = \left\{(u,v) \in \mathrm{Ker}(L) | \lambda J(u,v) + (1-\lambda)\theta QN(u,v) = (0,0,\ldots,0), \lambda \in [0,1]\right\}$$

is bounded in Y, where $J : \mathrm{Ker}(L) \to \mathrm{Im}(Q)$ is the linear isomorphism given by

$$J\left(h_1 t^{\alpha-1} + h_2 t^{\alpha-2}\right) = \left(\frac{\sigma_{11}h_1 - \sigma_{12}h_2}{\sigma_1} + \frac{(-\sigma_{13}h_1 + \sigma_{14}h_2)t}{\sigma_1}, 0, \ldots, 0\right),$$
$$J\left(h_3 t^{\beta-1} + h_4 t^{\beta-2}\right) = \left(\frac{\sigma_{21}h_3 - \sigma_{22}h_4}{\sigma_2} + \frac{(-\sigma_{23}h_3 + \sigma_{24}h_4)t}{\sigma_2}, 0, \ldots, 0\right)$$

and

$$\theta = \begin{cases} 1, & \textit{if } (\mathrm{H}_4)\ (1)\ \textit{hold}, \\ -1, & \textit{if } (\mathrm{H}_4)\ (2)\ \textit{hold}. \end{cases}$$

Proof For $(u,v) \in \mathrm{Ker}(L)$, set $u^* = h_1 t^{\alpha-1} + h_2 t^{\alpha-2}$, $v^* = h_3 t^{\beta-1} + h_4 t^{\beta-2}$, then $\lambda J(u,v) + (1-\lambda)\theta QN(u,v) = (0,0,\ldots,0)$ implies that

$$\lambda(h_1, 0, \ldots, 0) + (1-\lambda)\theta T_1 N_1\left(v^*\right) = (0,0,\ldots,0), \tag{3.16}$$
$$\lambda(h_2, 0, \ldots, 0) + (1-\lambda)\theta T_2 N_1\left(v^*\right) = (0,0,\ldots,0), \tag{3.17}$$
$$\lambda(h_3, 0, \ldots, 0) + (1-\lambda)\theta T_1 N_2\left(u^*\right) = (0,0,\ldots,0), \tag{3.18}$$
$$\lambda(h_4, 0, \ldots, 0) + (1-\lambda)\theta T_2 N_2\left(u^*\right) = (0,0,\ldots,0). \tag{3.19}$$

From (3.16) and (3.17), we have

$$\lambda\left(h_1^2 + h_2^2, 0, \ldots, 0\right) + (1-\lambda)\theta\left[h_1 T_1 N_1\left(v^*\right) + h_2 T_2 N_1\left(v^*\right)\right] = (0,0,\ldots,0),$$

the condition (H_4) gives that

$$\lambda\left(h_1^2 + h_2^2\right) = -(1-\lambda)\theta s < 0,$$

where

$$s = \begin{cases} s_1, & \text{if } (\mathrm{H}_4)\ (1)\ \text{hold}, \\ s_3, & \text{if } (\mathrm{H}_4)\ (2)\ \text{hold}, \end{cases}$$

which is a contradiction. As a result, there exist positive constants g_1, g_2 such that $|h_1| \le g_1$, $|h_2| \le g_2$. Similarly, from (3.18)-(3.19) and the second part of (1) or (2) of (H_4), there exist two positive constants g_3, g_4 such that $|h_3| \le g_3$, $|h_4| \le g_4$. It follows that $\|u^*\|_{Y_1}$, $\|v^*\|_{Y_1}$ are bounded, that is, Ω_3 is bounded in Y. □

Theorem 3.1 *Suppose that* (H_1)-(H_4) *hold. Then the problem* (1.1) *has at least one solution in* Y.

