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Solvability of Sturm-Liouville boundary value problems with impulses

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Abstract

In this paper, we consider a kind of Sturm-Liouville boundary value problems with impulsive effects. By using the mountain pass theorem and Ekeland's variational principle, the existence of two positive solutions and two negative solutions is established. Moreover, we do not assume that the nonlinearity satisfies the well-known AR-condition.

MSC: 34B15; 34B37; 58E30

Keywords: *p*-Laplacian; boundary value problem; variational; impulsive

1 Introduction

Impulsive effects exist widely in many evolution processes, in which their states are changed abruptly at a certain moment of time. Impulsive differential equations have become more important in recent years in mathematical models of real processes and phenomena studied in control theory [1, 2], population dynamics and biotechnology [3, 4], physics and mechanics problems [5]. There has been a significant development in the area of impulsive differential equations with fixed moments. We refer the reader to [6, 7] and the references therein. Fixed-point theorems in cones [8–10] and the method of lower and upper solutions with monotone iterative technique [11–13], have been used to study impulsive differential equations.

Moreover, the Sturm-Liouville boundary value problems (for short BVPs) have received a lot of attention. Many works have been carried out to discuss the existence of at least one solution, multiple solutions. The methods used therein mainly depend on the Leray-Schauder continuation theorem, Mawhin's continuation theorem. Since it is very difficult to give the corresponding Euler functional for Sturm-Liouville BVPs and verify the existence of the critical points for the Euler functional, few people consider the existence of solutions for Sturm-Liouville BVPs by critical point theory, and many works considered the existence of solutions for Dirichlet BVPs [14]. Recently, few researchers have used variational methods to study the existence of solutions for impulsive differential equations with Dirichlet boundary conditions [15, 16]. In [17], by mountain pass theorem, Tian and Ge considered the existence of positive solutions of a kind of Sturm-Liouville boundary value problems with impulsive effects. The authors require that the nonlinearity $f(t,x):[0,1]\times[0,+\infty)\to[0,+\infty)$ and $f(t,0)\not\equiv 0$. They have not obtained the existence of both positive solutions and negative solutions.

Based on the knowledge mentioned above, in this paper, we consider the constant-sign solutions of the following BVP

$$\begin{cases} -(\phi_p(x'(t)))' = -a(t)\phi_p(x(t)) + f(t,x(t)), & \text{a.e. } t \in [0,1], t \neq t_1, \dots, t_k, \\ -\Delta\phi_p(x'(t_i)) = I_i(x(t_i)), & i = 1,2,\dots,k, \\ \alpha_1 x(0) - \alpha_2 x'(0) = 0, \\ \beta_1 x(1) + \beta_2 x'(1) = 0, \end{cases} \tag{1.1}$$

where $p > 1$, $\phi_p(x) = |x|^{p-2}x$, $\alpha_1, \beta_1 \geq 0$, $\alpha_2, \beta_2 > 0$, $0 = t_0 < t_1 < \cdots < t_k < t_{k+1} = 1$, $\Delta(\phi_p(x'(t_i))) = \phi_p(x'(t_i^+)) - \phi_p(x'(t_i^-))$. Here $x'(t_i^+)$ and $x'(t_i^-)$ denote the right and left limits, respectively. Assume that $F(t,x) = \int_0^x f(t,s)\,ds$, $f(t,x)$ is continuous, $I_i(x)$ is continuous on R, $i = 1,\dots,k$, $a(t) \in C([0,1],(0,+\infty))$.

Ambrosetti and Rabinowitz [18] established the existence of nontrival solutions for Dirichlet problems under the well-known Ambrosetti-Rabinowitz condition: there exist some $\mu > 2$ and $R > 0$ such that

$$0 < \mu \int_0^x f(t,s)\,ds \leq f(t,x)x \tag{1.2}$$

for all $t \in [0,T]$ and $|x| \geq R$. Since then, the AR-condition has been used extensively. By the usual AR-condition, it is easy to show that the Euler-Lagrange functional associated with the system has the mountain pass geometry, and the Palais-Smale sequence is bounded. For example, in [16, 17], based on (1.2), the authors considered the boundary value problems with impulsive effects.

In this paper, we study the existence of constant-sign solutions of BVP (1.1) without the AR-condition. The paper is organized as follows. In the forthcoming section, we give the Euler functional of BVP (1.1) and some basic lemmas. The aim of Section 3 is to prove the existence of at least two positive solutions of BVP (1.1) based on the mountain pass theorem and Ekeland's variational principle. At last, we give some results of the existence of at least two negative solutions.

2 Preliminary

The Sobolev space $W^{1,p}[0,1]$ is defined by

$$W^{1,p}[0,1] = \{x : [0,1] \to R \mid x \text{ is absolutely continuous and } x' \in L^p(0,1;R)\}$$

and is endowed with the norm

$$\|x\| = \left(\int_0^1 |x(t)|^p\,dt + \int_0^1 |x'(t)|^p\,dt\right)^{\frac{1}{p}}.$$

Then, from [19], $W^{1,p}[0,1]$ is a sparable and reflexive Banach space.

Definition 2.1 We say that x is a classical solution of BVP (1.1) if it satisfies the equation of BVP (1.1) a.e. on $[0,1]$, the limits $x'(t_i^+)$ and $x'(t_i^-)$, $i = 1,2,\dots,k$, exist and the Sturm-Liouville boundary conditions hold.

However, if $x \in W^{1,p}[0,1]$, then x is absolutely continuous and $x' \in L^p[0,1]$. In this case, the one-sided derivatives $x'(t_i^+)$, $x'(t_i^-)$ may not exist. As a consequence, we need to introduce a different concept of solution.

Definition 2.2 We say that $x \in W^{1,p}[0,1]$ is a weak solution of BVP (1.1) if it satisfies

$$\begin{aligned}&\int_0^1 a(t)\phi_p(x)y\,dt + \int_0^1 \phi_p(x')y'\,dt \\ &\quad= \int_0^1 f(t,x)y\,dt + \sum_{i=1}^k I_i\bigl(x(t_i)\bigr)y(t_i) - \phi_p\biggl(\frac{\alpha_1 x(0)}{\alpha_2}\biggr)y(0) - \phi_p\biggl(\frac{\beta_1 x(1)}{\beta_2}\biggr)y(1)\end{aligned} \tag{2.1}$$

for $y \in W^{1,p}[0,1]$.