Proof Let Ω be a bounded open set of Y such that $\bigcup_{i=1}^{3} \overline{\Omega}_i \subset \Omega$. It follows from Lemma 2.2 that N is L-compact on $\overline{\Omega}$. By means of above Lemmas 3.1-3.3, one obtains that

(i) $L(u,v) \neq \lambda N(u,v)$ for every $((u,v),\lambda) \in [(\mathrm{dom}(L) \setminus \mathrm{Ker}(L)) \cap \partial\Omega] \times (0,1)$;

(ii) $N(u,v) \notin \mathrm{Im}(L)$ for every $(u,v) \in \mathrm{Ker}(L) \cap \partial\Omega$.

Then we need only to prove

(iii) $\deg(QN|_{\mathrm{Ker}(L)}, \Omega \cap \mathrm{Ker}(L), (0,0,\dots,0)) \neq 0$.

Take

$$H(u,v,\lambda) = \pm\lambda J + (1-\lambda)N(u,v).$$

According to Lemma 3.3, we know $H((u,v),\lambda) \neq (0,0,\dots,0)$ for all $(u,v) \in \partial\Omega \cap \mathrm{Ker}(L)$. Thus, the homotopy invariance property of degree theory gives that

$$\begin{aligned} \deg\bigl(QN|_{\mathrm{Ker}(L)}, \Omega \cap \mathrm{Ker}(L), (0,0,\dots,0)\bigr) &= \deg\bigl(H(\cdot,0), \Omega \cap \mathrm{Ker}(L), (0,0,\dots,0)\bigr) \\ &= \deg\bigl(H(\cdot,1), \Omega \cap \mathrm{Ker}(L), (0,0,\dots,0)\bigr) \\ &= \deg\bigl(\pm J, \Omega \cap \mathrm{Ker}(L), (0,0,\dots,0)\bigr) \neq 0. \end{aligned}$$

Then, by Theorem 2.1, $L(u,v) = N(u,v)$ has at least one solution in $\mathrm{dom}(L) \cap \overline{\Omega}$, *i.e.*, the problem (1.1) has at least one solution in Y, which completes the proof. □

4 An example

Example 4.1 Consider the following boundary value problem for coupled systems of impulsive fractional differential equations:

$$\begin{cases} D_{0^+}^{\frac{3}{2}} u(t) = f(t, v(t), D_{0^+}^{\frac{1}{6}} v(t)), & D_{0^+}^{\frac{4}{3}} v(t) = g(t, u(t), D_{0^+}^{\frac{1}{4}} u(t)), \quad 0 < t < 1; \\ \Delta u(\frac{3}{4}) = A_1(v(\frac{3}{4}), D_{0^+}^{\frac{1}{6}} v(\frac{3}{4})), & \Delta D_{0^+}^{\frac{1}{4}} u(\frac{3}{4}) = B_1(v(\frac{3}{4}), D_{0^+}^{\frac{1}{6}} v(\frac{3}{4})); \\ \Delta v(\frac{2}{3}) = C_1(u(\frac{2}{3}), D_{0^+}^{\frac{1}{4}} u(\frac{2}{3})), & \Delta D_{0^+}^{\frac{1}{6}} v(\frac{2}{3}) = D_1(u(\frac{2}{3}), D_{0^+}^{\frac{1}{4}} u(\frac{2}{3})); \\ D_{0^+}^{\frac{1}{2}} u(0) = 3D_{0^+}^{\frac{1}{2}} u(\frac{1}{6}) - 2D_{0^+}^{\frac{1}{2}} u(\frac{1}{4}), & u(1) = -\sqrt{5} u(\frac{1}{5}) + 2\sqrt{3} u(\frac{1}{3}); \\ D_{0^+}^{\frac{1}{3}} v(0) = 4D_{0^+}^{\frac{1}{3}} v(\frac{1}{5}) - 3D_{0^+}^{\frac{1}{3}} v(\frac{1}{4}), & v(1) = -3(\frac{1}{3})^{\frac{2}{3}} v(\frac{1}{3}) + 4(\frac{1}{2})^{\frac{2}{3}} v(\frac{1}{2}), \end{cases} \tag{4.1}$$