Consider $\varphi : W^{1,p}[0,1] \to \mathbb{R}$ defined by

$$\begin{aligned}\varphi(x) &= \frac{1}{p}\int_0^1 a(t)|x|^p\,dt + \frac{1}{p}\int_0^1 \bigl|x'\bigr|^p\,dt - \int_0^1 F(t,x)\,dt \\ &\quad- \sum_{i=1}^k \int_0^{x(t_i)} I_i(t)\,dt + \frac{1}{p}\phi_p\biggl(\frac{\alpha_1}{\alpha_2}\biggr)\bigl|x(0)\bigr|^p + \frac{1}{p}\phi_p\biggl(\frac{\beta_1}{\beta_2}\biggr)\bigl|x(1)\bigr|^p.\end{aligned} \tag{2.2}$$

It is clear φ is continuously differentiable on $W^{1,p}[0,1]$ and by computation, one has

$$\begin{aligned}\bigl\langle\varphi'(x),y\bigr\rangle &= \int_0^1 a(t)\phi_p(x)y\,dt + \int_0^1 \phi_p(x')y'\,dt - \int_0^1 f(t,x)y\,dt - \sum_{i=1}^k I_i\bigl(x(t_i)\bigr)y(t_i) \\ &\quad+ \phi_p\biggl(\frac{\alpha_1 x(0)}{\alpha_2}\biggr)y(0) + \phi_p\biggl(\frac{\beta_1 x(1)}{\beta_2}\biggr)y(1), \quad x,y \in W^{1,p}[0,1].\end{aligned} \tag{2.3}$$

Hence, a critical point of φ gives us a weak solution of BVP (1.1).

Lemma 2.1 [20] *There exists a positive constant c_p such that*

$$\bigl(|x|^{p-2}x - |y|^{p-2}y, x-y\bigr) \geq \begin{cases} c_p|x-y|^p, & p \geq 2, \\ c_p\frac{|x-y|^2}{(|x|+|y|)^{2-p}}, & 1 < p < 2 \end{cases} \tag{2.4}$$

for any $x, y \in R^N$, $|x| + |y| \neq 0$. Here, $(x,y) = x \cdot y^T$.

For $x \in C[0,1]$, suppose that $\|x\|_\infty = \max_{t\in[0,1]}|x(t)|$, $|x|_m = \min_{t\in[0,1]}|x|$.

Lemma 2.2 *If $x \in W^{1,p}[0,1]$, then, $\|x\|_\infty \leq 2\|x\|$.*

Lemma 2.3 [17] *For $x \in X$, let $x^\pm = \max\{\pm x, 0\}$, then, the following properties hold:*
(i) $x \in X \Rightarrow x^+, x^- \in X$;
(ii) $x = x^+ - x^-$;
(iii) $\|x^+\|_X \leq \|x\|_X$;

(iv) *if* $(x_n)_{n\in N}$ *uniformly converges to* x *in* $C([0,1])$, *then*, $(x_n^+)_{n\in N}$ *uniformly converges to* x^+;

(vi) $\phi_p(x)x^+ = |x^+|^p$, $\phi_p(x)x^- = -|x^-|^p$.

In the following, let H be a Banach space, let φ be continuously differentiable, and we state (C) condition [21].

(C) Every sequence $(x_n)_{n\in N} \subset H$ such that the following conditions hold:
 (i) $(\varphi(x_n))_{n\in N}$ is bounded,
 (ii) $(1 + \|x_n\|_H)\|\varphi'(x_n)\|_{H^*} \to 0$ as $n \to \infty$
 has a subsequence, which converges strongly in H.

This condition is weaker than the usual Palais-Smale condition, but can be used in place of it when constructing deformations of sublevel sets *via* negative pseudo-gradient flows, and, therefore, also in minimax theorems such as the mountain pass lemma and the saddle point theorem.

Lemma 2.4 *If* $x(t) \in W^{1,p}[0,1]$ *is a weak solution of BVP* (1.1), *then* $x(t)$ *is a classical solution of BVP* (1.1).

Proof The proof of this lemma is similar to that of [22]. For the sake of completeness, we give a simple proof here.

Choose $y \in W_0^{1,p}[0,1]$ with $y(t) = 0$ for every $t \in [0,t_i] \cup [t_{i+1},1]$, then

$$\int_{t_i}^{t_{i+1}} a(t)\phi_p(x)y\,dt - \int_{t_i}^{t_{i+1}} f(t,x)y\,dt - \int_{t_i}^{t_{i+1}} \big(\phi_p(x')\big)'y\,dt = 0.$$

Whence, by the fundamental lemma,

$$-\big(\phi_p\big(x'(t)\big)\big)' = -a(t)\phi_p(x) + f(t,x), \quad \text{a.e. } t \in [t_i,t_{i+1}].$$

Hence, $x \in W^{2,p}(t_i,t_{i+1})$, that is, $x'(t_i^+)$, $x'(t_{i+1}^-)$ exist, and x satisfies the equation of BVP (1.1) a.e. on $[0,1]$. Moreover,

$$\begin{aligned}
\int_0^1 \phi_p\big(x'\big)y'\,dt &= \sum_{i=0}^{k}\int_{t_i}^{t_{i+1}} \phi_p\big(x'\big)\,dy \\
&= \sum_{i=0}^{k}\left[\phi_p\big(x'\big(t_{i+1}^-\big)\big)y(t_{i+1}) - \phi_p\big(x'\big(t_i^+\big)\big)y(t_i) - \int_{t_i}^{t_{i+1}}\big(\phi_p\big(x'\big)\big)'y\,dt\right] \\
&= -\phi_p\big(x'(0)\big)y(0) + \phi_p\big(x'(1)\big)y(1) - \sum_{i=1}^{k}\big(\phi_p\big(x'\big(t_i^+\big)\big) - \phi_p\big(x'\big(t_i^-\big)\big)\big)y(t_i) \\
&\quad - \int_0^1\big(\phi_p\big(x'\big)\big)y\,dt \\
&= -\phi_p\big(x'(0)\big)y(0) + \phi_p\big(x'(1)\big)y(1) - \sum_{i=1}^{k}\Delta\phi_p\big(x'(t_i)\big)y(t_i) \\
&\quad - \int_0^1\big(\phi_p\big(x'\big)\big)'y\,dt.
\end{aligned}$$

Now multiplying the equation by $y \in W^{1,p}[0,1]$ and integrating between 0 and 1, together with (2.1), we get

$$\begin{aligned} 0 &= \left(\phi_p\left(\frac{\alpha_1 x(0)}{\alpha_2}\right) - \phi_p\big(x'(0)\big)\right)y(0) + \left(\phi_p\left(\frac{\beta_1 x(1)}{\beta_2}\right) + \phi_p\big(x'(1)\big)\right)y(1) \\ &\quad - \sum_{i=1}^{k}\big(\Delta\phi_p\big(x'(t_i)\big) + I_i\big(x(t_i)\big)\big)y(t_i). \end{aligned} \tag{2.5}$$