where

$$f(t,x,y) = \begin{cases} \frac{t^3+1}{10} + \frac{3}{2\pi} t \arctan y, & t \in [0, \frac{1}{6}], |y| > 1; \\ \frac{t^3+1}{10} + \frac{3}{8} t y^3, & t \in [0, \frac{1}{6}], |y| \le 1; \\ \frac{t^3+1}{10}, & t \in (\frac{1}{6}, \frac{1}{3}]; \\ \frac{t^3+1}{10} + \frac{1}{24}\sqrt{t} x^3, & t \in (\frac{1}{3}, 1], |x| \le 1; \\ \frac{t^3+1}{10} + \frac{1}{24}\sqrt{t} x, & t \in (\frac{1}{3}, 1], |x| > 1, \end{cases}$$

$$g(t,x,y)=\begin{cases}\frac{t^2+1}{40}+\frac{3}{20}ty & t\in[0,\frac{1}{5}],|y|>1;\\ \frac{t^2+1}{40}+\frac{3}{20}ty^3, & t\in[0,\frac{1}{5}],|y|\le 1;\\ \frac{t^2+1}{40}, & t\in(\frac{1}{5},\frac{1}{2}];\\ \frac{t^2+1}{40}+\frac{1}{20}t^{\frac{2}{3}}x^3, & t\in(\frac{1}{2},1],|x|\le 1;\\ \frac{t^2+1}{40}+\frac{1}{20}t^{\frac{2}{3}}x, & t\in(\frac{1}{2},1],|x|>1,\end{cases}$$

and

$$A_1(x,y)=\frac{1}{50}\sin x+\frac{1}{40}\arctan y,\qquad B_1(x,y)=\frac{1}{40}\sin x+\frac{1}{25}\arctan y,$$
$$C_1(x,y)=\frac{1}{800}\cos x+\frac{1}{160}\operatorname{arccot} y,\qquad D_1(x,y)=\frac{1}{160}\cos x+\frac{1}{800}\operatorname{arccot} y.$$

Due to the coupled problem (1.1), we have that $\alpha=\frac{3}{2}$, $\beta=\frac{4}{3}$, $p=\frac{1}{6}$, $q=\frac{1}{4}$, $a_1=3$, $a_2=-2$, $b_1=-5$, $b_2=6$, $c_1=4$, $c_2=-3$, $d_1=-3$, $d_2=4$. $\xi_1=\frac{1}{6}$, $\xi_2=\frac{1}{4}$; $\eta_1=\frac{1}{5}$, $\eta_2=\frac{1}{3}$; $\zeta_1=\frac{1}{5}$, $\zeta_2=\frac{1}{4}$; $\theta_1=\frac{1}{3}$, $\theta_2=\frac{1}{2}$. Obviously, $a_1+a_2=b_1+b_2=c_1+c_2=d_1+d_2=1$ and $b_1\eta_1+b_2\eta_2=d_1\theta_1+d_2\theta_2=1$. By direct calculation, we obtain that

$$\sigma_{11}=\frac{736}{3375},\qquad \sigma_{12}=-\frac{1}{48},\qquad \sigma_{13}=\frac{16}{45},\qquad \sigma_{14}=0,\qquad \sigma_1=\sigma_{11}\sigma_{14}-\sigma_{13}\sigma_{12}\neq 0;$$
$$\sigma_{21}=\frac{99}{504},\qquad \sigma_{22}=-\frac{11}{800},\qquad \sigma_{23}=\frac{1}{4},$$
$$\sigma_{24}=\frac{1}{20},\qquad \sigma_2=\sigma_{21}\sigma_{24}-\sigma_{23}\sigma_{22}\neq 0.$$

It is easy to see that

$$\bigl|f(t,x,y)\bigr|\le\bigl|\varphi_1(t)\bigr|+t^{2-\alpha}\bigl|\psi_1(t)\bigr|\cdot|x|+\bigl|\gamma_1(t)\bigr|\cdot|y|,$$
$$\bigl|g(t,x,y)\bigr|\le\bigl|\varphi_2(t)\bigr|+t^{2-\beta}\bigl|\psi_2(t)\bigr|\cdot|x|+\bigl|\gamma_2(t)\bigr|\cdot|y|,$$

where

$$\varphi_1=\frac{t^3+1}{10},\qquad \psi_1(t)=\begin{cases}0, & t\in[0,\frac{1}{3}];\\ \frac{1}{24}, & t\in(\frac{1}{3},1],\end{cases}\qquad \gamma_1(t)=\begin{cases}\frac{3}{2\pi}t, & t\in[0,\frac{1}{6}];\\ 0, & t\in(\frac{1}{6},1],\end{cases}$$
$$\varphi_2=\frac{t^2+1}{40},\qquad \psi_2(t)=\begin{cases}0, & t\in[0,\frac{1}{2}];\\ \frac{1}{20}, & t\in(\frac{1}{2},1],\end{cases}\qquad \gamma_2(t)=\begin{cases}\frac{3}{20}t, & t\in[0,\frac{1}{5}];\\ 0, & t\in(\frac{1}{5},1].\end{cases}$$

So, $\|\psi_1\|_\infty=\frac{1}{24}$, $\|\gamma_1\|_\infty=\frac{1}{4\pi}$, $\|\psi_2\|_\infty=\frac{1}{20}$, $\|\gamma_2\|_\infty=\frac{3}{100}$. And

$$\begin{aligned}\frac{16\|\psi_1\|_\infty\|\psi_2\|_\infty}{\Gamma(\alpha)\Gamma(\beta)}+\frac{8\|\gamma_1\|_\infty\|\psi_2\|_\infty}{\Gamma(\alpha)\Gamma(\beta-p)}&=\frac{1}{30\Gamma(\frac{3}{2})\Gamma(\frac{4}{3})}+\frac{1}{10\pi\Gamma(\frac{3}{2})\Gamma(\frac{7}{6})}\\&=0.0808362<1,\\ \frac{16\|\psi_1\|_\infty\|\psi_2\|_\infty}{\Gamma(\alpha)\Gamma(\beta)}+\frac{8\|\gamma_2\|_\infty\|\psi_1\|_\infty}{\Gamma(\beta)\Gamma(\alpha-q)}&=\frac{1}{30\Gamma(\frac{3}{2})\Gamma(\frac{4}{3})}+\frac{1}{100\Gamma(\frac{4}{3})\Gamma(\frac{5}{4})}\\&=0.0544752<1,\end{aligned}$$

$$\begin{aligned}&\frac{4}{\Gamma(\alpha-q)\Gamma(\beta-p)}\bigl(\|\psi_1\|_\infty A'+\|\gamma_1\|_\infty\bigr)\bigl(\|\psi_2\|_\infty A+\|\gamma_2\|_\infty\bigr)\\&\quad=0.0124261<1,\end{aligned}$$

where $A=0.0527672$, $A'=0.0110034$. Thus, the condition (H_2) holds.

Taking $M_1=1$, for any $v\in\operatorname{dom}(L_2)$, assume that $|D_{0^+}^{\frac{1}{6}}v(t)|>1$ holds for any $t\in[\frac{1}{12},\frac{1}{6}]$. Thus either $D_{0^+}^{\frac{1}{6}}v(t)>1$ or $D_{0^+}^{\frac{1}{6}}v(t)<-1$ for any $t\in[\frac{1}{12},\frac{1}{6}]$. If $D_{0^+}^{\frac{1}{6}}v(t)>1$, $t\in[\frac{1}{12},\frac{1}{6}]$, then

$$\begin{aligned}&\sum_{i=1}^m a_i\biggl(\int_0^{\xi_i}f\bigl(s,v(s),D_{0^+}^{\frac{1}{6}}v(s)\bigr)\,ds+\Gamma(\alpha-q)\sum_{t_i<\xi_i}B_it_i^{q+1-\alpha}\biggr)\\&\quad\ge\int_0^{\frac{1}{12}}\biggl(\frac{s^3+1}{10}-\frac{3}{4}s\biggr)\,ds+\int_{\frac{1}{12}}^{\frac{1}{6}}\biggl(\frac{s^3+1}{10}+\frac{3}{8}s\biggr)\,ds-2\int_{\frac{1}{6}}^{\frac{1}{4}}\biggl(\frac{s^3+1}{10}\biggr)\,ds\\&\quad=0.00116464>0.\end{aligned}$$