Assume that $y(t) = t(t-1)\Pi_{j=1,j\neq i}^{k}(t-t_j)$, then, $y(t_i) \neq 0$ and $-\Delta\phi_p(x'(t_i)) = I(x(t_i))$ $(i \neq 0, k+1)$. Let $i = 1,\dots,k$, we arrive x satisfies the impulsive condition and

$$0 = \left(\phi_p\left(\frac{\alpha_1 x(0)}{\alpha_2}\right) - \phi_p\big(x'(0)\big)\right)y(0) + \left(\phi_p\left(\frac{\beta_1 x(1)}{\beta_2}\right) + \phi_p\big(x'(1)\big)\right)y(1) \tag{2.6}$$

by (2.5). Let $y(t) = t-1$, then, $\phi_p(\frac{\alpha_1 x(0)}{\alpha_2}) = \phi_p(x'(0))$, that is, $\frac{\alpha_1 x(0)}{\alpha_2} = x'(0)$. Let $y(t) = t$, then, $\phi_p(\frac{\beta_1 x(1)}{\beta_2}) = -\phi_p(x'(1))$, that is, $\frac{\beta_1 x(1)}{\beta_2} = -x'(1)$. Hence, x is a solution of BVP (1.1). □

3 Existence of constant-sign solutions

Assume that $H(t,x) = xf(t,x) - pF(t,x)$, $G_i(x) = I_i(x)x - p\int_0^x I_i(t)\,dt$, $f(t,0) = 0$ a.e. on $[0,1]$, $f(t,x) \geq 0$ for a.e. $t \in [0,1]$ and $x \geq 0$; $f(t,x) \leq 0$ for a.e. $t \in [0,1]$ and $x \leq 0$; $I_i(0) = 0$, $I_i(x) \geq 0$ for $x \geq 0$, $I_i(x) \leq 0$ for $x \leq 0$, $i = 1,2,\dots,k$. Define $x^{\pm} = \max\{\pm x, 0\}$, $f^+(t,x) = \begin{cases} 0, & x \leq 0, \\ f(t,x), & x > 0, \end{cases}$ $I_i^+(x) = \begin{cases} 0, & x \leq 0, \\ I_i(x), & x > 0, \end{cases}$ $F^+(t,x) = \int_0^x f^+(t,s)\,ds$, and

$$\begin{aligned} \varphi_+(x) &= \frac{1}{p}\int_0^1 a(t)|x|^p\,dt + \frac{1}{p}\int_0^1 \left|x'\right|^p\,dt - \int_0^1 F^+(t,x)\,dt \\ &\quad - \sum_{i=1}^{k}\int_0^{x(t_i)} I_i^+(t)\,dt + \frac{1}{p}\phi_p\left(\frac{\alpha_1}{\alpha_2}\right)\left|x(0)\right|^p + \frac{1}{p}\phi_p\left(\frac{\beta_1}{\beta_2}\right)\left|x(1)\right|^p. \end{aligned} \tag{3.1}$$

It is obvious that φ_+ is continuously differentiable and $f^+(t,x) = f(t,x^+)$, $I_i^+(x) = I_i(x^+)$, $i = 1,2,\dots,k$.

Lemma 3.1 *Assume that*

(A_1) $I_i(x) \leq b_i + c_i x^{\tau-1}$, $b_i, c_i \geq 0$, $x \geq 0$, $\tau > p$, $i = 1,2,\dots,k$;
(A_2) *there exits a constant* $a_0 \geq 0$ *such that for a.e.* $t \in [0,1]$, $0 < x \leq y$, $H(t,x) \leq H(t,y) + a_0$, $G_i(x) \leq G_i(y) + a_0$, $i = 1,2,\dots,k$;
(A_3) $\lim_{x\to+\infty}\frac{f(t,x)}{x^{p-1}} = +\infty$ *for* $t \in [0,1]$.

Then, φ_+ *satisfies* (C) *condition.*

Remark 3.1 Let

$$f(t,x) = \begin{cases} 0, & x < 0, \\ cx^{p-1}(\ln(1+x^p) + 1 - \sin x^p), & 0 \leq x \leq 1, \\ cx^{p-1}(\ln(1+x^p) + 1 - \sin 1), & 1 < x. \end{cases}$$

Then, $f(t,x)$ satisfies (A_1)-(A_3). However, it does not satisfy the AR-condition while x is large.

Remark 3.2 The condition of $H(t,x) \leq H(t,y) + a_0$ for $a_0 \geq 0$, $0 < x \leq y$, a.e. $t \in [0,1]$, is weaker than the following condition:

there is $x_0 > 0$ such that $H(t,x)$ is increasing in $x \geq x_0 > 0$,

which is equivalent to the condition:

$\frac{f(t,x)}{x}$ is increasing in $x \geq x_0 > 0$.

Proof Let $(x_n)_{n\geq 1} \subset W^{1,p}[0,1]$ be a sequence such that

$$\bigl|\varphi_+(x_n)\bigr| \leq c, \qquad \bigl(1 + \|x_n\|\bigr)\bigl\|\varphi_+'(x_n)\bigr\|_{(W^{1,p})^*} \to 0, \quad \text{as } n \to \infty. \tag{3.2}$$

In order to prove that $(x_n)_{n\geq 1}$ is bounded in $W^{1,p}[0,1]$, there are several steps.

Step 1. $(x_n^-)_{n\in N} \subset W^{1,p}[0,1]$ is bounded.

From (3.2), for $\varepsilon > 0$, one has

$$\bigl|\bigl\langle \varphi_+'(x_n), u\bigr\rangle\bigr| < \varepsilon, \quad u \in W^{1,p}[0,1]. \tag{3.3}$$

We know that x_n^- is an absolutely continuous function on $[0,1]$, and so, the fundamental theorem of calculus ensures the existence of a set $E_0 \subset [0,1]$ such that $\operatorname{meas}([0,1] \setminus E_0) = 0$ and x_n^- is differentiable on E_0, then, let $u = -x_n^-$,

$$\begin{aligned}
\varepsilon &> \bigl|\bigl\langle \varphi_+'(x_n), -x_n^-\bigr\rangle\bigr| \\
&= \Biggl| -\int_0^1 a(t)\phi_p(x_n)x_n^-\,dt + \int_0^1 \phi_p\bigl(x_n'\bigr)\bigl(-x_n^-\bigr)'\,dt + \int_0^1 f^+(t,x_n)x_n^-\,dt \\
&\quad + \sum_{i=1}^k I_i^+\bigl(x_n(t_i)\bigr)x_n^-(t_i) - \phi_p\biggl(\frac{\alpha_1 x_n(0)}{\alpha_2}\biggr)x_n^-(0) - \phi_p\biggl(\frac{\beta_1 x_n(1)}{\beta_2}\biggr)x_n^-(1)\Biggr| \\
&= \int_0^1 a(t)\bigl|x_n^-\bigr|^p\,dt + \int_0^1 \bigl|\bigl(x_n^-\bigr)'\bigr|^p\,dt + \phi_p\biggl(\frac{\alpha_1}{\alpha_2}\biggr)\bigl|x_n^-(0)\bigr|^p + \phi_p\biggl(\frac{\beta_1}{\beta_2}\biggr)\bigl|x_n^-(1)\bigr|^p \\
&\geq \min\bigl\{\bigl|a(t)\bigr|_m, 1\bigr\}\bigl\|x_n^-\bigr\|^p + \phi_p\biggl(\frac{\alpha_1}{\alpha_2}\biggr)\bigl|x_n^-(0)\bigr|^p + \phi_p\biggl(\frac{\beta_1}{\beta_2}\biggr)\bigl|x_n^-(1)\bigr|^p.
\end{aligned}$$

Then, $(x_n^-)_{n\in N} \subset W^{1,p}[0,1]$ is bounded.

Step 2. $(x_n^+)_{n\in N} \subset W^{1,p}[0,1]$ is bounded.

Suppose that $\|x_n^+\| \to \infty$ as $n \to \infty$. Set $y_n = \frac{x_n^+}{\|x_n^+\|}$ for all $n \geq 1$. Obviously, $\|y_n\| = 1$, that is, $(y_n)_{n\in N}$ is a bounded sequence in $W^{1,p}[0,1]$. Going to a subsequence if necessary, we may assume that

$$y_n \rightharpoonup y \quad \text{in } W^{1,p}[0,1], \qquad y_n \to y \quad \text{in } C[0,1]. \tag{3.4}$$

It is clear that $y \geq 0$ and from the inequality $|\langle \varphi_+'(x_n), x_n^+\rangle| \leq \|\varphi_+'(x_n)\|_{(W^{1,p})^*} \cdot \|x_n^+\| \leq \|\varphi_+'(x_n)\|_{(W^{1,p})^*} \cdot \|x_n\| \to 0$ as $n \to \infty$, there exists a sequence $(\varepsilon_n)_{n\in N}$, $\varepsilon_n \geq 0$ and $\varepsilon_n \to 0$ as $n \to \infty$ such that

$$\bigl|\bigl\langle \varphi_+'(x_n), x_n^+\bigr\rangle\bigr| \leq \varepsilon_n \quad \text{for large } n.$$

Hence,

$$\begin{aligned}\left|\frac{\langle\varphi_+'(x_n),x_n^+\rangle}{\|x_n^+\|^p}\right| &= \left|\int_0^1 a(t)y_n^p\,dt + \int_0^1 \left|y_n'\right|^p\,dt - \int_0^1 \frac{f^+(t,x_n)y_n}{\|x_n^+\|^{p-1}}\,dt - \sum_{i=1}^k \frac{I_i^+(x_n(t_i))x_n^+(t_i)}{\|x_n^+\|^p}\right.\\ &\quad \left. + \phi_p\left(\frac{\alpha_1}{\alpha_2}\right)\frac{|x_n^+(0)|^p}{\|x_n^+\|^p} + \phi_p\left(\frac{\beta_1}{\beta_2}\right)\frac{|x_n^+(1)|^p}{\|x_n^+\|^p}\right|\\ &\le \frac{\varepsilon_n}{\|x_n^+\|^p} \quad \text{for large } n. \end{aligned} \tag{3.5}$$

From $\|y_n\| = 1$, $0 \le \int_0^1 |y_n|^p\,dt \le 1$, one has

$$0 \le \int_0^1 a(t)y_n^p\,dt \le \left\|a(t)\right\|_\infty.$$

Moreover, $\phi_p(\frac{\alpha_1}{\alpha_2})\frac{|x_n^+(0)|^p}{\|x_n^+\|^p} \le 2^p\phi_p(\frac{\alpha_1}{\alpha_2})$, $\phi_p(\frac{\beta_1}{\beta_2})\frac{|x_n^+(1)|^p}{\|x_n^+\|^p} \le 2^p\phi_p(\frac{\beta_1}{\beta_2})$, and from (A_1), one has

$$\begin{aligned} 0 &\ge -\sum_{i=1}^k \frac{I_i^+(x_n(t_i))x_n^+(t_i)}{\|x_n^+\|^p} \ge -\sum_{i=1}^k \frac{b_i|x_n^+(t_i)|}{\|x_n^+\|^p} - \sum_{i=1}^k \frac{c_i|x_n^+(t_i)|^\tau}{\|x_n^+\|^p}\\ &\ge -2\sum_{i=1}^k \frac{b_i}{\|x_n^+\|^{p-1}} - 2^\tau\sum_{i=1}^k c_i\left\|x_n^+\right\|^{\tau-p} \to -\infty, \quad \text{as } n\to\infty. \end{aligned}$$

Let $[0,1]_+ = \{t\in[0,1], y(t) > 0\}$, then, $x_n^+(t) \to +\infty$ as $n\to\infty$ for $t\in[0,1]_+$. By the hypothesis,

$$\frac{f(t,x_n^+(t))}{(x_n^+(t))^{p-1}} \to +\infty, \quad t\in[0,1]_+, \text{ as } n\to\infty.$$

Let $\chi_n(t) = \chi_{\{x_n^+>0\}}(t) = \chi_{\{y_n>0\}}(t)$, then, $\chi_n(t)y_n(t)^p \to \chi_{[0,1]_+}(t)y(t)^p$ for all $t\in[0,1]$. If $\operatorname{meas}[0,1]_+ > 0$, then,

$$\chi_n(t)y_n(t)^p\frac{f(t,x_n^+(t))}{(x_n^+(t))^{p-1}} \to +\infty, \quad t\in[0,1]_+, \text{ as } n\to\infty.$$

Hence, by Fatou's lemma,

$$\int_0^1 \frac{f^+(t,x_n)y_n}{\|x_n^+\|^{p-1}}\,dt = \int_0^1 \chi_n(t)y_n(t)^p\frac{f(t,x_n^+(t))}{(x_n^+(t))^{p-1}}\,dt \to +\infty, \quad \text{as } n\to\infty.$$

Then, from (3.5), we reach a contradiction, that is, $\operatorname{meas}[0,1]_+ = 0$. Since $y \ge 0$, we conclude that $y(t) = 0$ for a.e. $t\in[0,1]$. Then, $y(t)\equiv 0$ for $t\in[0,1]$.