If $D_{0^+}v(t)<-1$, $t\in[\frac{1}{12},\frac{1}{6}]$, then

$$\begin{aligned}&\sum_{i=1}^m a_i\biggl(\int_0^{\xi_i}f\bigl(s,v(s),D_{0^+}^{\frac{1}{6}}v(s)\bigr)\,ds+\Gamma(\alpha-q)\sum_{t_i<\xi_i}B_it_i^{q+1-\alpha}\biggr)\\&\quad\le\int_0^{\frac{1}{12}}\biggl(\frac{s^3+1}{10}+\frac{3}{4}s\biggr)\,ds+\int_{\frac{1}{12}}^{\frac{1}{6}}\biggl(\frac{s^3+1}{10}-\frac{3}{8}s\biggr)\,ds-2\int_{\frac{1}{6}}^{\frac{1}{4}}\biggl(\frac{s^3+1}{10}\biggr)\,ds\\&\quad=-0.00143953<0.\end{aligned}$$

Similarly, assume that $|D_{0^+}^{\frac{1}{4}}v(t)|>1$ holds for any $t\in[\frac{1}{10},\frac{1}{5}]$. Thus either $D_{0^+}^{\frac{1}{4}}v(t)>1$ or $D_{0^+}^{\frac{1}{4}}v(t)<-1$ for any $t\in[\frac{1}{10},\frac{1}{5}]$. If $D_{0^+}^{\frac{1}{6}}v(t)>1$, $t\in[\frac{1}{10},\frac{1}{5}]$, then

$$\begin{aligned}&\sum_{i=1}^m c_i\biggl(\int_0^{\zeta_i}g\bigl(s,u(s),D_{0^+}^{\frac{1}{4}}u(s)\bigr)\,ds+\Gamma(\beta-p)\sum_{t_i<\zeta_i}D_it_i^{p+1-\beta}\biggr)\\&\quad\ge\int_0^{\frac{1}{10}}\biggl(\frac{s^2+1}{40}-\frac{3}{20}s\biggr)\,ds+\int_{\frac{1}{10}}^{\frac{1}{5}}\biggl(\frac{s^2+1}{40}+\frac{3}{20}s\biggr)\,ds-3\int_{\frac{1}{5}}^{\frac{1}{4}}\biggl(\frac{s^2+1}{40}\biggr)\,ds\\&\quad=0.00262604>0.\end{aligned}$$

If $D_{0^+}^{\frac{1}{6}}v(t)<-1$, $t\in[\frac{1}{10},\frac{1}{5}]$, then

$$\begin{aligned}&\sum_{i=1}^m c_i\biggl(\int_0^{\zeta_i}g\bigl(s,u(s),D_{0^+}^{\frac{1}{4}}u(s)\bigr)\,ds+\Gamma(\beta-p)\sum_{t_i<\zeta_i}D_it_i^{p+1-\beta}\biggr)\\&\quad\le\int_0^{\frac{1}{10}}\biggl(\frac{s^2+1}{40}+\frac{3}{20}s\biggr)\,ds+\int_{\frac{1}{10}}^{\frac{1}{5}}\biggl(\frac{s^2+1}{40}-\frac{3}{20}s\biggr)\,ds-3\int_{\frac{1}{5}}^{\frac{1}{4}}\biggl(\frac{s^2+1}{40}\biggr)\,ds\\&\quad=-0.000373958<0.\end{aligned}$$

So, from the above arguments, the first part of the condition (H_3) is true for $M_1=1$, $t\in[\frac{1}{12},\frac{1}{6}]$.