Assume that $(t_n)_{n\ge1}\subset[0,1]$ be such that

$$\varphi_+\left(t_nx_n^+\right) = \max_{t\in[0,1]}\varphi_+\left(tx_n^+\right).$$

Fix an integer $m\ge1$ and define

$$z_n = \left(2p\left\|x_m^+\right\|^p\right)^{\frac{1}{p}}y_n, \quad n\ge1, \tag{3.6}$$

that is, $z_n = \frac{(2p\|x_m^+\|^p)^{\frac{1}{p}}}{\|x_n^+\|} x_n^+$. Since $\|x_n^+\| \to \infty$, there exists an integer n_0, for $n \geq n_0$, one has $\frac{(2p\|x_m^+\|^p)^{\frac{1}{p}}}{\|x_n^+\|} \leq 1$. Whence,

$$\begin{aligned}
\varphi_+(t_n x_n^+) &\geq \varphi_+(z_n) \\
&= \frac{1}{p}\int_0^1 a(t)|z_n|^p\,dt + \frac{1}{p}\int_0^1 |z_n'|^p\,dt - \int_0^1 F^+(t,z_n)\,dt - \sum_{i=1}^k \int_0^{z_n(t_i)} I_i^+(t)\,dt \\
&\quad + \frac{1}{p}\phi_p\left(\frac{\alpha_1}{\alpha_2}\right)|z_n(0)|^p + \frac{1}{p}\phi_p\left(\frac{\beta_1}{\beta_2}\right)|z_n(1)|^p \\
&= 2\|x_m^+\|^p \left(\int_0^1 a(t)|y_n|^p\,dt + \int_0^1 |y_n'|^p\,dt\right) - \int_0^1 F^+(t,z_n)\,dt \\
&\quad - \sum_{i=1}^k \int_0^{z_n(t_i)} I_i^+(t)\,dt + \frac{1}{p}\phi_p\left(\frac{\alpha_1}{\alpha_2}\right)|z_n(0)|^p + \frac{1}{p}\phi_p\left(\frac{\beta_1}{\beta_2}\right)|z_n(1)|^p \\
&\geq 2\min\{|a(t)|_m, 1\}\|x_m^+\|^p\|y_n\|^p - \int_0^1 F^+(t,z_n)\,dt - \sum_{i=1}^k \int_0^{z_n(t_i)} I_i^+(t)\,dt.
\end{aligned}$$

Since $y_n \to 0$ uniformly for $t \in [0,1]$, then, $z_n(t) \to 0$ uniformly for $t \in [0,1]$, and $z_n(t_i) \to 0$ for $i = 1, \dots, k$ as $n \to \infty$. Hence,

$$\varphi_+(t_n x_n^+) \geq 2\min\{|a(t)|_m, 1\}\|x_m^+\|^p, \quad n > n_0 > m.$$

Therefore, we have $\varphi_+(t_n x_n^+) \to \infty$ as $m \to \infty$. Since $\varphi_+(x_n)$ and $(x_n^-)_{n\in N} \subset W^{1,p}[0,1]$ are bounded, then, $(\varphi_+(x_n^+))_{n\in N} \subset R$ is bounded. Together with $\varphi_+(0) = 0$, one has $t_n \in (0,1)$ for all $n \geq 1$. Then,

$$\begin{aligned}
0 &= t_n\left(\frac{d}{dt}\varphi_+(t x_n^+)\Big|_{t=t_n}\right) = \langle \varphi_+'(t_n x_n^+), t_n x_n^+\rangle \\
&= \int_0^1 a(t)\phi_p(t_n x_n^+) t_n x_n^+\,dt + \int_0^1 \phi_p((t_n x_n^+)')(t_n x_n^+)'\,dt - \int_0^1 f^+(t, t_n x_n^+) t_n x_n^+\,dt \\
&\quad - \sum_{i=1}^k I_i^+(t_n x_n^+(t_i)) t_n x_n^+(t_i) + \phi_p\left(\frac{\alpha_1}{\alpha_2}\right)|t_n x_n^+(0)|^p + \phi_p\left(\frac{\beta_1}{\beta_2}\right)|t_n x_n^+(1)|^p \\
&= t_n^p \int_0^1 a(t)(x_n^+)^p\,dt + t_n^p \int_0^1 |(x_n^+)'|^p\,dt - \int_0^1 f^+(t, t_n x_n^+) t_n x_n^+\,dt \\
&\quad - \sum_{i=1}^k I_i^+(t_n x_n^+(t_i)) t_n x_n^+(t_i) + \phi_p\left(\frac{\alpha_1}{\alpha_2}\right)|t_n x_n^+(0)|^p + \phi_p\left(\frac{\beta_1}{\beta_2}\right)|t_n x_n^+(1)|^p.
\end{aligned}$$

Moreover,

$$\begin{aligned}
&\frac{1}{p}\int_0^1 H(t, t_n x_n^+)\,dt + \frac{1}{p}\sum_{i=1}^k G_i(t_n x_n^+(t_i)) \\
&\quad = \frac{1}{p}\int_0^1 f(t, t_n x_n^+) t_n x_n^+\,dt + \frac{1}{p}\sum_{i=1}^k I_i(t_n x_n^+(t_i)) t_n x_n^+(t_i)
\end{aligned}$$

$$
\begin{aligned}
&- \int_0^1 F\big(t, t_n x_n^+\big)\,dt - \sum_{i=1}^{k} \int_0^{t_n x_n^+(t_i)} I_i(t)\,dt \\
&= \frac{1}{p} t_n^p \int_0^1 a(t)\big(x_n^+\big)^p\,dt + \frac{1}{p} t_n^p \int_0^1 \big|\big(x_n^+\big)'\big|^p\,dt + \frac{1}{p}\phi_p\bigg(\frac{\alpha_1}{\alpha_2}\bigg)\big|t_n x_n^+(0)\big|^p \\
&\quad + \frac{1}{p}\phi_p\bigg(\frac{\beta_1}{\beta_2}\bigg)\big|t_n x_n^+(1)\big|^p - \int_0^1 F^+\big(t, t_n x_n^+\big)\,dt - \sum_{i=1}^{k}\int_0^{t_n x_n^+(t_i)} I_i^+(t)\,dt \\
&= \varphi_+\big(t_n x_n^+\big) \\
&\geq 2\min\big\{\big|a(t)\big|_m, 1\big\}\big\|x_m^+\big\|^p, \quad n > n_0 > m.
\end{aligned}
$$

Since $\varphi_+(x_n^+)$ is bounded, there exists $\eta > 0$ such that

$$
\begin{aligned}
\eta &\geq p\varphi_+\big(x_n^+\big) - \big\langle \varphi_+'(x_n), x_n^+\big\rangle \\
&= \int_0^1 f^+(t, x_n)x_n^+\,dt + \sum_{i=1}^{k} I_i^+\big(x_n(t_i)\big)x_n^+(t_i) - p\int_0^1 F^+\big(t, x_n^+\big)\,dt - p\sum_{i=1}^{k}\int_0^{x_n^+(t_i)} I_i^+(t)\,dt \\
&= \int_0^1 H\big(t, x_n^+\big)\,dt + \sum_{i=1}^{k} G_i\big(x_n^+(t_i)\big).
\end{aligned}
$$

Since $0 < t_n x_n^+ \leq x_n^+$, then,

$$
\begin{aligned}
(k+1)a_0 + \eta &\geq (k+1)a_0 + \int_0^1 H\big(t, x_n^+\big)\,dt + \sum_{i=1}^{k} G_i\big(x_n^+(t_i)\big) \\
&\geq \int_0^1 H\big(t, t_n x_n^+\big)\,dt + \sum_{i=1}^{k} G_i\big(t_n x_n^+(t_i)\big) \\
&\geq 2p\min\big\{\big|a(t)\big|_m, 1\big\}\big\|x_m^+\big\|^p, \quad n > n_0 > m.
\end{aligned}
$$

Since $m \geq 1$ is an arbitrary integer, let $m \to \infty$, we have a contradiction. This proves that $(x_n^+)_{n\in N} \subset W^{1,p}[0,1]$ is bounded.