Taking $M_{21} = 16$, assume that $|v| > 16$ holds for any $t \in [\frac{1}{3}, 1]$. Then either $v > 16$ or $v < -16$ for $t \in [\frac{1}{3}, 1]$. If $v > 16$ for $t \in [\frac{1}{3}, 1]$, then

$$\begin{aligned}
&\int_0^1 (1-s)^{\alpha-1} f\big(s, v(s), D_{0^+}^{\frac{1}{6}} v(s)\big)\, ds - \sum_{i=1}^m b_i \eta_i^{2-\alpha} \int_0^{\eta_i} (\eta_i - s)^{\alpha-1} f\big(s, v(s), D_{0^+}^{\frac{1}{6}} v(s)\big)\, ds \\
&\quad + \Gamma(\alpha) \sum_{i=1}^m b_i \sum_{\eta_i < t_i < 1} A_i t_i^{2-\alpha} + \Gamma(\alpha - q) \sum_{i=1}^m b_i \eta_i \sum_{\eta_i < t_i < 1} B_i t_i^{q+1-\alpha} \\
&\quad - \Gamma(\alpha - q) \sum_{i=1}^m b_i \sum_{\eta_i < t_i < 1} B_i t_i^{q+2-\alpha} \\
&\geq 0.194319 - 0.06725 = 0.127069 > 0.
\end{aligned}$$

If $v < -16$ for $t \in [\frac{1}{3}, 1]$, then

$$\begin{aligned}
&\int_0^1 (1-s)^{\alpha-1} f\big(s, v(s), D_{0^+}^{\frac{1}{6}} v(s)\big)\, ds - \sum_{i=1}^m b_i \eta_i^{2-\alpha} \int_0^{\eta_i} (\eta_i - s)^{\alpha-1} f\big(s, v(s), D_{0^+}^{\frac{1}{6}} v(s)\big)\, ds \\
&\quad + \Gamma(\alpha) \sum_{i=1}^m b_i \sum_{\eta_i < t_i < 1} \delta_i t_i^{2-\alpha} + \Gamma(\alpha - q) \sum_{i=1}^m b_i \eta_i \sum_{\eta_i < t_i < 1} \omega_i t_i^{q+1-\alpha} \\
&\quad - \Gamma(\alpha - q) \sum_{i=1}^m b_i \sum_{\eta_i < t_i < 1} \omega_i t_i^{q+2-\alpha} \\
&\leq -0.106389 + 0.06725 = -0.039139 < 0.
\end{aligned}$$

By the same way, taking $M_{22} = 10/\sqrt[3]{3}$, assume that $|u(t)| > M_{22}$ holds for any $t \in [\frac{1}{2}, 1]$. Then either $v > 10/\sqrt[3]{3}$ or $v < -10/\sqrt[3]{3}$ for $t \in [\frac{1}{2}, 1]$. If $v > 10/\sqrt[3]{3}$ for $t \in [\frac{1}{2}, 1]$, then

$$\begin{aligned}
&\int_0^1 (1-s)^{\beta-1} g\big(s, u(s), D_{0^+}^{\frac{1}{4}} u(s)\big)\, ds - \sum_{i=1}^m d_i \theta_i^{2-\beta} \int_0^{\theta_i} (\theta_i - s)^{\beta-1} g\big(s, u(s), D_{0^+}^{\frac{1}{4}} u(s)\big)\, ds \\
&\quad + \Gamma(\beta) \sum_{i=1}^m d_i \sum_{\theta_i < t_i < 1} C_i t_i^{2-\beta} + \Gamma(\beta - p) \sum_{i=1}^m d_i \eta_i \sum_{\theta_i < t_i < 1} D_i t_i^{p+1-\beta} \\
&\quad - \Gamma(\beta - p) \sum_{i=1}^m d_i \sum_{\theta_i < t_i < 1} D_i t_i^{p+2-\beta} \\
&\geq 0.0703184 - 0.0120714 = 0.058247 > 0.
\end{aligned}$$