From step 1 and step 2, we obtain that $(x_n)_{n\in N}$ is bounded. Hence, we may assume that

$$
x_n \rightharpoonup x \quad \text{in } W^{1,p}[0,1], \qquad x_n \to x \quad \text{in } C[0,1].
$$

Moreover, for $m, n \in N$, one has

$$
\begin{aligned}
&\big\langle \varphi_+'(x_n) - \varphi_+'(x_m), x_n - x_m\big\rangle \\
&= \int_0^1 a(t)\big(\phi_p(x_n) - \phi_p(x_m)\big)(x_n - x_m)\,dt + \int_0^1 \big(\phi_p\big((x_n)'\big) \\
&\quad - \phi_p\big((x_m)'\big)\big)\big(x_n' - x_m'\big)\,dt - \int_0^1 \big(f^+(t, x_n) - f^+(t, x_m)\big)(x_n - x_m)\,dt \\
&\quad - \sum_{i=1}^{k}\big(I_i^+\big(x_n(t_i)\big) - I_i^+\big(x_m(t_i)\big)\big)\big(x_n(t_i) - x_m(t_i)\big) + \bigg(\phi_p\bigg(\frac{\alpha_1 x_n(0)}{\alpha_2}\bigg)
\end{aligned}
$$

$$\begin{aligned}
&-\phi_p\left(\frac{\alpha_1 x_m(0)}{\alpha_2}\right)\Big)\big(x_n(0)-x_m(0)\big) \\
&+\left(\phi_p\left(\frac{\beta_1 x_n(1)}{\beta_2}\right)-\phi_p\left(\frac{\beta_1 x_m(1)}{\beta_2}\right)\right)\big(x_n(1)-x_m(1)\big).
\end{aligned}$$

Since $(x_n)_{n\in N}$ is a Cauchy sequence in $C[0,1]$, $|\langle\varphi'_+(x_n)-\varphi'_+(x_m),x_n-x_m\rangle| \le (\|\varphi'_+(x_n)\| + \|\varphi'_+(x_m)\|)(\|x_n\| + \|x_m\|)$, $(x_n)_{n\in N}$ is bounded in $W^{1,p}[0,1]$, $\varphi'_+(x_n)\to 0$, $\varphi'_+(x_m)\to 0$ as $m,n\to\infty$, one has $\langle\varphi'_+(x_n)-\varphi'_+(x_m),x_n-x_m\rangle\to 0$ as $m,n\to\infty$. Moreover, $f^+(t,x)$ is continuous in x, $I_i^+(x)$ is continuous, $x_n\to x$ uniformly in $[0,1]$, whence, $(\phi_p(\frac{\alpha_1 x_n(0)}{\alpha_2})-\phi_p(\frac{\alpha_1 x_m(0)}{\alpha_2}))(x_n(0)-x_m(0))\to 0$, $(\phi_p(\frac{\beta_1 x_n(1)}{\beta_2})-\phi_p(\frac{\beta_1 x_m(1)}{\beta_2}))(x_n(1)-x_m(1))\to 0$, and

$$\int_0^1 \big(\phi_p(x'_n)-\phi_p(x'_m)\big)\big(x'_n-x'_m\big)\,dt\to 0,\quad \text{as } n,m\to\infty. \tag{3.7}$$

If $p\ge 2$, from Lemma 2.1, there exists a positive constant c_p such that

$$\int_0^1 \big(\phi_p(x'_n)-\phi_p(x'_m)\big)\big(x'_n-x'_m\big)\,dt \ge c_p\int_0^1 \big|x'_n-x'_m\big|^p\,dt. \tag{3.8}$$

If $p<2$, by Lemma 2.1, the Hölder inequality and the boundedness of $(x_n)_{n\in N}$ in $W^{1,p}[0,1]$, one has

$$\begin{aligned}
\int_0^1 \big|x'_n-x'_m\big|^p\,dt &= \int_0^1 \frac{|x'_n-x'_m|^p}{(|x'_n|+|x'_m|)^{\frac{p(2-p)}{2}}}\big(|x'_n|+|x'_m|\big)^{\frac{p(2-p)}{2}}\,dt \\
&\le \left(\int_0^1 \frac{|x'_n-x'_m|^2}{(|x'_n|+|x'_m|)^{2-p}}\,dt\right)^{\frac{p}{2}}\left(\int_0^1 \big(|x'_n|+|x'_m|\big)^p\,dt\right)^{\frac{2-p}{2}} \\
&\le c_p^{-\frac{p}{2}}\left(\int_0^1 \big(\phi_p(x'_n)-\phi_p(x'_m),x'_n-x'_m\big)\,dt\right)^{\frac{p}{2}} 2^{\frac{(p-1)(2-p)}{2}} \\
&\quad\times\left(\int_0^1 \big(|x'_n|^p+|x'_m|^p\big)\,dt\right)^{\frac{2-p}{2}} \\
&\le c_p^{-\frac{p}{2}}\left(\int_0^1 \big(\phi_p(x'_n)-\phi_p(x'_m),x'_n-x'_m\big)\,dt\right)^{\frac{p}{2}} 2^{\frac{(p-1)(2-p)}{2}} \\
&\quad\times\big(\|x_n\|^p+\|x_m\|^p\big)^{\frac{2-p}{2}}.
\end{aligned} \tag{3.9}$$

Then, we have $\int_0^1|x'_n-x'_m|^p\,dt\to 0$ as $n,m\to\infty$. Hence, $\|x_n-x_m\|\to 0$, that is, $(x_n)_{n\in N}$ is a Cauchy sequence in $W^{1,p}[0,1]$. By the completeness of $W^{1,p}[0,1]$, one has that $(x_n)_{n\in N}$ is a convergence sequence. □

Theorem 3.1 *Assume that* (A_1)-(A_3) *and*

(A_4) $f(t,x)\le b_0(t)+c_0(t)x^{\tau-1}$, $x\ge 0$, $b_0(t),c_0(t)\in C([0,1],[0,+\infty))$;
(A_5) $\frac{1}{p}\min\{|a|_m,1\}-2(\int_0^1 b_0(t)\,dt+\sum_{i=1}^k b_i)\varrho^{1-p}-\frac{2^\tau}{\tau}(\int_0^1 c_0(t)\,dt+\sum_{i=1}^k c_i)\varrho^{\tau-p}>0$, $\varrho=(\frac{\tau(p-1)(\int_0^1 b_0(t)\,dt+\sum_{i=1}^k b_i)}{2^{\tau-1}(\tau-p)(\int_0^1 c_0(t)\,dt+\sum_{i=1}^k c_i)})^{\frac{1}{\tau-1}}$

hold, then, BVP (1.1) *has at least one positive solution.*

Proof From (A_4), one has $F^+(t,x) \le b_0(t)x^+ + \frac{c_0(t)}{\tau}(x^+)^\tau$ and

$$\begin{aligned}
\varphi_+(x) &\ge \frac{1}{p}\int_0^1 a(t)|x|^p\,dt + \frac{1}{p}\int_0^1 \left|x'\right|^p dt - \int_0^1 F^+(t,x)\,dt - \sum_{i=1}^k \int_0^{x(t_i)} I_i^+(t)\,dt \\
&\ge \frac{1}{p}\int_0^1 a(t)|x|^p\,dt + \frac{1}{p}\int_0^1 \left|x'\right|^p dt - \int_0^1 b_0(t)|x|\,dt - \frac{1}{\tau}\int_0^1 c_0(t)|x|^\tau\,dt \\
&\quad - \sum_{i=1}^k b_i\left|x(t_i)\right| - \sum_{i=1}^k \frac{c_i}{\tau}\left|x(t_i)\right|^\tau \\
&\ge \frac{1}{p}\min\left\{|a|_m,1\right\}\|x\|^p - 2\|x\|\int_0^1 b_0(t)\,dt - \frac{2^\tau}{\tau}\|x\|^\tau\int_0^1 c_0(t)\,dt \\
&\quad - 2\|x\|\sum_{i=1}^k b_i - \frac{2^\tau}{\tau}\|x\|^\tau\sum_{i=1}^k c_i \\
&= \left(\frac{1}{p}\min\left\{|a|_m,1\right\} - 2\left(\int_0^1 b_0(t)\,dt + \sum_{i=1}^k b_i\right)\|x\|^{1-p}\right. \\
&\quad \left. - \frac{2^\tau}{\tau}\left(\int_0^1 c_0(t)\,dt + \sum_{i=1}^k c_i\right)\|x\|^{\tau-p}\right)\|x\|^p.
\end{aligned} \tag{3.10}$$

Let $h(x) = 2(\int_0^1 b_0(t)\,dt + \sum_{i=1}^k b_i)x^{1-p} + \frac{2^\tau}{\tau}(\int_0^1 c_0(t)\,dt + \sum_{i=1}^k c_i)x^{\tau-p}$, then, $\lim_{x\to 0^+} h(x) = \lim_{x\to+\infty} h(x) = +\infty$. Hence, there exists $\bar{x} \in (0,+\infty)$ such that $0 < h(\bar{x}) = \min_{x\in(0,+\infty)} h(x)$. Obviously, $0 = h'(\bar{x}) = 2(1-p)(\int_0^1 b_0(t)\,dt + \sum_{i=1}^k b_i)\bar{x}^{-p} + 2^\tau\frac{\tau-p}{\tau}(\int_0^1 c_0(t)\,dt + \sum_{i=1}^k c_i)\bar{x}^{\tau-p-1}$, then, $\bar{x} = (\frac{\tau(p-1)(\int_0^1 b_0(t)\,dt+\sum_{i=1}^k b_i)}{2^{\tau-1}(\tau-p)(\int_0^1 c_0(t)\,dt+\sum_{i=1}^k c_i)})^{\frac{1}{\tau-1}}$. We infer that there exists an $\eta' > 0$ such that $\varphi_+(x) \ge \eta' > 0$ for all $x \in \{x \in W^{1,p}[0,1], \|x\| = \bar{x}\}$.

Moreover, choose $x(t) > 0$, $t \in (0,1)$, $x \in W^{1,p}[0,1]$, $\int_0^1 |x|^p\,dt = 1$. For, $\forall N_1 > 0$, there exists $M > 0$ such that $\frac{f(t,x)}{x^{p-1}} \ge N_1$ for $x > M$. Choose $N_2 = \|b_0(t)\|_\infty + \|c_0(t)\|_\infty M^{\tau-1}$, one has $f^+(t,x) \ge N_1x^{p-1} - N_2$. Hence, $F^+(t,x) \ge \frac{1}{p}N_1x^p - N_2x$ and

$$\begin{aligned}
\frac{\varphi_+(\lambda x)}{\lambda^p} &\le \frac{1}{p}\int_0^1 a(t)|x|^p\,dt + \frac{1}{p}\int_0^1 \left|x'\right|^p dt - \frac{1}{p}N_1 + \frac{N_2}{\lambda^{p-1}}\int_0^1 x\,dt + \frac{1}{p}\phi_p\left(\frac{\alpha_1}{\alpha_2}\right)\left|x(0)\right|^p \\
&\quad + \frac{1}{p}\phi_p\left(\frac{\beta_1}{\beta_2}\right)\left|x(1)\right|^p.
\end{aligned} \tag{3.11}$$

Since $N_1 > 0$ is arbitrary, we have $\lim_{\lambda\to+\infty}\frac{\varphi_+(\lambda x)}{\lambda^p} = -\infty$, that is, $\lim_{\lambda\to+\infty}\varphi_+(\lambda x) = -\infty$. Hence, from the mountain pass theorem, we obtain $x_0 \in W^{1,p}[0,1]$, such that

$$\varphi_+'(x_0) = 0 \quad\text{and}\quad \varphi_+(x_0) \ge \eta' > 0 = \varphi_+(0). \tag{3.12}$$

It follows $x_0 \not\equiv 0$. If $x_0 \le 0$ for a.e. $t \in [0,1]$, then, $0 = \langle\varphi_+'(x_0), x_0^-\rangle = -\int_0^1 a(t)|x_0^-|^p\,dt - \int_0^1 |(x_0^-)'|^p\,dt - \phi_p(\frac{\alpha_1}{\alpha_2})|x_0^-(0)|^p - \phi_p(\frac{\beta_1}{\beta_2})|x_0^-(1)|^p$. Hence, $x_0^-(t) = 0$ a.e. $t \in [0,1]$, that is $x_0(t) \ge 0$ and $x_0 \not\equiv 0$. This implies that x_0 is a positive solution of BVP (1.1). □

Theorem 3.2 *Assume* (A_1)-(A_5) *and*

(A_6) $I_i(x) \ge d_ix^{\gamma-1}$, $0 < \gamma < p$, $x \ge 0$, $d_i \ge 0$, $i = 1,2,\dots,k$

hold, then, BVP (1.1) *has two positive solutions.*

Proof Assume $B_\rho = \{x \in W^{1,p}[0,1] : \|x\| \le \tilde{x}\}$. Obviously, $\inf_{\bar{B}_\rho} \varphi_+(x) > -\infty$ and

$$\varphi_+(\lambda x) \le \frac{1}{p}\lambda^p \int_0^1 a(t)|x|^p\,dt + \frac{1}{p}\lambda^p \int_0^1 |x'|^p\,dt - \frac{\lambda^\gamma}{\gamma}\sum_{i=1}^k d_i\bigl(x^+(t_i)\bigr)^\gamma + \frac{\lambda^p}{p}\phi_p\left(\frac{\alpha_1}{\alpha_2}\right)\bigl|x(0)\bigr|^p + \frac{\lambda^p}{p}\phi_p\left(\frac{\beta_1}{\beta_2}\right)\bigl|x(1)\bigr|^p.$$