If $v > 10/\sqrt[3]{3}$ for $t \in [\frac{1}{2}, 1]$, then

$$\begin{aligned}
&\int_0^1 (1-s)^{\beta-1} g\big(s, u(s), D_{0^+}^{\frac{1}{4}} u(s)\big)\, ds - \sum_{i=1}^m d_i \theta_i^{2-\beta} \int_0^{\theta_i} (\theta_i - s)^{\beta-1} g\big(s, u(s), D_{0^+}^{\frac{1}{4}} u(s)\big)\, ds \\
&\quad + \Gamma(\beta) \sum_{i=1}^m d_i \sum_{\theta_i < t_i < 1} C_i t_i^{2-\beta} + \Gamma(\beta - p) \sum_{i=1}^m d_i \eta_i \sum_{\theta_i < t_i < 1} D_i t_i^{p+1-\beta} \\
&\quad - \Gamma(\beta - p) \sum_{i=1}^m d_i \sum_{\theta_i < t_i < 1} D_i t_i^{p+2-\beta} \\
&\leq -0.0231755 + 0.0120714 = -0.0111041 < 0.
\end{aligned}$$

So, from the above arguments, the second part of the condition (H_3) holds for $M_2 = \max\{M_{21}, M_{22}\} = 16$, $t \in [\frac{1}{3}, 1]$.

On the other hand, for $(u^*, v^*) = (h_1 t^{\alpha-1} + h_2 t^{\alpha-2}, h_3 t^{\beta-1} + h_4 t^{\beta-2}) \in \operatorname{Ker}(L)$, taking $g = 8$, assume that $h_i < -8$, $i = 1, 2, 3, 4$, then $h_3 t^{\beta-1} + h_4 t^{\beta-2} < -16$ for $t \in [\frac{1}{3}, 1]$, $h_1 t^{\alpha-1} + h_2 t^{\alpha-2} < -\frac{10}{\sqrt[3]{3}}$ for $t \in [\frac{1}{2}, 1]$. And $D_{0^+}^{\frac{1}{4}} u^* < -1$ for $t \in [\frac{1}{10}, \frac{1}{5}]$, $D_{0^+}^{\frac{1}{6}} u^* < -1$ for $t \in [\frac{1}{12}, \frac{1}{6}]$. Then we can see, from the above arguments, that $T_1 N_1 v^* = (r_1, 0, \ldots, 0)$, $T_2 N_1 v^* = (r_2, 0, \ldots, 0)$, $T_1 N_2 u^* = (r_3, 0, \ldots, 0)$, $T_2 N_2 u^* = (r_4, 0, \ldots, 0)$, where $r_i < 0$, $i = 1, 2, 3, 4$. Thus,

$$h_1 T_1 N_1 (h_3 t^{\beta-1} + h_4 t^{\beta-2}) + h_2 T_2 N_1 (h_3 t^{\beta-1} + h_4 t^{\beta-2}) = (s_1, 0, \ldots, 0),$$
$$h_3 T_1 N_2 (h_1 t^{\alpha-1} + h_2 t^{\alpha-2}) + h_4 T_2 N_2 (h_1 t^{\alpha-1} + h_2 t^{\alpha-2}) = (s_2, 0, \ldots, 0),$$

where $s_1 > 0$, $s_2 > 0$. So, the condition (H_4) holds. Hence, from Theorem 3.1, the coupled problem (4.1) has at least one solution in $\{u_{\frac{3}{2}}, D_{0^+}^{\frac{1}{2}} u \in PC[0,1]\} \times \{v_{\frac{4}{3}}, D_{0^+}^{\frac{1}{3}} v \in PC[0,1]\}$.

Competing interests

The authors declare that they have no competing interests.

Authors' contributions

CZ conceived the main idea of the study, XZ carried out the main parts of the draft. CZ gave many valuable suggestions and corrected the main theorems in the discussion. All authors read and approved the final manuscript.

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