If $\lambda \in (0,1)$ is small enough and x is positive, we have $\varphi_+(\lambda x) < 0$, then,

$$-\infty < \inf_{\bar{B}_\rho} \varphi_+(x) < 0.$$

Let $\varepsilon \in [0, \bar{\rho})$ with $\bar{\rho} = \inf_{\partial B_\rho} \varphi_+ - \inf_{\bar{B}_\rho} \varphi_+$ and consider the functional $\varphi_+ : \bar{B}_\rho \to R$, we can apply Ekeland's variational principle [19] and obtain $x_\varepsilon \in \bar{B}_\rho$ such that

$$\inf_{\bar{B}_\rho} \varphi_+(x) \le \varphi_+(x_\varepsilon) \le \inf_{\bar{B}_\rho} \varphi_+(x) + \varepsilon < \inf_{\bar{B}_\rho} \varphi_+(x) + \bar{\rho} = \inf_{\partial B_\rho} \varphi_+ \tag{3.13}$$

and

$$\varphi_+(x_\varepsilon) \le \varphi_+(y) + \varepsilon\|y - x_\varepsilon\| \quad \text{for all } y \in \bar{B}_\rho.$$

From (3.13), we have $x_\varepsilon \in B_\rho$. Define $\psi_\varepsilon(y) = \varphi_+(y) + \varepsilon\|y - x_\varepsilon\|$, then, $x_\varepsilon \in B_\rho$ is a minimizer of ψ_ε on $\bar{B}_\rho$. Therefore, for small $\lambda > 0$ and all $h \in W^{1,p}[0,1]$ with $\|h\| = 1$, we have

$$\frac{\psi_\varepsilon(x_\varepsilon + \lambda h) - \psi_\varepsilon(x_\varepsilon)}{\lambda} \ge 0,$$

then,

$$\frac{\varphi_+(x_\varepsilon + \lambda h) - \varphi_+(x_\varepsilon)}{\lambda} + \varepsilon\|h\| \ge 0,$$

that is,

$$\langle \varphi_+'(x_\varepsilon), h\rangle \ge -\varepsilon\|h\|. \tag{3.14}$$

Define $\psi_\varepsilon(y) = \varphi_+(y) - \varepsilon\|y - x_\varepsilon\|$, then, $\psi_\varepsilon(y) \le \psi_\varepsilon(x_\varepsilon)$, that is, $x_\varepsilon \in B_\rho$ is a maximum of ψ_ε on $\bar{B}_\rho$. Therefore, for small $\lambda > 0$ and all $h \in W^{1,p}[0,1]$ with $\|h\| = 1$, with the same discussion above, one has

$$\bigl\langle \varphi_+'(x_\varepsilon), h\bigr\rangle \le \varepsilon\|h\|. \tag{3.15}$$

Hence,

$$\bigl\|\varphi_+'(x_\varepsilon)\bigr\| \le \varepsilon. \tag{3.16}$$

Let $\varepsilon_n = \frac{1}{n}$ and set $x_n = x_{\varepsilon_n} \in B_\rho$. Then, $\varphi_+(x_{\varepsilon_n}) \to \inf_{\bar{B}_\rho} \varphi_+(x)$ and $\varphi_+'(x_{\varepsilon_n}) \to 0$. Since $\varphi_+(x)$ satisfies (C) condition, we may assume that $x_n \to \tilde{x}$ in $W^{1,p}[0,1]$. Hence, $\varphi_+'(\tilde{x}) = 0$

$$\varphi_+(\tilde{x}) = \inf_{\bar{B}_\rho} \varphi_+(x) < 0 = \varphi_+(0),$$

which implies that $\tilde{x} \not\equiv 0$ and $\tilde{x}$ is a critical point of φ_+. Moreover,

$$\varphi_+(\tilde{x}) = \inf_{\bar{B}_\rho} \varphi_+(x) < 0 < \eta \le \varphi_+(x_0),$$

so, $\tilde{x} \not\equiv x_0$. If $\tilde{x} \le 0$ a.e. $t \in [0,1]$, with the same discussion in Theorem 3.1, $\tilde{x}^- = 0$ a.e. $t \in [0,1]$. Hence, $\tilde{x} \ge 0$ and $\tilde{x} \not\equiv 0$, which implies $\tilde{x}$ is another positive solution of BVP (1.1). □

With the similar discussion above, we have the following result.

Theorem 3.3 *Assume* (A_1), (A_3)-(A_6) *and*

(A_2') *there exists* $\mu > 1$ *such that for all* $s \in [0,1]$, *we have* $\mu H(t,x) \ge H(t,sx)$ *for a.e.* $t \in [0,1]$, *all* $x \ge 0$, $\mu G_i(x) \ge G_i(sx)$, $i = 1,2,\dots,k$, $x \ge 0$

hold, *then*, *BVP* (1.1) *has at least two positive solutions*.

Theorem 3.4 *Assume that* (A_5) *and*

(B_1) $I_i(x) \ge -b_i - c_i|x|^{\tau-1}$, $b_i, c_i \ge 0$, $i = 1,2,\dots,k$, $x \le 0$, $\tau > p$;
(B_2) $H(t,x) \le H(t,y) + a_0$, $G_i(x) \le G_i(y) + a_0$, $y \le x \le 0$, $a_0 \ge 0$;
(B_3) $\lim_{x\to-\infty} \frac{f(t,x)}{\phi_p(x)} = +\infty$ *for a.e.* $t \in [0,1]$;
(B_4) $f(t,x) \ge -b_0(t) - c_0(t)|x|^{\tau-1}$, $b_0(t), c_0(t) \in C([0,1],[0,\infty))$, $x \le 0$;
(B_5) $I_i(x) \le -d_i|x|^{\gamma-1}$, $\gamma < p$, $x \le 0$, $i = 1,2,\dots,k$

hold, *then*, *BVP* (1.1) *has at least two negative solutions*.

Theorem 3.5 *Assume that* (B_1), (B_3)-(B_5), (A_5) *and*

(B_2') *there exists* $\mu > 1$ *such that for all* $s \in [0,1]$, *we have* $\mu H(t,x) \ge H(t,sx)$ *for a.e.* $t \in [0,1]$, *all* $x \ge 0$, $\mu G_i(x) \ge G_i(sx)$, $i = 1,2,\dots,k$, $x \le 0$

hold, *then*, *BVP* (1.1) *has at least two negative solutions*.

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
All authors contributed equally to the manuscript and read and approved the final manuscript.

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